ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 17, No. 1, 2022, *pp*.003-018

The nonexistence of global solutions for a damped time fractional diffusion equation with nonlinear memory

Qian Qian and Yaning Li +

School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

(Received October 05, 2021, accepted January 11, 2022)

Abstract: In this paper, we study the non-global existence of solutions to the following time fractional nonlinear diffusion equations

$$\begin{cases} {}^C D^{\alpha}_{0|t} u - \triangle u + (1+t)^r u_t = I^{\beta}_{0|t}(|u|^{p-1}u), \ x \in \mathbb{R}^n, \ t>0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \ x \in \mathbb{R}^n, \end{cases}$$

where $1<\alpha<2$, $\beta\in(0,1)$, $1<\alpha+\beta<2$, $r\in(-1,1)$, $p>1$, $u_0,u_1\in L^q(\mathbb{R}^n)(q>1)$ and ${}^C D^{\alpha}_{0|t}u$ denotes left Caputo fractional derivative of order α . By using the test function method, we prove that the problem admits no global weak solution with suitable initial data when p falls in different intervals. Our results generalize that in [4].

Keywords: fractional derivative, blow-up, test function, nonlinear memory.

1. Introduction

Fractional differential equations are widely used to describe abnormal diffusion, Hamiltonian chaos, dynamical systems with chaotic mechanical behavior and so on, ect. see [5, 10, 12] and the references therein. In addition, the fractional evolution equations of time appearing in electromagnetic, acoustic and mechanical phenomena have also attracted much attention. Such equations replace the first time derivative with the fractional derivative of α, where α belongs to (0,1). In recent years, time fractional differential equations yield many different results, see [1, 7, 8, 13, 14, 15] and the references therein. For instance, in [11], the existence and properties of solutions of time fractional equations in bounded domains are considered by using the expansion of characterisitic functions. In [2], the quasilinear abstract time fractional development equation in continuous interpolation space is studied. In [13], the L^p-type maximum regularity results for abstract parabolic Voltera equations with inhomogeneous boundary data problems are established by using pure operator theory. In [6], the $L^p(L^q)$ theory of semilinear time fractional equations with variable coefficients is given by using the classical theory of partial differential equation theory, such as the Marcinkiewicz interpolation theorem, the Calderon-Zygmund theorem, and the perturbation arguments.

This paper is concerned with the non-global existence of solutions to the Cauchy problem for a nonlinear time-fractional with nonlinear memory

$$\begin{cases} {}^C D^{\alpha}_{0|t} u - \triangle u + (1+t)^r u_t = I^{1-\gamma}_{0|t}(|u|^{p-1}u), \ x \in \mathbb{R}^N, \ t>0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \ x \in \mathbb{R}^N, \end{cases} \tag{0. 1}$$

where $1<\alpha<2$, $\beta\in(0,1)$, $1<\alpha+\beta<2$, $r\in(-1,1)$, $p>1$, $u_0,u_1\in L^q(R^n)(q>1)$ and ${}^C D^{\alpha}_{0|t}u$

+ Corresponding author. Tel:17705183752
E-mail address: liyn11@126.com.

denotes left Riemann-Liouville fractional derivative of order α, $I_{0|t}^{\beta}$ denotes left Riemann-Liouville fractional integrals of order β and is defined by

$$I_{0|t}^{\beta}u=\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{1-\beta}u(s)ds.$$

In the case $\alpha=2$, $\beta=1-\gamma$, the damped wave equation

$$\begin{cases} u_{tt}-\triangle u+(1+t)^r u_t = I_{0|t}^{1-\gamma}(|u|^{p-1}u), \ (t,x)\in\mathbb{R}^n, \\ u(0,x)=u_0(x), u_t(0,x)=u_1(x), \ x\in\mathbb{R}^n. \end{cases} \tag{0.2}$$

has been studied by [4], they proved the blow-up results for local (in time) Sobolev solutions. And applied the test function method to showed that when $\int_{R^n} u_0(x)dx>0$ and $\int_{R^n} u_1(x)dx>0$, if

