A Computational Algorithm to Obtain Positive Solutions for Classes of Competitive Systems

G. A. Afrouzi *, S. Mahdavi, Z. Naghizadeh

Department of Mathematics, Faculty of Basic Sciences, Mazandaran University, Babolsar, Iran

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Abstract. Using a numerical method based on sub-super solution, we will obtain positive solution to the coupled-system of boundary value problems of the form

\[- \Delta u(x) = \lambda f(x,u,v) \quad x \in \Omega \]
\[- \Delta v(x) = \lambda g(x,u,v) \quad x \in \Omega \]
\[u(x) = v(x) = 0 \quad x \in \partial \Omega \]

where \(f, g\) are \(C^1\) functions with at least one of \(f(x_0,0,0)\) or \(g(x_0,0,0)\) being negative for some \(x_0 \in \Omega\) (semipositone).

Keywords: positive solutions; sub and super-solutions

AMS Subject Classification: 35J60, 35B30

1. Introduction

Consider positive solutions to the coupled-system of boundary value problems

\[- \Delta u(x) = \lambda f(x,u,v) \quad x \in \Omega \]
\[- \Delta v(x) = \lambda g(x,u,v) \quad x \in \Omega \]
\[u(x) = v(x) = 0 \quad x \in \partial \Omega \]

(1)

Where \(\lambda > 0\) is a parameter, \(\Delta\) is the Laplacian operator, \(\Omega\) is a bounded region in \(\mathbb{R}^N\), \(N \geq 1\) with a smooth boundary \(\partial \Omega\), and \(f, g\) are \(C^1\) functions with at least one of \(f(x_0,0,0)\) or \(g(x_0,0,0)\) being negative for some \(x_0 \in \Omega\) (semipositone).

In this paper, we want to investigate numerically positive solution of (1) by using the method of sub-super solutions. A super solution to (1) is defined as an ordered pair of smooth functions \((\overline{u}, \overline{v})\) on \(\Omega\) satisfying

\[- \Delta \overline{u}(x) \geq \lambda f(x,\overline{u},\overline{v}) \quad x \in \Omega \]
\[- \Delta \overline{v}(x) \geq \lambda g(x,\overline{u},\overline{v}) \quad x \in \Omega \]
\[\overline{u}(x) \geq 0; \overline{v}(x) \geq 0 \quad x \in \partial \Omega \]

Sub solutions are similarly defined with inequalities reversed. Let \(D = [\underline{\rho}_1, \overline{\rho}_1] \times [\underline{\rho}_2, \overline{\rho}_2]\), where

\[\underline{\rho}_1 = \inf \{u(x) : x \in \overline{\Omega}\}, \overline{\rho}_1 = \sup \{u(x) : x \in \overline{\Omega}\}, \underline{\rho}_2 = \inf \{v(x) : x \in \overline{\Omega}\}, \overline{\rho}_2 = \sup \{v(x) : x \in \overline{\Omega}\}.\]

Theorem 1. Let \((\overline{u}, \overline{v}), (u, v)\) be ordered pairs of smooth functions such that \((\overline{u}, \overline{v})\) satisfies
\[-\Delta \overline{u}(x) \geq \lambda f(x, \overline{u}, \overline{v}) \quad x \in \Omega \]
\[-\Delta \overline{v}(x) \leq \lambda g(x, \overline{u}, \overline{v}) \quad x \in \Omega \]
\[\overline{u}(x) \geq 0; \overline{v}(x) \leq 0; \quad x \in \partial \Omega.\]

And \((\overline{u}, \overline{v})\) satisfies the corresponding reserved inequalities. Suppose that
\[u \leq \overline{u} \quad \text{and} \quad v \leq \overline{v} \quad \text{on} \quad \overline{\Omega},\]
then there is a solution \((u, v)\) of (1) such that \(u \leq u \leq \overline{u} \quad \text{and} \quad v \leq v \leq \overline{v} \quad \text{on} \quad \Omega.\]

In [1] for the first time in the literature, the authors consider a class of semipositone systems. In particular they extend many of the results discussed for the positive solutions of single equation in [2] to semipositone systems. It was shown positive solutions to (1) for either \(\lambda\) near the first eigenvalue \(\lambda_1\) of the operator \(-\Delta\) subject to Dirichlet boundary conditions, or for \(\lambda\) large exists. We consider following assumptions:

