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Analytic solutions of a class of matrix minimization model with unitary constraints

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Abstract: In this paper we present analytic solutions of a class of matrix minimization model with unitary constraints as follows:

$$\min_{U_k \in \mathbf{U}_n, W_k \in \mathbf{U}_t, V_k \in \mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V)|$$

$$\min_{U_k \in \mathbf{U}_n, W_k \in \mathbf{U}_t, V_k \in \mathbf{U}_m} |\mathrm{tr}(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V)|,$$

where $A_k \in \mathbf{C}^{m\times n}$, $B_k \in \mathbf{C}^{n\times t}$, $C_k \in \mathbf{C}^{t\times m}$, $\mathbf{C}^{m\times n}$ denotes $m\times n$ complex matrix set, and c is a complex number, I_m denotes the m-order identity matrix, $\det(\cdot)$ and $\mathrm{tr}(\cdot)$ denote matrix determinant and trace function, respectively. The proposed results improve some existing ones in Xu (2019) [1]. Numerical examples are given to verify the validity of the theoretical results.

Keywords: constrained matrix minimization model, determinant function, trace function, unitary constraints.

1. Introduction

The matrix optimization model with unitary constraints has important applications in Kronecker canonical form of a general matrix pencil, linearly constrained least-squares problem, test signals of mechanical systems, and aero engine fault diagnosis, see [2,3,4,5].

The latest significant application of matrix optimization model with unitary constraints is in the data analysis of DNA micro-array analysis [6,7,8,9]. Xu [1] in 2019 considered the upper bound of chordal metric between generalized singular values of Grassman matrix pairs with the same number of columns, which can be applied in comparing two sets of DNA micro-arrays of different organisms. Motivated by the applications, in this paper we consider analytic solutions of a class of matrix minimization model with different dimensional unitary constraints. The considered matrix minimization model are as follows:

$$\min_{U_k \in \mathbf{U}_n, W_k \in \mathbf{U}_t, V_k \in \mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V)|, \tag{1.1}$$

$$\min_{U_k \in \mathbf{U}_n, W_k \in \mathbf{U}_t, V_k \in \mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V)|, \tag{1.2}$$

where c is a complex number and A_k, B_k, C_k are $m\times n, n\times t, t\times m$ complex matrices, respectively. In this paper we will discuss their analytic solutions.

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1.1. Literature review

The existing works related to the constrained matrix problems (1.1) and (1.2) are summarized as follows. John von Neumann in 1937 [10] and K. Fan in 1951 [11] studied the maximum value problem of trace function for the same dimensional matrix. They presented

$$\max_{U_1,\cdots,U_m\in\mathbf{U_n}} Re\left[\operatorname{tr}\prod_{j=1}^{m}U_jA_j\right]=\max_{U_1,\cdots,U_m\in\mathbf{U_n}}\left|\operatorname{tr}\prod_{j=1}^{m}U_jA_j\right|=\sum_{i=1}^{n}\prod_{j=1}^{m}\sigma_i(A_j), \tag{1.3}$$

where $Re[x]$ denotes the real part of the complex number x. A special case of (1.3) was studied by Lu [12], where $m=2$, and both A_1 and A_2 are positive diagonal matrices with the main diagonal elements between 0 and 1 descending simultaneously or in ascending order. Moreover, Sun provided the Hoffman-Wielandt-type theorem for generalized singular values of Grassman matrix pairs [13,14,15]. Xu et al. [16] also considered the constrained optimization problems of Grassman matrix pairs and they presented

$$\min_{U_kU_k^H=I_n,V_kV_k^H=I_m}\left|\det\left(cI_m\pm\prod_{k=1}^{s}\Gamma_kU_k\Delta_kV_k\right)\right|=\begin{cases}|c|^{m-r}\prod_{i=1}^{r}\left(|c|-\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|\right), & \prod_{k=1}^{s}|\ddot{\gamma}_r^k||\ddot{\delta}_r^k|\le|c|,\\ |c|^{m-r}\prod_{i=1}^{r}\left(\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|-|c|\right), & \prod_{k=1}^{s}|\ddot{\gamma}_1^k||\ddot{\delta}_1^k|\ge|c|,\\ 0, & otherwise,\end{cases}$$

$$\min_{U_kU_k^H=I_n,V_kV_k^H=I_m}\left|\operatorname{tr}\left(cI_m\pm\prod_{k=1}^{s}\Gamma_kU_k\Delta_kV_k\right)\right|=\begin{cases}m|c|-\sum_{i=1}^{r}\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|, & \frac{1}{m}\sum_{i=1}^{r}\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|\le|c|,\\ 0, & \frac{1}{m}\sum_{i=1}^{r}\prod_{k=1}^{s}|\ddot{\gamma}_i^k||\ddot{\delta}_i^k|\ge|c|,\end{cases}$$

where $r=\min\{m,n\}$, $\ddot{\gamma}_i^k$ and $\ddot{\delta}_i^k$ denote the i-th singular value of Γ_k and Δ_k, which are $m\times n$ and $n\times m$ complex matrices, respectively. Compared with the above special cases, (1.1) and (1.2) are more complicated because they involved more unitary constrained conditions. These motivated us to use new technique for giving the analytic solutions of (1.1) and (1.2).