$$\frac{p}{p-1}>\inf_{d>0}\max\{\frac{\frac{nd}{2}+1}{1-\gamma+d},\sqrt{\frac{\frac{nd}{2}+1}{(1-\gamma)(1-r)}+(\frac{1-\gamma-(1-r)\frac{nd}{2}}{2(1-\gamma)(1-r)})^2}-\frac{1-\gamma-(1-r)\frac{nd}{2}}{2(1-\gamma)(1-r)}\}$$

for $r\in(-1,0)$ or $r\in(0,1)$, or $\frac{p}{p-1}>\frac{\frac{n}{2}+1}{2-\gamma}$ for $r=0$, the weak solution of (1.2) do not exist global in time. In the results of our paper, when $\alpha=2$, that is what the [4] says.

Recently, there are many papers which considered the existence and nonexistence of the global solution to semilinear time fractional diffusion equation and diffusion equation with nonlinear memory.

In [3], using the test function method, Fino and Kirane considered a heat equation with nonlinear memory. They generalized test function method to fractional case and determined the Fujita critical exponent of the problem.

For the nonlinear time fractional diffusion equation (i.e. (1.1) with $\gamma=1$ and the damped term $(1+t)^r u_t$ do not exist),

$$\begin{cases} {}^{C}D_{0|t}^{\alpha}u-\triangle u=|u|^{p-1}u, \ x\in\mathbb{R}^N, \ t>0, \\ u(0,x)=u_0(x)\geq 0, \ x\in\mathbb{R}^N. \end{cases} \tag{0.3}$$

Zhang and Sun [16] studied the local existence of this problem, where $u_0\in C_0(R^N)$, they obtained that if $1<p<1+\frac{2}{N}$, u blows up in finite time, and if $p\geq 1+\frac{2}{N}$, the problem (1.3) exists a global solution for small initial data. It should be noted that in the critical case $p=1+\frac{2}{N}$, the solution of (1.3) can exist globally. In [15], Zhang and Li studied the local existence and uniqueness of mild solutions of problem (1.3), and used test function method to show the blow-up and global existence of the solutions to (1.3).

As far as we know, there are few paper consider the existence and non-existence for solutions of damped fractional diffusion equation. Motivated by the above results, in this paper, we study the non-global existence of problem (1.1). Particularly, for $u_0,u_1\in L^q(R^n)(q>1)$, $p+q=1$, $u_0>0$, $u_0\not\equiv 0$, $\chi(x)=(\int_{R^n}e^{-\sqrt{n^2+|x|^2}}dx)^{-1}e^{-\sqrt{n^2+|x|^2}}$, we will show that with the following two conditions, when $q > q_1$ for $r \in (0,1)$, or $q > \frac{1+\beta}{\beta}$ for $r \in (-1,0)$ or $r = 0$, or $q > q_2$ for $r \in (0,1)$, or $q > \frac{n\alpha+2}{2(\beta+1)}$ for $r = 0$, or $q > q_3$ for $r \in (-1,0)$, the weak solution of (1.1) admit no global weak solution.

(i) $\int_{R^n} u_0(x)\chi(x)dx > 0$ and $\int_{R^n} u_1(x)\chi(x)dx > 0$.

(ii) $\int_{R^n} u_0(x)dx > 0$ and $\int_{R^n} u_1(x)dx > 0$.

where

$$q_1 = \max\left\{\frac{1+\beta}{\beta}; \frac{-\beta r + \sqrt{(\beta r)^2 - 4(\beta-1)}}{2\beta(1-r)}\right\},$$

$$q_2 = \frac{n\alpha(1-r) - 2\beta}{4\beta(1-r)} + \sqrt{\left(\frac{2\beta + (1-r)n\alpha}{4\beta(1-r)}\right)^2 + \frac{1}{\beta(1-r)}},$$

$$q_3 = \inf \max\left\{\frac{n\alpha+2}{2(\alpha+\beta)}; \frac{(1-r)n\alpha - 2\beta}{4\beta(1-r)} + \sqrt{\left(\frac{2\beta + (1-r)n\alpha}{4\beta(1-r)}\right)^2 + \frac{1}{\beta(1-r)}}\right\}.$$

This paper is organized as follows. In Section 2, some preliminaries are presented and the main results are listed. Section 3 is devoted to giving the proof of our main results.