- \(f, g\) are \(C^1\) functions satisfying:
  - either \(f(x_0, 0, 0) < 0\) or \(g(x_0, 0, 0) < 0\) for some \(x_0 \in \Omega\)
  - \(\lim_{u \to +\infty} \frac{f(x, u, v)}{u} = 0\) uniformly in \(x, v\)
  - \(\lim_{v \to +\infty} \frac{g(x, u, v)}{v} = 0\) uniformly in \(x, u\)

To introduce additional hypotheses to prove existence results near \(\lambda_1\), first we recall the anti-maximum principal by Clement Pletier (see [4]), namely, if \(z_\lambda\) is the unique solution of
\[-\Delta z - \lambda z = -1 \quad x \in \Omega \]
\[z = 0 \quad x \in \partial \Omega \]
for \((\lambda_1, \lambda_1 + \delta)\), where \(\lambda_1\) is the smallest eigenvalue of the problem
\[-\Delta \phi(x) = \lambda \phi(x) \quad x \in \Omega \]
\[\phi(x) = 0 \quad x \in \partial \Omega.\]

Let \(I = [\alpha, \gamma]\) where \(\alpha > \lambda_1\) and \(\gamma < \lambda_1 + \delta\), and let
\[\sigma := \max_{\lambda \in I} ||z_\lambda||\]
Where \(||.||\) denotes the supremum norm. Now assuming that there exists a \(m_1 > 0\) such that
\[f(x, u, v) \geq u - m_1 \quad \forall x \in \overline{\Omega}, u \in [0, m_1 \gamma \sigma], \quad v \geq 0\]
and exists a \(m_2 > 0\) such that
\[g(x, u, v) \geq v - m_2 \quad \forall x \in \overline{\Omega}, v \in [0, m_2 \gamma \sigma], \quad u \geq 0.\]

Finally to prove existence results for \(\lambda\) large, in addition to (3)-(5), we assume \(f_1(u) \leq f(x, u, v) \forall x \in \overline{\Omega}, u \geq 0, v \geq 0\) such that \(f_1(r_1) = 0, f_1'(r_1) < 0,\)
\[\int_0^{r_1} f_1(s) ds > 0 \quad \text{for some} \quad r_1 > 0\]
And \(g_2(v) \leq g(x, u, v) \forall x \in \overline{\Omega}, u \geq 0, v \geq 0\) such that \(g_2(r_2) = 0, g_2'(r_2) < 0,\)
\[
\int_0^t g_2(s)ds > 0 \quad \text{for some } r_2 > 0
\]  
(11)

2. Existence results

**Theorem 2.** Let \( \tilde{\lambda}_i \in I \) and assume (3)-(5), and (8)-(9) hold, then (1) has a positive solution.

It was shown in [1] \((u, v)\) is a subsolution of (1) where \( u(x) = \gamma_1 z_\lambda \) and \( v(x) = \gamma_2 z_\lambda \).

Now let \( w(x) \) to be the unique positive solution of

\[
-\Delta w(x) = 1 \quad x \in \Omega \\
w(x) \leq 0 \quad x \in \partial \Omega
\]  
(12)

\((\tilde{u}, \tilde{v})\) is a supersolution that \( \tilde{u} = Jw(x) \) and \( \tilde{v} = \tilde{J}w(x) \) where \( J, \tilde{J} > 0 \), are sufficiently large, such that

\[
\frac{1}{\lambda \| w \|} \geq \frac{f(x, J \| w \|, \| v \|)}{J \| w \|} = \frac{g(x, u, \tilde{J} \| w \|)}{\tilde{J} \| w \|}
\]  
(13)

and

\[
\tilde{u}(x) \geq u(x) \quad \text{on } \Omega \quad \text{and} \quad \tilde{v}(x) \geq v(x) \quad \text{on } \Omega
\]  
(13)'

**Theorem 3.** Assume (3)-(5) and (10)-(11) hold. Then there exists a \( \lambda^* > 0 \) such that for every \( \lambda > \lambda^* \), (1) has a positive solution.

Here we give a simple example that satisfies the hypotheses of theorem 2 and 3. Consider

\[
h(x, u, v) = m\sqrt{u + 1} - \frac{3m}{2} + e^{-v} \quad \forall u \geq 0, v \geq 0
\]  
(14)

where \( m > 0 \) is a constant. Let

\[
f(x, u, v) = h(x, u, v) \\
g(x, u, v) = h(x, u, v)
\]

Here \( f(x, 0, 0) = 1 - \frac{m}{2} < 0 \quad \text{for } m > 2 \), \( f \) is increasing in \( u, v \), and \( \lim_{u \to \infty} \frac{f(x, u, v)}{u} = 0 \) uniformly in \( v \).

