

Inverse Eigenvalue Problem of Generalized Centro-anti-symmetric Matrices

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Abstract. The inverse eigenvalue problem of constructing centro-anti-symmetric matrices M, C and K of size n for the quadratic pencil $Q(\Lambda) = MX\Lambda^2 + CX\Lambda + KX$ so that has a prescribed subset of eigenvalues and eigenvectors is discussed. A general expression of solution to the problem is provided. The set of such solutions is denoted by S_L . The optimal approximation problem associated with S_L is posed, that is: to find the nearest triple matrix $[\widehat{M}, \widehat{C}, \widehat{K}]$ from S_L . The existence and uniqueness of the optimal approximation problem is discussed and the expression is provided for the nearest triple matrix. A numerical method for solving the problem is given.

Keywords: centro-anti-symmetric matrix, matrix equation, quadratic eigenvalue problem, inverse problem, SVD

1. Introduction

Let $C^{n \times m}$ denote the set of all $n \times m$ complex matrices, $UC^{n \times m}$ denote the set of all $n \times m$ unitary matrices,

A^H is conjugate transpose of matrix A , A^+ is Moore-Penrose generalized inverse matrix of A , I_k is $k \times k$ unit matrix, is Frobenius norm of matrix.

Let $R \in C^{n \times n}$ satisfying $R = R^H = R^{-1} \neq \pm I_n$ be a nontrivial generalized reflexive matrix. $A \in C^{n \times n}$ is said to be generalized centro-anti-symmetric matrices if $RAR = -A$. The set of all $n \times n$ generalized centro-anti-symmetric matrices is denoted by D_n .

In this paper, we discuss the inverse quadratic eigenvalue problem for generalized centro-anti-symmetric matrices as following:

Problem I. Given matrices $X \in C^{n \times m}$, $\Lambda \in C^{m \times m}$, find $M, C, K \in D_n$, such that

$$MX\Lambda^2 + CX\Lambda + KX = 0. \quad (1)$$

Problem II. Given triple matrix $[\widetilde{M}, \widetilde{C}, \widetilde{K}] \in C^{n \times 3n}$, find $[\widehat{M}, \widehat{C}, \widehat{K}] \in S_L$, such that

$$\left\| [\widehat{M}, \widehat{C}, \widehat{K}] - [\widetilde{M}, \widetilde{C}, \widetilde{K}] \right\| = \min_{\forall [M, C, K] \in S_L} \left\| [M, C, K] - [\widetilde{M}, \widetilde{C}, \widetilde{K}] \right\|. \quad (2)$$

where $S_L = \{[M, C, K] \mid MX\Lambda^2 + CX\Lambda + KX = 0, M, C, K \in D_n\}$ is set of solution for Problem I.

2. The solution Problem I

Lemma 1. Suppose $R \in C^{n \times n}$ be a nontrivial generalized reflexive matrix, the spectral decomposition of R as follow:^[1,2]

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$$R = [P, Q] \begin{bmatrix} I_k & O \\ O & -I_{n-k} \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix}. \quad (3)$$

where $[P, Q] \in UC^{n \times n}$. Then the necessary and sufficient conditions of $A \in D_n$ as follow:

$$A = [P, Q] \begin{bmatrix} O & A_{12} \\ A_{21} & O \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix}. \quad (4)$$

where $A_{12} = Q^H AP$, $A_{21} = P^H AQ$.

Proof. Suppose $A \in D_n$, by (3) and $RAR = -A$, we have

$$\begin{bmatrix} P^H \\ Q^H \end{bmatrix} A [P, Q] = - \begin{bmatrix} I_k & O \\ O & -I_{n-k} \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix} A [P, Q] \begin{bmatrix} I_k & O \\ O & -I_{n-k} \end{bmatrix}. \quad (5)$$

Let

$$\begin{bmatrix} P^H \\ Q^H \end{bmatrix} A [P, Q] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \quad (6)$$

By (5) and (6), we have

$$A_{11} = 0, A_{22} = 0, A_{12} = P^H AQ, A_{21} = Q^H AP. \quad (7)$$

Therefore by (6) and (7), we have

$$A = [P, Q] \begin{bmatrix} O & A_{12} \\ A_{21} & O \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix}. \quad (8)$$

where $A_{12} = P^H AQ$, $A_{21} = Q^H AP$.

