

# The Computation of a Common Quadratic Lyapunov Function for a Linear Control System\*

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**Abstract.** In this paper, we consider the existence of common quadratic Lyapunov functions for a linear control system. For stable LTI systems whose system matrices are upper triangular form, or lower triangular form, it is proved that there exist common quadratic Lyapunov functions. Then, two algorithms of computing a common Lyapunov function are given. Finally, two examples are listed.

**Key words:** common quadratic Lyapunov functions, switched systems

## 1. Introduction

In recent years, there has been increasing attention given to the stability and stabilization of switched systems. The stability of a switched system can be assured by a common Lyapunov function of the different models for arbitrary switching. When the switching models are linear time-invariant, the work of determining conditions for the existence of a common quadratic Lyapunov function, or the methods of computing a common quadratic Lyapunov function is done in [1,2]. In [3], it is proved that a set of block upper triangular matrices share a common quadratic Lyapunov function, if and only if each set of diagonal blocks share a common quadratic Lyapunov function.

In this note, we will show that when the system matrices are upper triangular form or lower triangular form, there exists a common quadratic Lyapunov function  $V(x) = x^T P x$  where  $P > 0$  is a diagonal matrix. Based on this result, the algorithms of a common quadratic Lyapunov function are given.

## 2. Preliminaries

We start with some notations which will be used later on. Throughout,  $R^{n \times n}$  denotes the space of  $n \times n$  matrices with real entries.  $x_i$  is  $i$ th component of the vector  $x$  in  $R^n$ .  $a_{ij}^i$  stands for entry in the  $(i, j)$  position of the matrix  $A_i$  in  $R^{n \times n}$ .

Let us consider the set of LTI systems

$$\Sigma_{A_i} : \dot{x} = A_i x, \quad i = 1, 2, \dots, m, \quad (1)$$

where  $m$  is finite and the  $A_i$ ,  $i = 1, 2, \dots, m$  are constant Hurwitz matrices in  $R^{n \times n}$  (i.e., the eigenvalues of  $A_i$  lie in the open left half of the complex plane and hence the  $\Sigma_{A_i}$  are stable LTI systems). Let the matrix  $P = P^T > 0$ ,  $P \in R^{n \times n}$ , be a simultaneous solution to the Lyapunov inequations

$$A_i^T P + P A_i < 0, \quad i = 1, 2, \dots, m, \quad (2)$$

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then  $V(x) = x^T Px$  is called a common quadratic Lyapunov function for the systems (1).

Since the system matrices are upper triangular form or lower triangular form, we can write

$$A_i = \begin{bmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{bmatrix} \text{ or } A_i = \begin{bmatrix} a_{11}^i & 0 & \cdots & 0 \\ a_{21}^i & a_{22}^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^i & a_{n2}^i & \cdots & a_{nn}^i \end{bmatrix}, i = 1, 2, \dots, m.$$

### 3. Main Results

**Theorem 3.1.** Let  $A_i, i = 1, 2, \dots, m$  be Hurwitz matrices in  $R^{n \times n}$  such that  $A_i$  are upper triangular matrices. Then the systems  $\Sigma_{A_i} : \dot{x} = A_i x, i = 1, 2, \dots, m$  share a common quadratic Lyapunov function  $V(x) = x^T Px$ , where

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{n-1} \end{bmatrix} > 0.$$

*Proof.* Noticing that  $A_i$  is Hurwitz, we get  $a_{jj}^i < 0, j = 1, 2, \dots, n$  and  $Rank(A_i) = n$ . We prove the theorem by mathematical induction. When  $n = 2$ , we have

$$A_i = \begin{bmatrix} a_{11}^i & a_{12}^i \\ 0 & a_{22}^i \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ 0 & p_1 \end{bmatrix} > 0$$

and

$$A_i^T P + PA_i = \begin{bmatrix} 2a_{11}^i & a_{12}^i \\ a_{12}^i & 2a_{22}^i p_1 \end{bmatrix} = D_2^i < 0, i = 1, 2, \dots, m. \text{ Let } \det D_2^i > 0,$$

it is obtained that  $p_1 > \frac{(a_{12}^i)^2}{4a_{11}^i a_{22}^i}$ . For any  $p_1 > \max_{1 \leq i \leq m} \frac{(a_{12}^i)^2}{4a_{11}^i a_{22}^i}$ ,  $A_i^T P + PA_i$  are always negative definite.