1.2. Organization

The rest of this paper is organized as follows. In Section 2 we will give some notations and lemmas, which are useful to deduce the main results. In Section 3 we will provide the analytic solutions of (1.1) and (1.2). In Section 4 numerical examples are given to illustrate the theoretical results. Finally, concluding remarks are drawn in Section 5.

1.3. Notation

Throughout this paper we always use the following notations and definitions. Let $\mathbf{R}$, $\mathbf{C}$, $\mathbf{C}^{m\times n}$ and $\mathbf{U}_n$ be the sets of real numbers, complex numbers, $m\times n$ complex matrix set and $n\times n$ unitary matrices, respectively. $|\cdot|$ and $Re[\cdot]$ stand for absolute value and real part of a complex number, respectively. The symbols I_m and $O_{m\times n}$ stand for the identity matrix of order n and $m\times n$ zero matrix, respectively. For a matrix $\Gamma\in\mathbf{C}^{n\times n}$, $\det(\Gamma)$ and $\operatorname{tr}(\Gamma)$ denote the determinant and trace of the matrix Γ, respectively. We denote by $\sigma_i(\Gamma)$ the set of its singular values, and throughout the paper we assume that its singular values are arranged in decreasing order, i.e., $\sigma_1(\Gamma)\ge\sigma_2(\Gamma)\ge\cdots\ge\sigma_n(\Gamma)\ge 0$.

2. Preliminaries

Lemma 2.1 [1] Let $A_1,\cdots,A_m\in\mathbf{C}^{n\times n}$ with singular values $\sigma_1(A_j)\ge\sigma_2(A_j)\ge\cdots\ge\sigma_n(A_j),\ j=1,\cdots,m$

and $c\in\mathbf{R}$. We have the following conclusions:

$$\min_{U_1,\cdots,U_m\in\mathbf{U}_n}|\det(cI_n\pm\prod_{j=1}^m A_jU_j)=\begin{cases}\prod_{i=1}^n(|c|-\prod_{j=1}^m\sigma_i(A_j)), & \prod_{j=1}^m\sigma_1(A_j)\le|c|,\\ \prod_{i=1}^n(\prod_{j=1}^m\sigma_i(A_j)-|c|), & \prod_{j=1}^m\sigma_n(A_j)\ge|c|,\\ 0, & otherwise,\end{cases}$$

$$\min_{U_1,\cdots,U_m\in\mathbf{U}_n}|\operatorname{tr}(cI_n\pm\prod_{j=1}^m U_jA_j)=\begin{cases}n|c|-\sum_{i=1}^n\prod_{j=1}^m\sigma_i(A_j), & \frac{1}{n}\sum_{i=1}^n\prod_{j=1}^m\sigma_i(A_j)\le|c|,\\ 0, & \frac{1}{n}\sum_{i=1}^n\prod_{j=1}^m\sigma_i(A_j)\ge|c|.\end{cases}$$

Lemma 2.2 Let

$$\Gamma_k=\begin{pmatrix}\Gamma_r^k & O_{r\times(n-r)}\\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)}\end{pmatrix}\in\mathbf{C}^{m\times n},$$
$$\Delta_k=\begin{pmatrix}\Delta_r^k & O_{r\times(t-r)}\\ O_{(n-r)\times r} & O_{(n-r)\times(t-r)}\end{pmatrix}\in\mathbf{C}^{n\times t},$$
$$H_k=\begin{pmatrix}H_r^k & O_{r\times(m-r)}\\ O_{(t-r)\times r} & O_{(t-r)\times(m-r)}\end{pmatrix}\in\mathbf{C}^{t\times m}$$

with

$$\Gamma_r^k=\operatorname{diag}\left(\gamma_1^k,\ldots,\gamma_r^k\right),\ \Delta_r^k=\operatorname{diag}\left(\delta_1^k,\ldots,\delta_r^k\right),\ H_r^k=\operatorname{diag}\left(h_1^k,\ldots,h_r^k\right), c\in\mathbf{C},$$
$$r=\min\{m,n,t\},\ p=\max\{m,n,t\},\gamma_i^k,\delta_i^k,h_i^k\in\mathbf{R},\ i=1,\ldots,r,\ k=1,\ldots,s.$$

Then the following issues hold true.

(i) Let $p=\max\{m,n,t\}, r=\min\{m,n,t\}$, then

$$\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}|\det(cI_m\pm\prod_{k=1}^s\Gamma_kU_k\Delta_kW_kH_kV)$$
$$=\begin{cases}|c|^{m-r}\prod_{i=1}^r(|c|-\prod_{k=1}^s\sigma_i(\tilde\Gamma_k)\sigma_i(\tilde\Delta_k)\sigma_i(\tilde H_k)), & \prod_{k=1}^s\sigma_1(\tilde\Gamma_k)\sigma_1(\tilde\Delta_k)\sigma_1(\tilde H_k)\le|c|,\\ |c|^{m-r}\prod_{i=1}^r(\prod_{k=1}^s\sigma_i(\tilde\Gamma_k)\sigma_i(\tilde\Delta_k)\sigma_i(\tilde H_k)-|c|), & \prod_{k=1}^s\sigma_p(\tilde\Gamma_k)\sigma_p(\tilde\Delta_k)\sigma_p(\tilde H_k)\ge|c|,\\ 0, & otherwise,\end{cases}$$

(ii)

$$\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}\left|\mathrm{tr}\left(cI_m\pm\prod_{k=1}^{s}\Gamma_kU_k\Delta_kW_kH_kV_k\right)\right|$$
$$=\begin{cases} m|c|-\sum_{i=1}^{r}\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k), & \frac{1}{m}\sum_{i=1}^{m}\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k)\le|c|,\\ 0, & \frac{1}{m}\sum_{i=1}^{r}\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k)\ge|c|,\end{cases}$$

where the $\tilde{\Gamma}_k$, $\tilde{\Delta}_k$, $\tilde{H}_k$ are $p\times p$ complex matrices. The singular values of $\tilde{\Gamma}_k$, $\tilde{\Delta}_k$, $\tilde{H}_k$ are arranged in decreasing order.