On the other hand, for any matrices $A_{12} \in C^{k \times (n-k)}$, $A_{21} \in C^{(n-k) \times k}$, we have

$$\begin{aligned} R \begin{bmatrix} O & A_{12} \\ A_{21} & O \end{bmatrix} R &= [P, Q] \begin{bmatrix} I_k & O \\ O & -I_{n-k} \end{bmatrix} \begin{bmatrix} O & A_{12} \\ A_{21} & O \end{bmatrix} \begin{bmatrix} I_k & O \\ O & -I_{n-k} \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix} \\ &= [P, Q] \begin{bmatrix} O & -A_{12} \\ -A_{21} & O \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix} = -A. \end{aligned}$$

Thus, we have $A \in D_n$.

Lemma 2 ^[2]. Given matrices $X, Y, Z, F \in C^{n \times m}$, $t = \text{rank} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$. Suppose

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = U \begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix} V^H = U_1 \Sigma V_1^H, \quad (9)$$

where

$$U = [U_1, U_2] \in UC^{3n \times 3n}, U_1 \in C^{3n \times t}, V = [V_1, V_2] \in UC^{m \times m}, V_1 \in C^{m \times t}, \Sigma = \text{diag}(\sigma_1, \dots, \sigma_t).$$

The necessary and sufficient condition of the solution exist for matrix equation $AX + BY + CZ = F$ is

$$FV_2 = 0. \quad (10)$$

and when the solution of matrix equation existence, the set of general solution can write as follow:

$$[A, B, C] = F \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}^+ + G U_2^H = F V_1 \Sigma^{-1} U_1^H + G U_2^H. \tag{11}$$

where $G \in C^{n \times (3n-t)}$ is any matrix.

Let $[P, Q] = N, [P, Q]^H = N^H$, then we have

Theorem 1. Given matrices $X, Y, Z, F \in C^{n \times m}$, let

$$N^H X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, N^H Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, N^H Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, N^H F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{12}$$

where $X_1, Y_1, Z_1, F_2 \in C^{k \times m}$, $X_2, Y_2, Z_2, F_1 \in C^{(n-k) \times m}$. Suppose the SVD of triple matrix

$\begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}$ ($i = 1, 2$) as follow:

$$\begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} = U_i \begin{pmatrix} \Sigma_i & O \\ O & O \end{pmatrix} V_i^H, i = 1, 2, \tag{13}$$

where

$$\Sigma_i = (\lambda_{1i}, \dots, \lambda_{t_i i}), t_i = \text{rank} \left(\begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} \right), U_1 = [U_{11}, U_{12}] \in UC^{3k \times 3k}, U_{11} \in C^{3k \times t_1},$$

$$U_2 = [U_{21}, U_{22}] \in UC^{3(n-k) \times 3(n-k)}, U_{21} \in C^{3(n-k) \times t_2}, V_i = [V_{i1}, V_{i2}] \in UC^{m \times m}, V_{i1} \in UC^{m \times t_i}.$$

Then the necessary and sufficient conditions of centro-anti-symmetric solution for matrix equation $AX + BY + CZ = F$ as follow:

$$F_1 V_{22} = 0, F_2 V_{12} = 0. \tag{14}$$

and when the solution exist for (1), the generalized solution can write as follow:

$$A = N \begin{bmatrix} O & A_2 \\ A_1 & O \end{bmatrix} N^H, B = N \begin{bmatrix} O & B_2 \\ B_1 & O \end{bmatrix} N^H, C = N \begin{bmatrix} O & C_2 \\ C_1 & O \end{bmatrix} N^H. \tag{15}$$

where

$$[A_1, B_1, C_1] = F_2 \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}^+ + G_1 U_{12}^H = E_1 + G_1 U_{12}^H, [A_2, B_2, C_2] = F_1 \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}^+ + G_2 U_{22}^H = E_2 + G_2 U_{22}^H, \tag{16}$$

where $G_1 \in C^{(n-k) \times (3(n-k)-t_1)}$, $G_2 \in C^{k \times (3k-t_2)}$ is any matrix.