Thus, we get a common quadratic Lyapunov function

$$V(x) = x^T Px,$$

where

$$P = \begin{bmatrix} 1 & 0 \\ 0 & p_1 \end{bmatrix} > 0$$

with

$$p_1 > \max_{1 \leq i \leq m} \frac{(a_{12}^i)^2}{4a_{11}^i a_{22}^i}.$$

Assume that for  $n = k \geq 2$ , the conclusion is true. When  $n = k + 1$ , we have

$$A_i = \begin{bmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1,k+1}^i \\ 0 & a_{22}^i & \cdots & a_{2,k+1}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{k+1,k+1}^i \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k \end{bmatrix} > 0$$

and

$$A_i^T P + PA_i = \begin{bmatrix} 2a_{11}^i & a_{12}^i & \cdots & a_{1,k+1}^i \\ a_{12}^i & 2a_{22}^i p_1 & \cdots & a_{2,k+1}^i \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,k+1}^i & a_{2,k+1}^i & \cdots & 2a_{k+1,k+1}^i p_k \end{bmatrix} = D_{k+1}^i < 0.$$

In addition, we denote  $E_k^i = (a_{1,k+1}^i, a_{2,k+1}^i, \dots, a_{k,k+1}^i)^T$ . By deduce, it follows

$$\begin{aligned} \det D_{k+1}^i &= \det \begin{bmatrix} D_k^i & E_k^i \\ (E_k^i)^T & a_{k+1,k+1}^i \end{bmatrix} + 2 \det \begin{bmatrix} D_k^i & 0 \\ (E_k^i)^T & a_{k+1,k+1}^i \end{bmatrix} p_k \\ &= -\det D_k^i (E_k^i)^T (D_k^i)^{-1} E_k^i + 2 \det D_k^i a_{k+1,k+1}^i p_k \end{aligned}$$

Assume  $k$  is odd, then  $\det D_k^i < 0$  and  $\det D_{k+1}^i > 0$ . From the equation above, we obtain  $p_k > \frac{(E_k^i)^T (D_k^i)^{-1} E_k^i}{2a_{k+1,k+1}^i}$ . For any  $p_k > \max_{i \leq i \leq m} \frac{(E_k^i)^T (D_k^i)^{-1} E_k^i}{2a_{k+1,k+1}^i}$ ,  $A_i^T P + PA_i$  are always negative define.

Thus, we obtain a common quadratic Lyapunov function

$$V(x) = x^T P x,$$

where  $P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k \end{bmatrix} > 0$  with  $p_k > \max_{i \leq i \leq m} \frac{(E_k^i)^T (D_k^i)^{-1} E_k^i}{2a_{k+1,k+1}^i}$ . This completes the proof of the

theorem.

**Theorem 3.2.** Let  $A_i, i = 1, 2, \dots, m$  be Hurwitz matrices in  $R^{n \times n}$  such that  $A_i$  are lower triangular matrices, then the systems  $\Sigma_{A_i} : \dot{x} = A_i x, i = 1, 2, \dots, m$  share a common quadratic Lyapunov function  $V(x) = x^T P x$ , where

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{n-1} \end{bmatrix}.$$

*Proof.* Since  $A_i$  is Hurwitz, we get  $a_{jj}^i < 0, j = 1, 2, \dots, n$  and  $Rank(A_i) = n$ . We prove the theorem by mathematical induction. When  $n = 2$ , We have

$$P = \begin{bmatrix} 1 & 0 \\ 0 & p_1 \end{bmatrix}, A_i^T P + PA_i < 0,$$

and  $D_2 = \begin{bmatrix} 2a_{11}^i & a_{21}^i p_1 \\ a_{21}^i p_1 & 2a_{22}^i p_1 \end{bmatrix} < 0, i = 1, 2, \dots, m$ . Let  $\det \begin{bmatrix} 2a_{11}^i & a_{21}^i p_1 \\ a_{21}^i p_1 & 2a_{22}^i p_1 \end{bmatrix} > 0$ . We obtain  $p_1 < \frac{(a_{21}^i)^2}{4a_{11}^i a_{22}^i}$ .