2. Preliminaries and main results

To derive the nonexistence of results for the problem (1.1), let us recall some notations and definitions, we state some results about fractional detrivative and fractional integral which will be used in the proof of our main results. For $T > 0$, $\alpha \in (1,2]$, the Riemann-Liouville fractional integrals are defined by

$$I_{0|t}^{\alpha} u = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds, \; I_{t|T}^{\alpha} u = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(s-t)^{1-\alpha}} ds.$$

For $\alpha \in (1,2]$ and T> 0, the Caputo fractional derivatives satisfy that if $g \in AC^2([0,T])$, then ${}^C D_{0|t}^{\alpha} g$ and ${}^C D_{t|T}^{\alpha} g$ a.e. exist on $[0,T]$ and

$${}^C D_{0|t}^{\alpha} g = \frac{d^2}{dt^2} I_{0|t}^{2-\alpha}[g(s) - g'(0)s - g(0)] = I_{0|t}^{2-\alpha} g'',$$

$${}^C D_{t|T}^{\alpha} g = \frac{d^2}{dt^2} I_{t|T}^{2-\alpha}[g(t) - g'(T)t - g(T)] = I_{t|T}^{2-\alpha} g''.$$

In addition, if $f \in C^1([0,T])$, ${}^C D_{t|T}^{\alpha} f \in L^1(0,T)$, $g \in AC^2([0,T])$ and $g(T) = g'(T) = 0$, then we have the following formula of integration by parts

$$\int_0^T {}^C D_{0|t}^{\alpha} f \cdot g dt = \int_0^T (f(t) - f'(0)t - f(0))^C D_{t|T}^{\alpha} g dt \tag{2.1}$$

Moreover, given $\beta \gg 1$ and $T > 0$. Let $\varphi(t) = (1 - \frac{t}{T})_+^k$, then for all $\alpha \in (1,2)$ we have

$${}^C D_{0|t}^{\alpha} \varphi_T \, {}^C D_{0|t}^{\beta} \varphi_T = {}^C D_{0|t}^{\alpha+\beta} \varphi_T \tag{2.2}$$

Furthermore, by [9], let $\alpha\in(n-1,n)$, $\beta\in(0,1)$, $\alpha+\beta\in(n-1,n]$, $\varphi_T\in AC^{n+1}([0,T])$, then we have

$$ {}^C D_{0|t}^{\alpha}\varphi_T \, {}^C D_{0|t}^{\beta}\varphi_T = {}^C D_{0|t}^{\alpha+\beta}\varphi_T \tag{2. 3}$$

Throughout the present paper we write $f\lesssim g$ when there exist a constant $C>0$ such that $f\leqslant Cg$.

Now, we give the definition of the weak solution for (1.1).

Definition 2.1 Let $T>0$, $\alpha\in(1,2)$, $\beta\in(0,1)$, $q\geq 1$, and $b(t)=(1+t)^r$ with $r\in(-1,1)$. A weak solution for the Cauchy problem (1.1) on $(0,T)\times R^n$ with the data $u_0,u_1\in L^1_{loc}(R^n)$ is a locally integrable function $u\in L^p((0,T),L^p_{loc}(R^n))$ satisfying the relation