Proof. By Lemma 1, we have easily matrix equation (1) equivalence with

$$N \begin{bmatrix} O & A_2 \\ A_1 & O \end{bmatrix} N^H X + N \begin{bmatrix} O & B_2 \\ B_1 & O \end{bmatrix} N^H Y + N \begin{bmatrix} O & C_2 \\ C_1 & O \end{bmatrix} N^H Z = F. \tag{17}$$

where $A_1, B_1, C_1 \in C^{(n-k) \times k}$, $A_2, B_2, C_2 \in C^{k \times (n-k)}$. Then equation (13) equivalence with

$$A_1 X_1 + B_1 Y_1 + C_1 Z_1 = F_2, A_2 X_2 + B_2 Y_2 + C_2 Z_2 = F_1. \tag{18}$$

therefore, Problem I equivalence with matrix equation (18). By Lemma 2, the solution exist for (18) equivalence with

$$F_1V_{22} = 0, F_2V_{12} = 0. \tag{19}$$

and when the solution exist for (18), the generalized solution can write as follow:

$$A = N \begin{bmatrix} O & A_2 \\ A_1 & O \end{bmatrix} N^H, B = N \begin{bmatrix} O & B_2 \\ B_1 & O \end{bmatrix} N^H, C = N \begin{bmatrix} O & C_2 \\ C_1 & O \end{bmatrix} N^H, \tag{20}$$

where

$$[A_1, B_1, C_1] = F_2 \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}^+ + G_1U_{12}^H = E_1 + G_1U_{12}^H, [A_2, B_2, C_2] = F_1 \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}^+ + G_2U_{22}^H = E_2 + G_2U_{22}^H. \tag{21}$$

where $G_1 \in C^{(n-k) \times (3(n-k)-t_1)}$, $G_2 \in C^{k \times (3k-t_2)}$, is any matrix.

$$\text{If let } F_2 \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}^+ + G_1U_{12}^H = E_1 + G_1U_{12}^H, F_1 \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}^+ + G_2U_{22}^H = E_2 + G_2U_{22}^H.$$

By matrix computation and Theorem 1, we have

$$[A, B, C] = N \begin{pmatrix} O & E_2 + G_2U_{22}^H \\ E_1 + G_1U_{12}^H & O \end{pmatrix} P^H \begin{pmatrix} N^H & O & O \\ O & N^H & O \\ O & O & N^H \end{pmatrix}. \tag{22}$$

where

$$P = \begin{pmatrix} I_s & O & O & O & O & O \\ O & O & O & I_r & O & O \\ O & I_s & O & O & O & O \\ O & O & O & O & I_r & O \\ O & O & I_s & O & O & O \\ O & O & O & O & O & I_r \end{pmatrix}.$$

Theorem 2. Given matrices $X \in C^{n \times m}$, $\Lambda \in C^{m \times m}$, let

$$N^H X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \tag{23}$$

where $X_1 \in C^{k \times m}$, $X_2 \in C^{(n-k) \times m}$. Suppose the SVD of triple matrix $\begin{bmatrix} X_i \Lambda^2 \\ X_i \Lambda \\ X_i \end{bmatrix}$ as follow:

$$\begin{bmatrix} X_i \Lambda^2 \\ X_i \Lambda \\ X_i \end{bmatrix} = U_i \begin{pmatrix} \Sigma_i & O \\ O & O \end{pmatrix} V_i^H, i = 1, 2, \tag{24}$$

where $\Sigma_i = (\lambda_{1i}, \dots, \lambda_{t_i i})$, $t_i = \text{rank} \left(\begin{bmatrix} X_i \Lambda^2 \\ X_i \Lambda \\ X_i \end{bmatrix} \right)$, $U_1 = [U_{11}, U_{12}] \in UC^{3k \times 3k}$, $U_{11} \in C^{3k \times t_1}$,