Proof. Since $p=\max\{m,n,t\}$, let

$$\tilde{\Gamma}_k=\begin{pmatrix}\Gamma_k & O_{m\times(p-n)}\\ O_{(p-m)\times n} & O_{(p-m)\times(p-n)}\end{pmatrix}\in\mathbf{C}^{p\times p},\tilde{\Delta}_k=\begin{pmatrix}\Delta_k & O_{n\times(p-t)}\\ O_{(p-n)\times t} & O_{(p-n)\times(p-t)}\end{pmatrix}\in\mathbf{C}^{p\times p},$$
$$\tilde{H}_k=\begin{pmatrix}H_k & O_{t\times(p-m)}\\ O_{(p-t)\times m} & O_{(p-t)\times(p-n)}\end{pmatrix}\in\mathbf{C}^{p\times p},\tilde{U}_k=\begin{pmatrix}U_k & O_{n\times(m-n)}\\ O_{(m-n)\times n} & I_{(m-n)}\end{pmatrix}\in\mathbf{C}^{p\times p},$$
$$\tilde{W}_k=\begin{pmatrix}W_k & O_{t\times(p-t)}\\ O_{(p-t)\times t} & I_{(p-t)}\end{pmatrix}\in\mathbf{C}^{p\times p},\tilde{V}_k=\begin{pmatrix}V_k & O_{m\times(p-m)}\\ O_{(p-m)\times m} & I_{(p-m)}\end{pmatrix}\in\mathbf{C}^{p\times p}.$$

Then we have

$$\begin{aligned}&\prod_{k=1}^{s}\tilde{\Gamma}_k\tilde{U}_k\tilde{\Delta}_k\tilde{W}_k\tilde{H}_k\tilde{V}_k\\ &=\prod_{k=1}^{s}\begin{pmatrix}\Gamma_k & O_{m\times(p-n)}\\ O_{(p-m)\times n} & O_{(p-m)\times(p-n)}\end{pmatrix}\begin{pmatrix}U_k & O_{n\times(m-n)}\\ O_{(m-n)\times n} & I_{(m-n)}\end{pmatrix}\begin{pmatrix}\Delta_k & O_{n\times(p-t)}\\ O_{(p-n)\times t} & O_{(p-n)\times(p-t)}\end{pmatrix}\\ &\quad\begin{pmatrix}W_k & O_{t\times(p-t)}\\ O_{(p-t)\times t} & I_{(p-t)}\end{pmatrix}\begin{pmatrix}H_k & O_{t\times(p-m)}\\ O_{(p-t)\times m} & O_{(p-t)\times(p-n)}\end{pmatrix}\begin{pmatrix}V_k & O_{m\times(p-m)}\\ O_{(p-m)\times m} & I_{(p-m)}\end{pmatrix}\\ &=\begin{pmatrix}\prod_{k=1}^{s}\Gamma_kU_k\Delta_kW_kH_kV_k & O_{m\times(p-m)}\\ O_{(p-m)\times m} & O_{(p-m)\times(p-m)}\end{pmatrix}.\end{aligned}\tag{2.1}$$

(i) Since $c\in\mathbf{C},$ we deduce that

$$\begin{aligned}&\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}\left|\det\left(cI_m\pm\prod_{k=1}^{s}\Gamma_kU_k\Delta_kW_kH_kV_k\right)\right|\\ &=\min_{\tilde{U}_k\in\mathbf{U}_p,\tilde{W}_k\in\mathbf{U}_p,\tilde{V}_k\in\mathbf{U}_p}\left|\det\left(cI_p\pm\prod_{k=1}^{s}\tilde{\Gamma}_k\tilde{U}_k\tilde{\Delta}_k\tilde{W}_k\tilde{H}_k\tilde{V}_k\right)\right|\cdot|c|^{m-p}\\ &=\min_{\tilde{U}_k\in\mathbf{U}_p,\tilde{W}_k\in\mathbf{U}_p,\tilde{V}_k\in\mathbf{U}_p}\frac{\left|\det\bar{c}\left(cI_p\pm\tilde{\Gamma}_k\tilde{U}_k\tilde{\Delta}_k\tilde{W}_k\tilde{H}_k\tilde{V}_k\right)\right|\cdot|c|^{m-p}}{|c|^{p}}\\ &=\min_{\tilde{U}_k\in\mathbf{U}_p,\tilde{W}_k\in\mathbf{U}_p,\tilde{V}_k\in\mathbf{U}_p}\left|\det\left(|c|I_p\pm\prod_{k=1}^{s}\tilde{\Gamma}_k\tilde{U}_k\tilde{\Delta}_k\tilde{W}_k\tilde{H}_k\tilde{V}_k\right)\right|\cdot|c|^{m-p}.\end{aligned}\tag{2.2}$$