$$\begin{aligned}&\int_{R^n}\int_0^T I_{0|t}^{\beta}(|u|^p)\varphi(t,x)dtdx+\int_{R^n}\int_0^T (u_1(x)t+u_0(x))^c D_{t|T}^{\alpha}\varphi(t,x)dtdx+b(0)\int_{R^n}u_0(x)\varphi(0,x)dx\\&=\int_{R^n}\int_0^T u(t,x)^c D_{t|T}^{\alpha}\varphi(t,x)dtdx-\int_{R^n}\int_0^T u(t,x)\Delta\varphi(t,x)dtdx\\&-\int_{R^n}\int_0^T u(t,x)b(t)\varphi_t(t,x)dtdx-\int_{R^n}\int_0^T u(t,x)b'(t)\varphi(t,x)dtdx.\end{aligned}\tag{2. 4}$$

for every $\varphi\in C^{2,2}_{x,t}(R^n\times[0.T])$, $\varphi_t\in C^{2,0}_{x,t}(R^n\times(0,T))$, $\varphi(T,\cdot)=\varphi_t(T,\cdot)=0$.

We have the following results about the no global existence of weak solution for (1.1).

Theorem 2.2 Let $\beta\in(0,1)$ and $p\in(1,\infty)$, $\alpha\in(1,2)$, $\alpha+\beta\in(1,2)$. Assume that the data $(u_0,u_1)\in L^q(R^n), q>1$ and satisfies the conditions

$$\int_{R^n}u_0(x)\chi(x)dx>0,\ \int_{R^n}u_1(x)\chi(x)dx>0,\tag{2. 5}$$

Where $\chi(x)=(\int_{R^n}e^{-\sqrt{n^2+|x|^2}}dx)^{-1}e^{-\sqrt{n^2+|x|^2}}$, then if p satisfies for $r\in(0,1)$ the condition

$$\frac{p}{p-1}>\max\Big\{\frac{1+\beta}{\beta};\frac{-\beta r+\sqrt{(\beta r)^2-4(\beta-1)}}{2\beta(1-r)}\Big\}\tag{2. 6}$$

or for $r\in(-1,0)$ and $r=0$ the condition

$$\frac{p}{p-1}>\frac{1+\beta}{\beta}\tag{2. 7}$$

then the problem (1.1) admits no global weak solution.

Theorem 2.3 Let $\beta\in(0,1)$, $r\in(-1,1)$, $\alpha\in(1,2)$, $\alpha+\beta\in(1,2)$, $p,q>1$. Assume that the data $(u_0,u_1)\in L^q(R^n)$ and satisfies the conditions

$$\int_{R^n}u_0(x)dx>0,\ \int_{R^n}u_1(x)dx>0.\tag{2. 8}$$

Then, if p satisfies for $r\in(0,1)$ the condition

$$\frac{p}{p-1}>\frac{n\alpha(1-r)-2\beta}{2\beta(1-r)}+\sqrt{\frac{1}{(1-r)^2}+(\frac{n\alpha}{2\beta})^2+\frac{n\alpha+4}{\beta(1-r)}},\tag{2. 9}$$

or for $r=0$ the condition

$$\frac{p}{p-1} > \frac{n\alpha+2}{2(\beta+1)}, \tag{2.10}$$

or for $r \in (-1,0)$ the condition

$$\frac{p}{p-1} > \inf \max\left\{\frac{n\alpha+2}{2\alpha+\beta}; \frac{(1-r)n\alpha-2\beta}{4\beta(1-r)} + \sqrt{\left(\frac{2\beta+(1-r)n\alpha}{4\beta(1-r)}\right)^2 + \frac{1}{\beta(1-r)}}\right\}, \tag{2.11}$$

then the problem (1.1) admits no global weak solution.

3. Proof of main results

In this section, we give the proof of Theorems 2.2 and 2.3. For the sake of convenience, we use C to denote a positive constant which may vary as a result of the estimation process, but it does not necessarily influence the analysis of the problem.