Also \( g(x, 0, 0) = 1 - \frac{m}{2} < 0 \quad \text{for } m > 2 \), \( g \) is increasing in \( u, v \), and \( \lim_{v \to \infty} \frac{g(x, u, v)}{v} = 0 \) uniformly in \( u \).

Now, to show that (8) and (9) are satisfied, it suffices to show that \( h_i(u) = m\sqrt{u + 1} - \frac{3m}{2} \) satisfies (8) since \( h(x, u, v) \geq h_i(u) \forall u \geq 0, v \geq 0 \). Let \( p > 0 \) be such that \( h_i(p) = p - m \). That is,

\[
m\sqrt{p + 1} - \frac{3m}{2} = p - m
\]

\[
m^2(p + 1) = \left\lfloor p + \frac{m}{2} \right\rfloor^2
\]

\[
p^2 + (m - m^2)p - \frac{3m^2}{4} = 0
\]

and

\[
p = \frac{(m^2 - m) + \sqrt{m^4 - 2m^3 + 4m^2}}{2} = \frac{(m^2 - m) + m\sqrt{(m-1)^2 + 3}}{2}
\]

Hence in order that (8) be satisfied, we must have

\[JIC\]email for contribution: editor@jic.org.uk
\[
\frac{m^2 - m + m \sqrt{(m-1)^2 + 3}}{2} \geq m(\sigma \alpha),
\]
that is,
\[
(m-1) + \sqrt{(m-1)^2 + 3} \geq 2(\sigma \alpha).
\] (15)

Since \(\sigma\) and \(\alpha\) are quantities that depend only on \(\Omega\), clearly for a given \(\sigma\) and \(\alpha\), there exists and sufficiently large such that if \(m > m_0\), then (15) is satisfied and equivalently (8) will be satisfied. Thus, (9) is also satisfied for \(m > m_0\).

Note that this example satisfies the hypotheses of theorem 3 also since \(h(x,u,v) \geq h_1(u)\forall u \geq 0, v \geq 0\) and one can construct a function \(f_1(u) \leq h_1(u)\) satisfying (10).

3. Numerical Results

We see in section 2 that there must always exists a solution for problems such as (1) between a sub-solution \((u,v)\) and a super-solution \((\tilde{u},\tilde{v})\) when \(\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} \leq 0\).

Consider the coupled-system boundary value problems
\[
\begin{align*}
-\Delta u(x) &= \lambda \tilde{f}(x,u,v) \quad x \in \Omega \\
-\Delta v(x) &= \lambda \tilde{g}(x,u,v) \quad x \in \Omega \\
u(x) &= v(x) = 0 \quad x \in \partial \Omega
\end{align*}
\] (16)

Since \(f, g\) are \(C^1\) functions, there exists positive constants \(k_1, k_2\) such that \(\frac{\partial f}{\partial u} \geq -k_1, \text{ and } \frac{\partial g}{\partial v} \geq -k_2\) on \(\overline{\Omega} \times D\). Thus we can study the equivalent system
\[
\begin{align*}
-\Delta u(x) + \lambda k_1 u(x) &= \lambda \tilde{f}(x,u,v) + \lambda k_1 u(x) = \lambda \tilde{f}(x,u,v) \quad x \in \Omega \\
-\Delta v(x) + \lambda k_2 v(x) &= \lambda \tilde{g}(x,u,v) + \lambda k_2 v(x) = \lambda \tilde{g}(x,u,v) \quad x \in \Omega \\
u(x) &= v(x) = 0 \quad x \in \partial \Omega
\end{align*}
\] (17)

The mapping \(T: (u_1,v_1) \rightarrow (u_2,v_2), \ (u_2,v_2) = T(u_1,v_1)\)
\[
(u_1,v_1) \in [\tilde{u},\tilde{u}] \times [\tilde{v},\tilde{v}] \quad \forall x \in \overline{\Omega}
\]

Where \((u_2,v_2)\) is the unique solution of the coupled-system
\[
\begin{align*}
-\Delta u_2(x) + \lambda k_1 u_2(x) &= \lambda \tilde{f}(x,u_1,v_1) + \lambda k_1 u_1(x) \quad x \in \Omega \\
-\Delta v_2(x) + \lambda k_2 v_2(x) &= \lambda \tilde{g}(x,u_1,v_1) + \lambda k_2 v_1(x) \quad x \in \Omega \\
u_2(x) &= v_2(x) = 0 \quad x \in \partial \Omega
\end{align*}
\] (18)
satisfied the hypotheses of Schauder fixed point theorem, and then we can conclude that \(\exists (u,v) \in D\) \(T(u,v) = (u,v)\) so \((u,v)\) is a solution of (1) (see [3]).