By Lemma 2.1, we have

$$\begin{aligned}
&\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s}\Gamma_k U_k \Delta_k W_k H_k V_k) \\
&= \min_{\tilde{U}_k\in\mathbf{U}_p, \tilde{W}_k\in\mathbf{U}_p, \tilde{V}_k\in\mathbf{U}_p} |\det(|c| I_p \pm \prod_{k=1}^{s}\tilde{\Gamma}_k \tilde{U}_k \tilde{\Delta}_k \tilde{W}_k \tilde{H}_k \tilde{V}_k)|\cdot|c|^{m-p} \qquad (2.3)\\
&= \begin{cases} |c|^{m-r}\prod_{i=1}^{r}(|c| - \prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k)), & \prod_{k=1}^{s}\sigma_1(\tilde{\Gamma}_k)\sigma_1(\tilde{\Delta}_k)\sigma_1(\tilde{H}_k) \le |c|, \\ |c|^{m-r}\prod_{i=1}^{r}(\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k) - |c|), & \prod_{k=1}^{s}\sigma_p(\tilde{\Gamma}_k)\sigma_p(\tilde{\Delta}_k)\sigma_p(\tilde{H}_k) \ge |c|, \\ 0, & otherwise, \end{cases}
\end{aligned}$$

where the singular values of $\tilde{\Gamma}_k, \tilde{\Delta}_k, \tilde{H}_k$ are arranged in decreasing order.

(ii) Since $c \in \mathbf{C}$, hence

$$\begin{aligned}
&\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\operatorname{tr}(cI_m \pm \prod_{k=1}^{s}\Gamma_k U_k \Delta_k W_k H_k V_k) \\
&= \min_{\tilde{U}_k\in\mathbf{U}_p, \tilde{W}_k\in\mathbf{U}_p, \tilde{V}_k\in\mathbf{U}_p} |\operatorname{tr}(cI_p \pm \prod_{k=1}^{s}\tilde{\Gamma}_k \tilde{U}_k \tilde{\Delta}_k \tilde{W}_k \tilde{H}_k \tilde{V}_k)| + |c|(m-p) \\
&= \min_{\tilde{U}_k\in\mathbf{U}_p, \tilde{W}_k\in\mathbf{U}_p, \tilde{V}_k\in\mathbf{U}_p} \frac{|\operatorname{tr}\bar{c}(cI_p \pm \prod_{k=1}^{s}\tilde{\Gamma}_k \tilde{U}_k \tilde{\Delta}_k \tilde{W}_k \tilde{H}_k \tilde{V}_k)|}{|c|} + |c|(m-p) \qquad (2.4)\\
&= \min_{\tilde{U}_k\in\mathbf{U}_p, \tilde{W}_k\in\mathbf{U}_p, \tilde{V}_k\in\mathbf{U}_p} |\operatorname{tr}(|c| I_p \pm \prod_{k=1}^{s}\frac{\bar{c}}{|c|}\tilde{\Gamma}_k \tilde{U}_k \tilde{\Delta}_k \tilde{W}_k \tilde{H}_k \tilde{V}_k)| + |c|(m-p).
\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
&\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\operatorname{tr}(cI_m \pm \prod_{k=1}^{s}\Gamma_k U_k \Delta_k W_k H_k V_k)| \\
&= \min_{\tilde{U}_k\in\mathbf{U}_p, \tilde{W}_k\in\mathbf{U}_p, \tilde{V}_k\in\mathbf{U}_p} |\operatorname{tr}(|c| I_p \pm \prod_{k=1}^{s}\tilde{\Gamma}_k \tilde{U}_k \tilde{\Delta}_k \tilde{W}_k \tilde{H}_k \tilde{V}_k)| + |c|(m-p) \qquad (2.5)\\
&= \begin{cases} m|c| - \sum_{i=1}^{r}(\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k)), & \frac{1}{m}\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k) \le |c|, \\ 0, & \frac{1}{m}\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k) \ge |c|, \end{cases}
\end{aligned}$$

where the singular values of $\tilde{\Gamma}_k, \tilde{\Delta}_k, \tilde{H}_k$ are arranged in decreasing order. This completes the proof.

3. Analytic solutions of problems (1.1) and (1.2)

In this section we provide analytic solutions of constrained matrix minimization models (1.1) and (1.2) as follows.

Theorem 3.1 Let $A_k \in \mathbf{C}^{m\times n}, B_k \in \mathbf{C}^{n\times t}, C_k \in \mathbf{C}^{t\times m}$, $\tilde{A}_k, \tilde{B}_k, \tilde{C}_k \in \mathbf{C}^{p\times p}$, $p = \max\{m,n,t\}$, $k = 1,\dots s$, $c \in \mathbf{C}$. The following issues hold true.