Proof of Theorem 2.2 Suppose that u is weak solution of (1.1). Take $\psi \in C_0^\infty(R^n)$ such that $0 \le \psi(x) \le 1$ and $\psi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2. \end{cases}$ Let $\psi_n(x) = \psi(\frac{x}{n})$, $n = 1,2,\dots$ and $\varphi_T \in L^q(0,T)$, $\varphi_T(T) = 0$, $\varphi_T^{'}(T) = 0$ and $\varphi_T \ge 0$. Taking $\varphi(x,t) = \chi(x)\psi_n(x)^C D_{t|T}^{\beta}\varphi_T(t)$ as a test function in definition of weak solution, using (2.3), then

$$\begin{aligned}
&\int_{R^n}\int_0^T |u|^p \chi(x)\varphi_T(t)dtdx + \int_{R^n}\int_0^T (u_1 t + u_0)\chi(x)\psi_n(x)^C D_{t|T}^{\alpha+\beta}\varphi_T(t)dtdx + b(0)C_1 T^{-\beta}\int_{R^n} u_0(x)\chi(x)\psi_n(x)dx \\
&= \int_{R^n}\int_0^T u\chi(x)\psi_n(x)^C D_{t|T}^{\alpha+\beta}\varphi_T(t)dtdx + \int_{R^n}\int_0^T u[-\Delta(\chi(x)\psi_n(x))]^C D_{t|T}^{\beta}\varphi_T(t)dtdx \\
&-\int_{R^n}\int_0^T ub(t)\chi(x)\psi_n(x)^C D_{t|T}^{\beta+1}\varphi_T(t)dtdx - \int_{R^n}\int_0^T ub'(t)\chi(x)\psi_n(x)^C D_{t|T}^{\beta}\varphi_T(t)dtdx.
\end{aligned} \tag{3.1}$$

A simple calculation shows that $\Delta\chi = \left(-\frac{n}{\sqrt{n^2+|x|^2}} + \frac{|x|^2}{n^2+|x|^2} + \frac{|x|^2}{(n^2+|x|^2)^{\frac{3}{2}}}\right)\chi$. Hence, $|\Delta\chi| \le 3\chi$, note that $\Delta(\chi\psi_n) = (\Delta\chi)\psi_n + 2\nabla\chi\cdot\Delta\psi_n + (\Delta\psi_n)\chi$. Then by (3.1), letting $n \to \infty$ and using the dominated convergence theorem, we have

$$\begin{aligned}
&\int_{R^n}\int_0^T |u|^p \chi\varphi_T dtdx + \int_{R^n}\int_0^T (u_1 t + u_0)\chi^C D_{t|T}^{\alpha+\beta}\varphi_T dtdx + b(0)C_1 T^{-\beta}\int_{R^n} u_0\chi dx \\
&\le \int_{R^n}\int_0^T |u|\chi\left|{}^C D_{t|T}^{\alpha+\beta}\varphi_T(t)\right|dtdx + \int_{R^n}\int_0^T 3|u|\chi\left|{}^C D_{t|T}^{\beta}\varphi_T\right|dtdx \\
&+\int_{R^n}\int_0^T |b(t)||u|\chi\left|{}^C D_{t|T}^{\beta+1}\varphi_T\right|dtdx + \int_{R^n}\int_0^T |b'(t)||u|\chi\left|{}^C D_{t|T}^{\beta}\varphi_T\right|dtdx.
\end{aligned} \tag{3.2}$$

Denote $f(t) = \int_{R^n}|u|\chi dx$, $f(0) = \int_{R^n} u_0\chi dx > 0$, $f(1) = \int_{R^n} u_1\chi dx \geqslant 0$, then $f(t) \ge 0$. Since $u \in C([0,T], L^q(R^n))$, we know $f \in C([0,T])$. So follows from Jensen's inequality and (3.2), we have

$$\begin{aligned}
&\int_0^T f^p\varphi_T dt + \int_0^T (f(1)t + f(0))^C D_{t|T}^{\alpha+\beta}\varphi_T dt + C_1 T^{-\beta} b(0) f(0) \\
&\leqslant \int_0^T f\left|{}^C D_{t|T}^{\alpha+\beta}\varphi_T(t)\right|dt + 3\int_0^T f\left|{}^C D_{t|T}^{\beta}\varphi_T(t)\right|dt + \int_0^T b(t) f\left|{}^C D_{t|T}^{\beta+1}\varphi_T(t)\right|dt + \int_0^T b'(t) f\left|{}^C D_{t|T}^{\beta}\varphi_T(t)\right|dt.
\end{aligned} \tag{3.3}$$