$U_2 = [U_{21}, U_{22}] \in UC^{3(n-k) \times 3(n-k)}$, $U_{21} \in C^{3(n-k) \times t_2}$, $V_i = [V_{i1}, V_{i2}] \in UC^{m \times m}$, $V_{i1} \in UC^{m \times t_i}$, the solution of matrix equation $MX\Lambda^2 + CX\Lambda + KX = 0$ can write as follow:

$$[M, C, K] = N \begin{pmatrix} O & G_2 U_{22}^H \\ G_1 U_{12}^H & O \end{pmatrix} P^H \begin{pmatrix} N^H & O & O \\ O & N^H & O \\ O & O & N^H \end{pmatrix}. \tag{25}$$

where $G_1 \in C^{k \times (3k-t_1)}$, $G_2 \in C^{(n-k) \times (3(n-k)-t_2)}$ is any matrices.

Proof. Given matrices $X \in C^{n \times m}$, $\Lambda \in C^{m \times m}$, let

$$N^H X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \tag{26}$$

where $X_1 \in C^{k \times m}$, $X_2 \in C^{(n-k) \times m}$. Suppose the svd of triple matrix $\begin{bmatrix} X_i \Lambda^2 \\ X_i \Lambda \\ X_i \end{bmatrix}$ as follow:

$$\begin{bmatrix} X_i \Lambda^2 \\ X_i \Lambda \\ X_i \end{bmatrix} = U_i \begin{pmatrix} \Sigma_i & O \\ O & O \end{pmatrix} V_i^H, i = 1, 2, \tag{27}$$

where $\Sigma_i = (\lambda_{1i}, \dots, \lambda_{t_i i})$ $t_i = rank \left(\begin{bmatrix} X_i \Lambda^2 \\ X_i \Lambda \\ X_i \end{bmatrix} \right)$, $U_1 = [U_{11}, U_{12}] \in UC^{3k \times 3k}$, $U_{11} \in C^{3k \times t_1}$,

$$U_2 = [U_{21}, U_{22}] \in UC^{3(n-k) \times 3(n-k)}$$
, $U_{21} \in C^{3(n-k) \times t_2}$, $V_i = [V_{i1}, V_{i2}] \in UC^{m \times m}$, $V_{i1} \in UC^{m \times t_i}$.

By Lemma 2, the matrix equation has solution, the solution can write as (25).

3. The solution Problem II

For given matrices $\tilde{M}, \tilde{C}, \tilde{K} \in C^{n \times n}$, we denote

$$N^H [\tilde{M}, \tilde{C}, \tilde{K}] \begin{pmatrix} N^H & O & O \\ O & N^H & O \\ O & O & N^H \end{pmatrix} P = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \tag{28}$$

Theorem 3. For given matrices $\tilde{M}, \tilde{C}, \tilde{K} \in C^{n \times n}$, $X \in C^{n \times m}$, and $\Lambda \in C^{m \times m}$, then exist unique $[\hat{M}, \hat{C}, \hat{K}] \in S_L$ such that (2) hold in certain, and $[\hat{M}, \hat{C}, \hat{K}]$ can write as follow:

$$[\hat{M}, \hat{C}, \hat{K}] = N \begin{pmatrix} O & \tilde{A}_{12} U_{22} U_{22}^H \\ \tilde{A}_{21} U_{12} U_{12}^H & O \end{pmatrix} P^H \begin{pmatrix} N^H & O & O \\ O & N^H & O \\ O & O & N^H \end{pmatrix}. \tag{29}$$