(i) If $p = \max\{m,n,t\}$, $r = \min\{m,n,t\}$, then

$$\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s}\Gamma_k U_k \Delta_k W_k H_k V_k)|$$
$$= \begin{cases} |c|^{m-r}\prod_{i=1}^{r}(|c| - \prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k)), & \prod_{k=1}^{s}\sigma_1(\tilde{\Gamma}_k)\sigma_1(\tilde{\Delta}_k)\sigma_1(\tilde{H}_k) \le |c|, \\ |c|^{m-r}\prod_{i=1}^{r}(\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k) - |c|), & \prod_{k=1}^{s}\sigma_p(\tilde{\Gamma}_k)\sigma_p(\tilde{\Delta}_k)\sigma_p(\tilde{H}_k) \ge |c|, \\ 0, & otherwise, \end{cases} \tag{3.1}$$

(ii)

$$\min_{U_k\in\mathbf{U}_n, W_k\in\mathbf{U}_t, V_k\in\mathbf{U}_m} |\mathrm{tr}(cI_m \pm \prod_{k=1}^{s}\Gamma_k U_k \Delta_k W_k H_k V_k)|$$
$$= \begin{cases} \mathrm{m}|c| - \sum_{i=1}^{r}(\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k)), & \frac{1}{m}\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k) \le |c|, \\ 0, & \frac{1}{m}\prod_{k=1}^{s}\sigma_i(\tilde{\Gamma}_k)\sigma_i(\tilde{\Delta}_k)\sigma_i(\tilde{H}_k) \ge |c|, \end{cases} \tag{3.2}$$

where the singular values of $\tilde{A}_k, \tilde{B}_k, \tilde{C}_k$ are arranged in decreasing order.

Proof. (i) If $p = \max\{m,n,t\}$, $r = \min\{m,n,t\}$ and by singular value decompositions of A_k, B_k, C_k we have

$$A_k = P_k\Gamma_k Q_k, \quad B_k = M_k\Delta_k N_k, \quad C_k = S_k H_k T_k, \tag{3.3}$$

where $T_k, P_{k+1} \in \mathbf{U}_m$, $Q_k, M_k \in \mathbf{U}_n$, $N_k, S_k \in \mathbf{U}_t$, $k = 1,\dots,s$ and assume that $P_{s+1} = P_1$, $\Gamma_k = diag(\Gamma_r^k,\cdots,O) \in C^{m\times n}$, $\Delta_k = diag(\Delta_r^k,\cdots,O) \in C^{n\times t}$, $H_k = diag(H_r^k,\cdots,O) \in C^{t\times m}$ and $r = \min\{m,n,t\}$ with

$$\Gamma_r^k = diag(\gamma_1^k,\cdots,\gamma_r^k), \quad \Delta_r^k = diag(\delta_1^k,\cdots,\delta_r^k), \quad H_r^k = diag(h_1^k,\cdots,h_r^k),$$
$$\gamma_1^k \ge \cdots \ge \gamma_r^k \ge 0, \quad \delta_1^k \ge \cdots \ge \delta_r^k \ge 0, \quad H_r^k = h_1^k \ge \cdots \ge h_r^k \ge 0.$$

It follows that

$$cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V_k = cI_m \pm \prod_{k=1}^{s} P_k\Gamma_k Q_k U_k M_k \Delta_k N_k W_k S_k H_k T_k V_k. \tag{3.4}$$

Let $Q_k U_k M_k \equiv U_k \in \mathbf{U}_n$, $N_k W_k S_k \equiv W_k \in \mathbf{U}_t$, $T_k V_k P_{k+1} \equiv V_k \in \mathbf{U}_m$, then we have

$$\begin{aligned}&\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}|\det(cI_m\pm\prod_{k=1}^{s}A_kU_kB_kW_kC_kV_k)|\\&=\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}|\det(P_1^H[cI_m\pm P_1\Gamma_1Q_1U_1M_1\Delta_1N_1W_1S_1H_1T_1V_1\cdots\Delta_sN_sW_sS_sH_sT_sV_s]P_1)|\\&=\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}|\det(cI_m\pm\prod_{k=1}^{s}\Gamma_kU_k\Delta_kW_kH_kV_k)|.\end{aligned}\tag{3.5}$$

which together with Lemma 2.1 lead to (3.1) hold true.

(ii) If $p=\max\{m,n,t\}$, then $r=\min\{m,n,t\}$, and by singular value decompositions of A_k,B_k,C_k we have

$$A_k=P_k\Gamma_kQ_k,\quad B_k=M_k\Delta_kN_k,\quad C_k=S_kH_kT_k,$$

where $T_k,P_{k+1}\in\mathbf{U}_m$, $Q_k,M_k\in\mathbf{U}_n$, $N_k,S_k\in\mathbf{U}_t$, $k=1,\ldots,s$ and assume that $P_{s+1}=P_1$, $\Gamma_k=diag(\Gamma_r^k,\cdots,O)\in C^{m\times n}$, $\Delta_k=diag(\Delta_r^k,\cdots,O)\in C^{n\times t}$, $H_k=diag(H_r^k,\cdots,O)\in C^{t\times m}$ and $r=\min\{m,n,t\}$ with

$$\Gamma_r^k=diag(\gamma_1^k,\cdots,\gamma_r^k),\quad \Delta_r^k=diag(\delta_1^k,\cdots,\delta_r^k),\quad H_r^k=diag(h_1^k,\cdots,h_r^k),$$
$$\gamma_1^k\ge\cdots\ge\gamma_r^k\ge0,\quad \delta_1^k\ge\cdots\ge\delta_r^k\ge0,\quad H_r^k=h_1^k\ge\cdots\ge h_r^k\ge0.$$