For any  $0 < p_1 < \min_{1 \leq i \leq m} \frac{(a_{21}^i)^2}{4a_{11}^i a_{22}^i}$ ,  $A_i^T P + PA_i$  are always negative define. Thus, we get a common quadratic

Lyapunov function  $V(x) = x^T P x$ , where  $P = \begin{bmatrix} 1 & 0 \\ 0 & p_1 \end{bmatrix} > 0$  with  $0 < p_1 < \min_{1 \leq i \leq m} \frac{(a_{21}^i)^2}{4a_{11}^i a_{22}^i}$ .

Assume that for  $n = k \geq 2$ , the theorem is true. When  $n = k + 1$ , we have that

$$A_i = \begin{bmatrix} a_{11}^i & 0 & \cdots & 0 \\ a_{21}^i & a_{22}^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1}^i & a_{k+1,2}^i & \cdots & a_{k+1,k+1}^i \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k \end{bmatrix} > 0$$

and

$$A_i^T P + PA_i < 0. \text{ i.e. } D_{k+1}^i = \begin{bmatrix} D_k^i & F_k^i p_k \\ (F_k^i)^T p_k & 2a_{k+1,k+1}^i p_k \end{bmatrix} < 0.$$

Denote  $F_k^i = (a_{k+1,1}^i, a_{k+1,2}^i, \dots, a_{k+1,k}^i)^T$ . Without loss of generality, we assume  $(F_k^i)^T (D_k^i)^{-1} F_k^i \neq 0$ .

Thus,

$$\begin{aligned} \det D_{k+1}^i &= 2 \det \begin{bmatrix} D_k^i & F_k^i \\ 0 & a_{k+1,k+1}^i \end{bmatrix} p_k + \det \begin{bmatrix} D_k^i & F_k^i \\ (F_k^i)^T & 0 \end{bmatrix} p_k^2 \\ &= 2 \det(D_k^i) a_{k+1,k+1}^i p_k - \det(D_k^i) ((F_k^i)^T (D_k^i)^{-1} F_k^i) p_k^2 \end{aligned}$$

Assume  $k$  is odd, then  $\det(D_k^i) < 0$  and  $\det D_{k+1}^i > 0$ . From the equation above, we obtain

$$p_k < \frac{2a_{k+1,k+1}^i}{(F_k^i)^T (D_k^i)^{-1} F_k^i}. \text{ Assume } k \text{ is even, then } \det(D_k^i) > 0 \text{ and } \det D_{k+1}^i < 0. \text{ We}$$

obtain  $p_k < \frac{2a_{k+1,k+1}^i}{(F_k^i)^T (D_k^i)^{-1} F_k^i}$ . For any  $0 < p_k < \min_{1 \leq i \leq m} \frac{2a_{k+1,k+1}^i}{(F_k^i)^T (D_k^i)^{-1} F_k^i}$ ,  $A_i^T P + PA_i$  are always negative

define. Thus we get a common quadratic Lyapunov function  $V(x) = x^T P x$ , where

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k \end{bmatrix} > 0 \text{ with } 0 < p_k < \min_{1 \leq i \leq m} \frac{2a_{k+1,k+1}^i}{(F_k^i)^T (D_k^i)^{-1} F_k^i}. \text{ This completes the proof of the theorem.}$$

### 4. Algorithms

Denote

$$M_k = \max_{i \leq i \leq m} \frac{(E_k^i)^T (D_k^i)^{-1} E_k^i}{2a_{k+1,k+1}^i}$$

$$U_k = (M_k, +\infty)$$

$$N_k = \min_{1 \leq i \leq m} \frac{2a_{k+1,k+1}^i}{(F_k^i)^T (D_k^i)^{-1} F_k^i}.$$

$$L_k = (0, N_k)$$

Based on the proof of Theorems 3.1 and 3.2, two algorithms of computing a common quadratic Lyapunov functions are presented.