Proof. We can proof easily S_L is closed convex set of real space $C^{n \times 3n}$, for given matrices $\tilde{M}, \tilde{C}, \tilde{K} \in C^{n \times n}$, $X \in C^{n \times m}$, and $\Lambda \in C^{m \times m}$, we have unique $[\hat{M}, \hat{C}, \hat{K}] \in S_L$, such that (2) hold, by (25), (28), and keep norm constant property of unitary matrix, we have

$$\| [M, C, K] - [\tilde{M}, \tilde{C}, \tilde{K}] \|^2$$

$$\begin{aligned}
 &= \left\| N \begin{pmatrix} O & G_2 U_{22}^H \\ G_1 U_{12}^H & O \end{pmatrix} P^H \begin{pmatrix} N^H & O & O \\ O & N^H & O \\ O & O & N^H \end{pmatrix} - [\widetilde{M}, \widetilde{C}, \widetilde{K}] \right\|^2 \\
 &= \left\| \begin{pmatrix} O & G_2 U_{22}^H \\ G_1 U_{12}^H & O \end{pmatrix} - \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix} \right\|^2 \\
 &= \left\| \begin{pmatrix} -\widetilde{A}_{11} & G_2 U_{22}^H - \widetilde{A}_{12} \\ G_1 U_{12}^H - \widetilde{A}_{21} & -\widetilde{A}_{22} \end{pmatrix} \right\|^2 \\
 &= \|G_1 U_{12}^H - \widetilde{A}_{21}\|^2 + \|G_2 U_{22}^H - \widetilde{A}_{12}\|^2 + \|\widetilde{A}_{11}\|^2 + \|\widetilde{A}_{22}\|^2 \\
 &= \|G_1 - A_{21} U_{12}\|^2 + \|G_2 - A_{12} U_{22}\|^2 + \|A_{11}\|^2 + \|A_{22}\|^2
 \end{aligned}$$

then $\min \{ \|[\widetilde{M}, \widetilde{C}, \widetilde{K}] - [M, C, K]\| \}$ equivalence with

$$\min \{ \|G_1 - \widetilde{A}_{21} U_{12}\| \}, \text{ and } \min \{ \|G_2 - \widetilde{A}_{12} U_{22}\| \}. \tag{30}$$

by (30), we have

$$G_1 = \widetilde{A}_{21} U_{12}, \text{ and } G_2 = \widetilde{A}_{12} U_{22}. \tag{31}$$

Therefore, we have easily (29) hold.

4. The numerical method and example

For Problem I and Problem II, we obtain the following encapsulation:

Algorithm1.

1. To partition the given matrix for X as (23);
2. Compute SVD of matrix $\begin{bmatrix} X_i \Lambda^2 \\ X_i \Lambda \\ X_i \end{bmatrix} (i = 1, 2)$ as (24);
3. For given matrices $\widetilde{M}, \widetilde{C}, \widetilde{K}$, by (28) to compute matrices $\widetilde{A}_{12}, \widetilde{A}_{21}$;
4. By (29) to compute matrices $\widehat{M}, \widehat{C}, \widehat{K}$.

Example 1. We random selection matrices $R, \widetilde{M}, \widetilde{C}, \widetilde{K}, X, \Lambda$ as follows:

$$R = \begin{bmatrix} 0.0970 & 0.8017 & 0.4754 & 0.3488 \\ 0.8017 & 0.1919 & -0.0760 & -0.5608 \\ 0.4754 & -0.0760 & -0.4981 & 0.7213 \\ 0.3488 & -0.5608 & 0.7213 & 0.2089 \end{bmatrix}, \widetilde{M} = \begin{bmatrix} 0.2722 & 0.4451 & 0.8462 & 0.8381 \\ 0.1988 & 0.9318 & 0.5252 & 0.0196 \\ 0.0153 & 0.4660 & 0.2026 & 0.6813 \\ 0.7468 & 0.4186 & 0.6721 & 0.3795 \end{bmatrix},$$