Similar to the derivation in (3.3) and (3.4), we can inferred

$$\begin{aligned}&\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}|\operatorname{tr}(cI_m\pm\prod_{k=1}^{s}A_kU_kB_kW_kC_kV_k)|\\&=\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}|\operatorname{tr}(P_1^H[cI_m\pm P_1\Gamma_1Q_1U_1M_1\Delta_1N_1W_1S_1H_1T_1V_1\cdots\Delta_sN_sW_sS_sH_sT_sV_s]P_1)|\\&=\min_{U_k\in\mathbf{U}_n,W_k\in\mathbf{U}_t,V_k\in\mathbf{U}_m}|\operatorname{tr}(cI_m\pm\prod_{k=1}^{s}\Gamma_kU_k\Delta_kW_kH_kV_k)|.\end{aligned}\tag{3.6}$$

which together with Lemma 2.1 and (3.6) lead to (3.2) hold true. This completes the proof.

Remark 1 A few comments are in order.

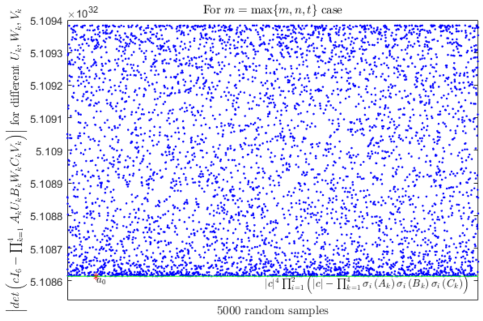
- When $m=n=t$, $k=1$ and $A_k=\Gamma_k,B_k=\Delta_k,C_k=H_k$ are nonnegative diagonal matrices with $0\le\gamma_1\le\cdots\le\gamma_n$, $0\le\delta_1\le\cdots\le\delta_n$, $0\le h_1\le\cdots\le h_n$ then $r=\min\{m,n,t\}$ and Theorem 3.1 reduces to Lemma 2.8 of [1] and Theorem 3.2 reduces to Lemma 3.1 of [1], respectively. This implies that Theorems 3.1 and 3.2 improve Lemmas 2.8 and 3.1 of [1] partially.
- In Theorems 3.1 and 3.2 we provide analytic solutions of extended constrained matrix minimization problems with different dimensional matrices. The results of this research improve Lemmas 2.8 and 3.1 of [1] partially to more general cases, which can be further applied in gene data analysis with different dimensional datasets.

4. Numerical examples

In this section we will give some synthetic examples to illustrate the efficiency of the proposed theoretical results.

(i) Case 1: for $m=\max\{m,n,t\}$, let $A_k=\begin{pmatrix}F^k\\ I_{2\times4}\end{pmatrix}\in\mathbf{C}^{6\times4}$, $B_k=\begin{pmatrix}kD+E\\ kD-E\end{pmatrix}\in\mathbf{C}^{4\times2}$, and

$C_k=\begin{pmatrix}D^{\sqrt{k}}+E & I_{2\times6}\end{pmatrix}\in\mathbf{C}^{2\times6}$, where $k=1,2,3,4$ and C,D are generated by MATLAB command *randn(2)+randn(2)*i* and F are generated by MATLAB command *randn(4)+randn(4)*i* . Let $c=2i$, we use the command $orth(\cdot)$ to generate 5000 groups of different random unitary matrices U_k, W_k, V_k, the results in Fig. 1 and Fig. 2.



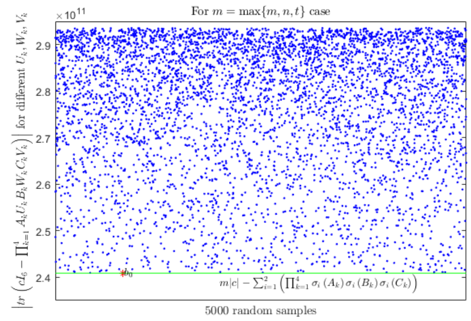


Fig. 1: Computing $|\det(cI_m \pm \prod_{k=1}^{4} A_k U_k B_k W_k C_k V_k)|$. Fig. 2: Computing $|\operatorname{tr}(cI_m \pm \prod_{k=1}^{4} A_k U_k B_k W_k C_k V_k)|$.

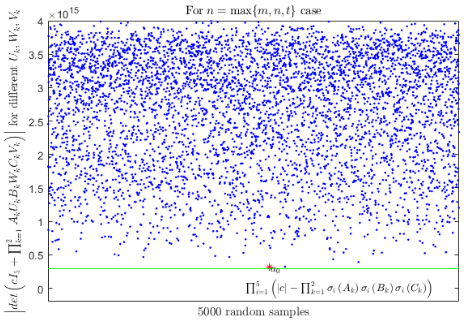
In Fig. 1 the green line represent the value of $\prod_{i=1}^{r}(\prod_{k=1}^{4}\sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k) - |c|)$. The blue dots indicate that the value of $|\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V_k)|$ with 5000 groups of different random unitary matrices U_k, W_k, V_k. If let $U_k = Q_k^H M_k^H$, $W_k = N_k^H S_k^H$, and $V_k = T_k^H P_{k+1}^H$, with Q_k, M_k, N_k, S_k, T_k, P_{k+1} given by (3.3), then $a_0 = |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V_k)|$ is computed marked by red "*". Seen in Fig. 1 we have

$$|\det(cI_m \pm A_k U_k B_k W_k C_k V_k)| \geq \prod_{i=1}^{r}(\prod_{k=1}^{4}\sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k) - |c|) = a_0.$$