Taking $\varphi_T(t)=(1-\frac{t}{T})^k(k\geq\frac{\beta p+1}{p-1};t\leq T)$, we have

$$\begin{aligned}
&\int_0^T f^p\varphi_2 dt+\int_0^T (f(1)t+f(0))^C D_{t|T}^{\alpha+\beta}\varphi_2 dt+C_1T^{-\beta}b(0)f(0)\\
&\leqslant\int_0^T f\left|^C D_{t|T}^{\alpha+\beta}\varphi_2(t)\right|dt+3\int_0^T f\left|^C D_{t|T}^{\beta}\varphi_2(t)\right|dt\\
&\quad+\int_0^T b(t)f\left|^C D_{t|T}^{\beta+1}\varphi_2(t)\right|dt+\int_0^T b'(t)f\left|^C D_{t|T}^{\beta}\varphi_2(t)\right|dt\\
&=\int_0^T f\varphi_2^{\frac{1}{p}}\varphi_2^{-\frac{1}{p}}{}^C D_{t|T}^{\alpha+\beta}\varphi_2 dt+3\int_0^T f\varphi_2^{\frac{1}{p}}\varphi_2^{-\frac{1}{p}}{}^C D_{t|T}^{\beta}\varphi_2 dt\\
&\quad+\int_0^T b(t)f\varphi_2^{\frac{1}{p}}\varphi_2^{-\frac{1}{p}}{}^C D_{t|T}^{\beta+1}\varphi_2 dt+\int_0^T b'(t)f\varphi_2^{\frac{1}{p}}\varphi_2^{-\frac{1}{p}}{}^C D_{t|T}^{\beta}\varphi_2 dt\\
&\leq\frac{1}{2}\int_0^T f^p\varphi_2 dt+C_2\int_0^T\varphi_2^{-\frac{1}{p-1}}({}^C D_{t|T}^{\alpha+\beta}\varphi_2)^{\frac{p}{p-1}}dt+C_3\int_0^T\varphi_2^{-\frac{1}{p-1}}({}^C D_{t|T}^{\beta}\varphi_2)^{\frac{p}{p-1}}dt\\
&\quad+C_4\int_0^T b(t)^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}({}^C D_{t|T}^{\beta+1}\varphi_2)^{\frac{p}{p-1}}dt+C_5\int_0^T|b'(t)|^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}({}^C D_{t|T}^{\beta}\varphi_2)^{\frac{p}{p-1}}dt\\
&\lesssim\frac{1}{2}\int_0^T f^p\varphi_2 dt+I_1+I_2+I_3+I_4.
\end{aligned}\tag{3.4}$$

The treatment of the second term of left-hand side (3.4) as follows:

$$\int_0^T (f(1)t+f(0))|{}^c D_{t|T}^{\alpha+\beta}\varphi_2|\ dt\leq CT^{1-\alpha-\beta}(f(0)+f(1)t),$$

Let $t=T\tau$,we have

$$\begin{aligned}
I_1&=\int_0^T\varphi_2^{-\frac{1}{p-1}}({}^C D_{t|T}^{\alpha+\beta}\varphi_2)^{\frac{p}{p-1}}dt=\int_0^1(1-\tau)^{-\frac{k}{p-1}}(T^{1-\alpha-\beta}(1-\tau)^{k-\alpha-\beta})^{\frac{p}{p-1}}Td\tau\\
&=T^{1-(\alpha+\beta)\frac{p}{p-1}}\int_0^1(1-\tau)^{(k-\alpha-\beta)\frac{p}{p-1}-\frac{k}{p-1}}d\tau=CT^{1-(\alpha+\beta)\frac{p}{p-1}},
\end{aligned}$$

Similarly,

$$I_2=CT^{1-\beta\frac{p}{p-1}}.$$

Now we estimate I_3 and I_4. For this reason, we distinguish between the three cases $r\in(0,1)$, $r=0$ and $r\in(-1,0)$ in $b(t)=(1+T)^r$.