$$\widetilde{C} = \begin{bmatrix} 0.8318 & 0.3046 & 0.3028 & 0.3784 \\ 0.5028 & 0.1897 & 0.5417 & 0.8600 \\ 0.7095 & 0.1934 & 0.1509 & 0.8537 \\ 0.4289 & 0.6822 & 0.6979 & 0.5936 \end{bmatrix}, \widetilde{K} = \begin{bmatrix} 0.4966 & 0.8180 & 0.3412 & 0.8385 \\ 0.8998 & 0.6602 & 0.5341 & 0.5681 \\ 0.8216 & 0.3420 & 0.7271 & 0.3704 \\ 0.6449 & 0.2897 & 0.3093 & 0.7027 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.8939 & 0.2844 & 0.5828 \\ 0.1991 & 0.4692 & 0.4235 \\ 0.2987 & 0.0648 & 0.5155 \\ 0.6614 & 0.9883 & 0.3340 \end{bmatrix}, \Lambda = \begin{bmatrix} 0.7948 & 0 & 0 \\ 0 & -0.2714 & 0.7373 \\ 0 & -0.7373 & -0.2714 \end{bmatrix}.$$

By algorithm 1, we have the solution for Problem II as follow:

$$\hat{M} = \begin{bmatrix} -0.5259 & 0.0982 & 0.2375 & 0.3873 \\ -0.1767 & 0.5748 & 0.0451 & -0.0647 \\ -0.1751 & 0.1145 & 0.0837 & 0.0986 \\ 0.0006 & 0.3245 & -0.0073 & -0.1325 \end{bmatrix}, \hat{C} = \begin{bmatrix} 0.0562 & 0.1112 & -0.3115 & 0.0911 \\ 0.0567 & -0.0970 & -0.2096 & 0.2138 \\ -0.0082 & 0.0749 & -0.0997 & 0.0400 \\ -0.0725 & -0.0220 & 0.0187 & 0.1405 \end{bmatrix},$$

$$\hat{K} = \begin{bmatrix} -0.0534 & -0.0049 & -0.0507 & 0.1669 \\ -0.0459 & 0.1303 & -0.0577 & -0.0141 \\ 0.0164 & 0.0375 & -0.0677 & 0.0334 \\ 0.0263 & 0.0335 & -0.1065 & -0.0092 \end{bmatrix}.$$

5. References

- [1] Chen H.C. Generalized reflexive matrices: special properties and applications. SIAM. J. Matrix Anal. Appl. 1998; 19: 140-153
- [2] Gui B., Dai H. The Centrosymmetric Solution of the Inverse Quadratic Eigenvalue Problem and its Optimal Approximation. Numerical Mathematics A journal of Chinese universities 2006; 28(4):367-373
- [3] Zhou S., Wu B. S. Inverse Generalized Eigenvalue Problems for Anti-centrosymmetric Matrices. Numerical Mathematics A journal of Chinese universities 2005; 27(1):53-59
- [4] Wang X. R., Yuan Y. D. The Optimal Approximation Solutions of Inverse Problem for Generalized Anti-Hamilton Matrices on the Linear Manifold. Proceedings of The 14th Conference of International Linear Algebra Society. World Academic Press 2007; 304-307. ISTP.
- [5] Yuan Y. D., Wang X. R. Inverse Eigenvalue Problem for Anti-centrosymmetric Matrices. Proceedings of The 14th Conference of International Linear Algebra Society. World Academic Press 2007; 398-401. ISTP.
- [6] Yuan Y. D., Wang X. R., Ai S. M. The Best Approximation Solutions of In Problem of Generalized Anti-Hamilton Matrices. Advances in Matrix Theory and Applications. World Academic Press 2006; 176-179. ISTP.
- [7] Tisseur F., Meerbergen K. The quadratic eigenvalue problem. SIAM Review 2001; 43:235-286
- [8] Mottershead J. E., Friswell M. I. Model updating in structural dynamics: a survey. J. of Sound vibration 1993; 167:347-375
- [9] Weaver J. R. Centrosymmetric matrices, their basic properties, eigenvalues, and eigenvectors. Amer. Math. Monthly 1985; 92:711-717
- [10] Wang Y. F. Computational Methods for Inverse Problems and Their Applications. HIGHER EDUCATION PRESS, 2007.
- [11] Chen J. L., Chen X. H. Special Matrices. Tsinghua Press, 2001.