By letting \(\tilde{f}(x,u,v) = \lambda \tilde{f}(x,u,v) + \lambda k_1 u(x)\) and \(\tilde{g}(x,u,v) = \lambda \tilde{g}(x,u,v) + \lambda k_2 v(x)\), we use the following iteration to obtain solution:
\[
\begin{align*}
u_0(x) &= u_0, v_0(x) = v \\
(\Delta - \lambda k_1) u_{n+1} &= -\tilde{f}(x,u_n,v_n) \quad x \in \Omega \\
(\Delta - \lambda k_2) v_{n+1} &= -\tilde{g}(x,u_{n+1},v_n) \quad x \in \Omega \\
u_{n+1} &= 0 = v_{n+1} \quad x \in \partial \Omega
\end{align*}
\] (19)
We can also use \( u_0(x) = \bar{u}, v_0(x) = \bar{v} \) as initial guesses. We use the following algorithm:

**Sub- and super-solution algorithm**

1. Find \( u_0(x) = u, v_0(x) = v \). Choose numbers \( k_1, k_2 > 0 \);
2. Solve the boundary value system (19);
3. If \( \| u_{n+1} - u_n \| \leq \epsilon \) and \( \| v_{n+1} - v_n \| \leq \epsilon \), output and stop. Else go to step 2.

Now we want to apply the algorithm for:

\[
\begin{align*}
-\Delta u(x) &= \lambda (m\sqrt{u+1} - \frac{3m}{2} + e^{-u}) & x \in \Omega \\
-\Delta v(x) &= \lambda (m\sqrt{v+1} - \frac{3m}{2} + e^{-v}) & x \in \Omega \\
u(x) &= v(x) = 0 & x \in \partial \Omega
\end{align*}
\] (20)

For doing step 1, we solve the problem:

\[
\begin{align*}
-\Delta z - \lambda z &= -1 & x \in \Omega \\
z &= 0 & x \in \partial \Omega
\end{align*}
\] (21)

to obtain \( u \). We know from section 2 that problem (21) has a positive solution for \( (\lambda_1, \lambda_1 + \delta) \). The obtained results show there is an array of positive solution for \( \lambda \in (17,35) \) so \( \lambda_1 \) is around 17.

For brevity we express just some of those numerical results:

**Approximation of \( z_\lambda \) for \( \lambda = 15 \)**

<table>
<thead>
<tr>
<th>( x/y )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>-0.268</td>
<td>0.423</td>
<td>-0.431</td>
<td>-0.283</td>
</tr>
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<td>0.4</td>
<td>-0.447</td>
<td>-0.701</td>
<td>-0.718</td>
<td>-0.493</td>
</tr>
<tr>
<td>0.6</td>
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<td>-0.753</td>
<td>-0.778</td>
<td>-0.636</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.505</td>
<td>-0.528</td>
<td>-0.497</td>
<td>-0.345</td>
</tr>
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</table>

**Approximation of \( z_\lambda \) for \( \lambda = 17 \)**

<table>
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<tr>
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<th>0.6</th>
<th>0.8</th>
</tr>
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<tbody>
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<td>1.895</td>
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<td>3.266</td>
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<td>5.818</td>
<td>4.172</td>
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<td>0.8</td>
<td>3.727</td>
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<td>3.676</td>
<td>2.514</td>
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**Approximation of \( z_\lambda \) for \( \lambda = 30 \)**

<table>
<thead>
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<th>0.2</th>
<th>0.4</th>
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<th>0.8</th>
</tr>
</thead>
<tbody>
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<td>0.002</td>
<td>0.017</td>
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<td>0.008</td>
</tr>
<tr>
<td>0.4</td>
<td>0.028</td>
<td>0.062</td>
<td>0.068</td>
<td>0.041</td>
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<tr>
<td>0.6</td>
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<td>0.070</td>
</tr>
<tr>
<td>0.8</td>
<td>0.054</td>
<td>0.060</td>
<td>0.056</td>
<td>0.033</td>
</tr>
</tbody>
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**Approximation of \( z_\lambda \) for \( \lambda = 36 \)**