In Fig. 2 the green line represent the value of $m|c| - \sum_{i=1}^{r}(\prod_{k=1}^{4}\sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k))$. The blue dots calculated from 5000 groups of samples represent $|\operatorname{tr}(cI_m \pm \prod_{k=1}^{4} A_k U_k B_k W_k C_k V_k)|$. If let $U_k = Q_k^H M_k^H$, $W_k = N_k^H S_k^H$, and $V_k = T_k^H P_{k+1}^H$, with Q_k, M_k, N_k, S_k, T_k, P_{k+1} given by (3.6), then $b_0 = |\operatorname{tr}(cI_m \pm \prod_{k=1}^{4} A_k U_k B_k W_k C_k V_k)|$ is computed marked by red "*". Seen in Fig. 2 we have

$$|\operatorname{tr}(cI_m \pm A_k U_k B_k W_k C_k V_k)| \geq m|c| - \sum_{i=1}^{r}(\prod_{k=1}^{4}\sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k)) = b_0.$$

(ii) Case 2: for $n = \max\{m,n,t\}$, let $A_k = \begin{pmatrix} D^k \\ I_5 \end{pmatrix} \in \mathbf{C}^{5\times 10}$, $B_k = \begin{pmatrix} \sqrt{k}E & I_{2\times 8} \end{pmatrix} \in \mathbf{C}^{4\times 12}$, and $C_k = \begin{pmatrix} D^{\sqrt{k}} \\ I_{3\times 5} \end{pmatrix} \in \mathbf{C}^{12\times 8}$, where $k = 1, 2$ and C, D are generated by MATLAB command *randn(5)+randn(5)*i* and F are generated by MATLAB command *randn(8)+randn(8)*i* . Let $c = 20 - 10i$, we use the command $orth(\cdot)$ to generate 5000 groups of different random unitary matrices U_k, W_k, V_k, the results in Fig. 3 and Fig. 4.



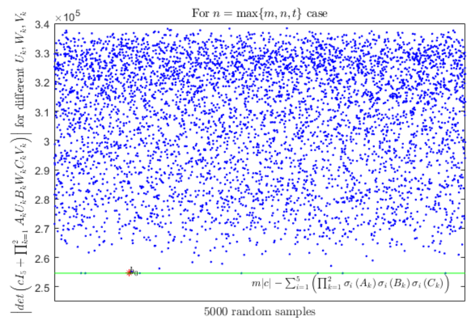


Fig. 3: Computing $|\det(cI_m \pm \prod_{k=1}^{2} A_k U_k B_k W_k C_k V_k)|$. Fig. 4: Computing $|\det(cI_m \pm \prod_{k=1}^{2} A_k U_k B_k W_k C_k V_k)|$.

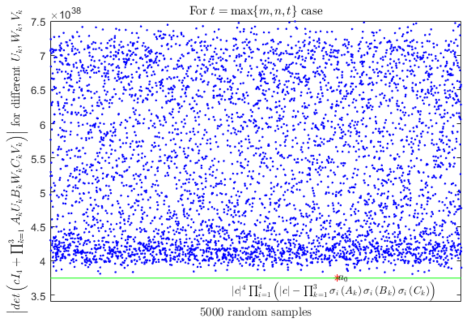
In Fig. 3 the green line represent the value of $|c|^{m-r}\prod_{i=1}^{r}(\prod_{k=1}^{2}\sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k)-|c|)$. The blue dots indicate that the value of $|\det(cI_m \pm \prod_{k=1}^{2} A_k U_k B_k W_k C_k V_k)|$ with 5000 groups of different random unitary matrices U_k, W_k, V_k. If let $U_k = Q_k^H M_k^H$, $W_k = N_k^H S_k^H$, and $V_k = T_k^H P_{k+1}^H$, with Q_k, M_k, N_k, S_k, T_k, P_{k+1} given by (3.3), then $a_0 = |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V_k)|$ is computed marked by red "*". Seen in Fig. 3 we have

$$|\det(cI_m \pm A_k U_k B_k W_k C_k V_k)| \geq |c|^4 \prod_{i=1}^{r}(\prod_{k=1}^{4}\sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k)-|c|) = a_0.$$

In Fig. 4 the green line represent the value of $m|c|-\sum_{i=1}^{r}(\prod_{k=1}^{2}\sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k))$. The blue dots calculated from 5000 groups of samples represent $|\mathrm{tr}(cI_m \pm \prod_{k=1}^{2} A_k U_k B_k W_k C_k V_k)|$. If let $U_k = Q_k^H M_k^H$, $W_k = N_k^H S_k^H$, and $V_k = T_k^H P_{k+1}^H$, with Q_k, M_k, N_k, S_k, T_k, P_{k+1} given by (3.6), then $b_0 = |\mathrm{tr}(cI_m \pm \prod_{k=1}^{2} A_k U_k B_k W_k C_k V_k)|$ is computed marked by red "*". Seen in Fig. 4 we have

$$|\mathrm{tr}(cI_m \pm A_k U_k B_k W_k C_k V_k)| \geq m|c| - \sum_{i=1}^{r}(\prod_{k=1}^{4}\sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k)) = b_0.$$