**Algorithm 4.1** ( $A_i$  is an upper triangular matrix)

Step 1 Compute  $\max_{1 \leq i \leq m} \frac{(a_{12}^i)^2}{4a_{11}^i a_{22}^i}$ . For any  $p_1 \in (\max_{1 \leq i \leq m} \frac{(a_{12}^i)^2}{4a_{11}^i a_{22}^i}, +\infty)$ , output  $p_1$ ;

Step 2 Set  $k = 2$ ;

Step 3 Compute  $M_k$ . For any  $p_k \in U_k$ , output  $p_k$ ;

Step 4 If  $k < n - 1$ , then goto step 3, otherwise stop;

Step 5 Output  $V(x) = x_1^2 + p_1 x_2^2 + \dots + p_{n-1} x_n^2$ .

**Algorithm 4.2** ( $A_i$  is a lower triangular matrix)

Step 1 Compute  $\min_{1 \leq i \leq m} \frac{(a_{21}^i)^2}{4a_{11}^i a_{22}^i}$ . For any  $p_1 \in (0, \min_{1 \leq i \leq m} \frac{(a_{21}^i)^2}{4a_{11}^i a_{22}^i})$ , output  $p_1$ ;

Step 2 Set  $k = 2$ ;

Step 3 Compute  $N_k$ . For any  $p_k \in L_k$ , output  $p_k$ ;

Step 4 If  $k < n - 1$ , then go to step 3, otherwise stop;

Step 5 Output  $V(x) = x_1^2 + p_1 x_2^2 + \dots + p_{n-1} x_n^2$ .

Now, we give examples to illustrate the main results.

**Example 4.1** In systems (1), let  $m = 2$ . Let

$$A_1 = \begin{bmatrix} -1 & -2 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Obviously,  $V(x) = x_1^2 + p_1 x_2^2 + p_2 x_2^2$ . Next we will find  $p_1$  and  $p_2$ . We compute

$$\max \left\{ \frac{(a_{12}^1)^2}{4a_{11}^1 a_{22}^1}, \frac{(a_{12}^2)^2}{4a_{11}^2 a_{22}^2} \right\} = \max \left\{ \frac{1}{2}, \frac{1}{8} \right\} \text{ and get } \left( \frac{1}{2}, +\infty \right). \text{ We let } p_1 = 1.$$

$$M_2 = \max \left\{ \frac{(E_2^1)^T (D_2^1)^{-1} E_2^1}{2a_{33}^1}, \frac{(E_2^2)^T (D_2^2)^{-1} E_2^2}{2a_{33}^2} \right\} = \max \left\{ \frac{1}{8}, \frac{8}{7} \right\}$$

and

$$U_2 = \left( \frac{8}{7}, +\infty \right).$$

Let  $p_2 = 2$ . Hence we find a common quadratic Lyapunov function

$$V(x) = x_1^2 + x_2^2 + 2x_2^2.$$

**Example 4.2** In systems (1), let  $m = 2$ . Let

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 3 & -1 \end{bmatrix}.$$

Evidently,  $V(x) = x_1^2 + p_1 x_2^2 + p_2 x_2^2$ . Next we will find  $p_1$  and  $p_2$ . We get

$$\min \left\{ \frac{(a_{21}^1)^2}{4a_{11}^1 a_{22}^1}, \frac{(a_{21}^2)^2}{4a_{11}^2 a_{22}^2} \right\} = \min \left\{ \frac{1}{2}, \frac{1}{16} \right\} \text{ and } L_1 = \left( 0, \frac{1}{16} \right), \text{ Let } p_1 = \frac{1}{20}.$$

$$N_2 = \min \left\{ \frac{2a_{33}^1}{(F_2^1)^T (D_2^1)^{-1} F_2^1}, \frac{2a_{33}^2}{(F_2^2)^T (D_2^2)^{-1} F_2^2} \right\} = \min \left\{ \frac{13}{40}, \frac{319}{8020} \right\} \text{ and } L_2 = \left( 0, \frac{319}{8020} \right). \text{ Let } p_2 = \frac{1}{401}.$$

Hence we find a common quadratic Lyapunov function  $V(x) = x_1^2 + \frac{1}{20} x_2^2 + \frac{1}{401} x_2^2$ .

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