<table>
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<th>0.4</th>
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</tr>
</thead>
<tbody>
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<td>0.2</td>
<td>-0.001</td>
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<td>0.005</td>
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<td>0.4</td>
<td>0.024</td>
<td>0.056</td>
<td>0.063</td>
<td>0.038</td>
</tr>
</tbody>
</table>

JIC email for contribution: editor@jic.org.uk
Let \( u = \gamma m \frac{z_i}{z_{i+1}} \) where \( \gamma \) and \( m \) obtained from section 1, 2 and to obtain \( \bar{v} \) for \( \lambda \in I \) (where \( \lambda > \lambda_i \) and \( \gamma < \lambda_i + \delta \)) we solve
\[
- \Delta v(x) = 1 \quad x \in \Omega \\
v(x) = 0 \quad x \in \partial \Omega
\]
by finite difference (see [5,6]). We choose \( J \) such that (13) and \( \alpha \delta^2 \) are satisfied.

We execute algorithm for \( \lambda \in [17.1,34.9] \). It is easy to see that \( u = v \) for problem (20).

For brevity we express just some of those numerical results:

### Approximation of \( u \) for \( \lambda = 17.1 \)

<table>
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<tr>
<th>( x )</th>
<th>( y )</th>
<th>0.2</th>
<th>0.4</th>
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<td>1.510 \times 10^4</td>
<td>1.009 \times 10^4</td>
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</tr>
<tr>
<td>0.4</td>
<td>1.510 \times 10^4</td>
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<td></td>
</tr>
<tr>
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<td>2.777 \times 10^4</td>
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</tr>
<tr>
<td>0.8</td>
<td>1.009 \times 10^4</td>
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<td>1.510 \times 10^4</td>
<td>1.009 \times 10^4</td>
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</tr>
</tbody>
</table>

### Approximation of \( u \) for \( \lambda = 25 \)

<table>
<thead>
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</thead>
<tbody>
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<td>3.253 \times 10^4</td>
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<tr>
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### Approximation of \( u \) for \( \lambda = 30 \)

<table>
<thead>
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<th>( y )</th>
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<th>0.6</th>
<th>0.8</th>
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### Approximation of \( u \) for \( \lambda = 34.9 \)

<table>
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<th>0.8</th>
</tr>
</thead>
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<td>6.368 \times 10^4</td>
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</tr>
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<td>9.596 \times 10^4</td>
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<td>6.368 \times 10^4</td>
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</tr>
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</tr>
<tr>
<td>0.8</td>
<td>4.256 \times 10^4</td>
<td>6.368 \times 10^4</td>
<td>6.368 \times 10^4</td>
<td>4.256 \times 10^4</td>
<td></td>
</tr>
</tbody>
</table>

Our numerical results (in following tables) show that there exist \( \lambda^* > 0 \) such that for every \( \lambda > \lambda^* \), (20) has a positive solution. In this case \( \lambda^* = 166.696 \) with decimal accuracy.

### Approximation of \( u \) for \( \lambda = 170 \)

JIC email for subscription: info@jic.org.uk
<table>
<thead>
<tr>
<th>x / y</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
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<tbody>
<tr>
<td>0.2</td>
<td>1.019×10^6</td>
<td>1.524×10^6</td>
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</tr>
<tr>
<td>0.4</td>
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<td>1.524×10^6</td>
<td>1.019×10^6</td>
</tr>
</tbody>
</table>

Approximation of $u$ for $\lambda = 500$

<table>
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<th>0.8</th>
</tr>
</thead>
<tbody>
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<td>0.2</td>
<td>0.883×10^7</td>
<td>1.321×10^7</td>
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<td>0.883×10^7</td>
</tr>
<tr>
<td>0.4</td>
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<td>1.990×10^7</td>
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<td>1.321×10^7</td>
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<tr>
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<td>1.990×10^7</td>
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<td>1.321×10^7</td>
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<tr>
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<td>1.321×10^7</td>
<td>1.321×10^7</td>
<td>0.883×10^7</td>
</tr>
</tbody>
</table>

Approximation of $u$ for $\lambda = 1000$

<table>
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<th>0.4</th>
<th>0.6</th>
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</tr>
</thead>
<tbody>
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<td>3.534×10^7</td>
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</tr>
</tbody>
</table>

4. References