(iii) Case 3: for $t = \max\{m,n,t\}$, let $A_k = \begin{pmatrix} \sqrt{k}D \\ I_4 \end{pmatrix} \in \mathbf{C}^{8\times 4}$, $B_k = \begin{pmatrix} D^{\sqrt{k}} & I_{4\times 8} \end{pmatrix} \in \mathbf{C}^{4\times 12}$, and $C_k = \begin{pmatrix} F^{\sqrt{k}} \\ I_{8\times 4} \end{pmatrix} \in \mathbf{C}^{12\times 8}$, where $k = 1,2,3$ and C, D are generated by MATLAB command *randn(4)+randn(4)*i* and F are generated by MATLAB command *randn(8)+randn(8)*i* . Let $c = 1+i$, we use the command $orth(\cdot)$ to generate 5000 groups of different random unitary matrices U_k, W_k, V_k, the results in Fig. 5 and Fig. 6.



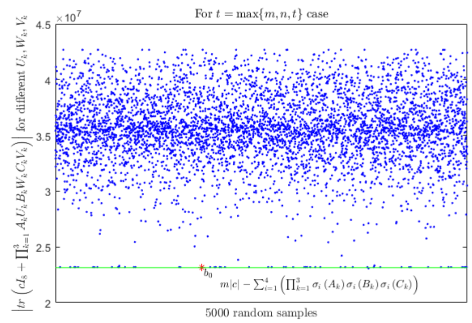


Fig. 5: Computing $|\det(cI_m \pm \prod_{k=1}^{3} A_k U_k B_k W_k C_k V_k)|$. Fig. 6: Computing $|\det(cI_m \pm \prod_{k=1}^{3} A_k U_k B_k W_k C_k V_k)|$.

In Fig. 5 the green line represent the value of $|c|^{m-r} \prod_{i=1}^{r} (\prod_{k=1}^{3} \sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k) - |c|)$. The blue dots indicate that the value of $|\det(cI_m \pm \prod_{k=1}^{3} A_k U_k B_k W_k C_k V_k)|$ with 5000 groups of different random unitary matrices U_k, W_k, V_k. If let $U_k = Q_k^H M_k^H$, $W_k = N_k^H S_k^H$, and $V_k = T_k^H P_{k+1}^H$, with Q_k, M_k, N_k, S_k, T_k, P_{k+1} given by (3.3), then $a_0 = |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V_k)|$ is computed marked by red "*". Seen in Fig. 5 we have

$$|\det(cI_m \pm A_k U_k B_k W_k C_k V_k)| \geq |c|^4 \prod_{i=1}^{r} (\prod_{k=1}^{3} \sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k) - |c|) = a_0.$$

In Fig. 6 the green line represent the value of $m|c| - \sum_{i=1}^{r} (\prod_{k=1}^{3} \sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k))$. The blue dots calculated from 5000 groups of samples represent $|\mathrm{tr}(cI_m \pm \prod_{k=1}^{3} A_k U_k B_k W_k C_k V_k)|$. If let $U_k = Q_k^H M_k^H$, $W_k = N_k^H S_k^H$, and $V_k = T_k^H P_{k+1}^H$, with Q_k, M_k, N_k, S_k, T_k, P_{k+1} given by (3.6), then $b_0 = |\mathrm{tr}(cI_m \pm \prod_{k=1}^{3} A_k U_k B_k W_k C_k V_k)|$ is computed marked by red "*". Seen in Fig. 6 we have

$$|\mathrm{tr}(cI_m \pm A_k U_k B_k W_k C_k V_k)| \geq m|c| - \sum_{i=1}^{r} (\prod_{k=1}^{3} \sigma_i(\tilde{A}_k)\sigma_i(\tilde{B}_k)\sigma_i(\tilde{C}_k)) = b_0.$$

Hence, case (i)-(iii) verify the efficiency of Theorem 3.1.

5. Concluding remarks

In this paper we give analytic solutions of a class of constrained matrix minimization problems as follows:

$$\min_{U_k \in \mathbf{U}_n, W_k \in \mathbf{U}_t, V_k \in \mathbf{U}_m} |\det(cI_m \pm \prod_{k=1}^{s} A_k U_k B_k W_k C_k V_k)|$$

$$\min_{U_k \in \mathbf{U}_n, W_k \in \mathbf{U}_t, V_k \in \mathbf{U}_m} |\mathrm{tr}(cI_m \pm \prod_{k=1}^{4} A_k U_k B_k W_k C_k V_k)|,$$

where $A_k \in \mathbf{C}^{m\times n}, B_k \in \mathbf{C}^{n\times t}, C_k \in \mathbf{C}^{t\times m}$, c is a complex number, I_m denotes the m-order identity matrix,

$\mathbf{C}^{m\times n}$ denotes $m\times n$ complex matrix set, $\det(\cdot)$ and $\mathrm{tr}(\cdot)$ denote matrix determinant and trace function, respectively. Our proposed results improve the corresponding existing ones in [1]. Numerical examples are given to illustrate the efficiency of the proposed theoretical results.

6. **References**

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