Case 1 When $r\in(0,1)$, the function $b(t)=(1+t)^r$ is strictly increasing. So

$$b(t)\leq b(T)=(1+T)^r,\quad t\in[0,T].$$

We have

$$\begin{aligned}
I_3&=\int_0^T b(t)^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}({}^C D_{t|T}^{\beta+1}\varphi_2)^{\frac{p}{p-1}}dt\leq\int_0^1(1+T)^{\frac{rp}{p-1}}(1-\tau)^{-\frac{k}{p-1}}(T^{-\beta-1}(1-\tau)^{k-\beta-1})^{\frac{p}{p-1}}Td\tau\\
&=(1+T)^{\frac{rp}{p-1}}T^{1-(\beta+1)\frac{p}{p-1}}\int_0^1(1-\tau)^{(k-\beta-1)\frac{p}{p-1}-\frac{k}{p-1}}d\tau=CT^{1-(\beta+1-r)\frac{p}{p-1}},
\end{aligned}$$

and

$$I_4=\int_0^T |b'(t)|^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}({}^C D_{t|T}^{\beta}\varphi_2)^{\frac{p}{p-1}}dt = C\int_0^1 (1+t)^{(r-1)\frac{p}{p-1}}(1-\tau)^{-\frac{k}{p-1}}(T^{-\beta}(1-\tau)^{k-\beta})^{\frac{p}{p-1}}Td\tau$$

$$=CT^{1-\beta\frac{p}{p-1}}\int_0^1(1+\tau T)^{(r-1)\frac{p}{p-1}}(1-\tau)^{(k-\beta)\frac{p}{p-1}-\frac{k}{p-1}}d\tau.$$

We split the integral

$$\int_0^1=\int_0^{T^{-m}}+\int_{T^{-m}}^1$$

into two integrals. Let us choose $m\in(0,1)$. If k is large, then

$$\int_0^{T^{-m}}(1+\tau T)^{(r-1)\frac{p}{p-1}}(1-\tau)^{(k-\beta)\frac{p}{p-1}-\frac{k}{p-1}}d\tau\leqslant CT^{-m}.$$

In the second integral we use the integrand the estimate

$$(1+\tau T)^{(r-1)\frac{p}{p-1}}\le(1+T^{1-m})^{(r-1)\frac{p}{p-1}}.$$

If k is large,

$$\int_{T^{-m}}^1(1+\tau T)^{(r-1)\frac{p}{p-1}}(1-\tau)^{\frac{(k-\beta)p-k}{p-1}}d\tau\leqslant(1+T^{1-m})^{\frac{(r-1)p}{p-1}}\int_{T^{-m}}^1(1-\tau)^{\frac{(k-\beta)p-k}{p-1}}d\tau\leqslant C(1+T^{1-m})^{\frac{(r-1)p}{p-1}}.$$

The optimal choice of m is given by the condition (for large T)

$$T^{-m}\sim T^{(1-m)(r-1)\frac{p}{p-1}},$$

that is,

$$m=\frac{(1-r)p}{p-1+(1-r)p}.$$

Therefore, we get the estimate

$$I_4\le CT^{1-\frac{\beta p}{p-1}-\frac{(1-r)p}{p-1+(1-r)p}}.$$

Based on the above estimate, we can conclude

$$\frac{1}{2}\int_0^T f^p\varphi_2 dt+CT^{1-\alpha-\beta}(f(0)+f(1)t)+C_1T^{-\beta}b(0)f(0)$$

$$\leqslant C(T^{1-(\alpha+\beta)\frac{p}{p-1}}+T^{1-\beta\frac{p}{p-1}}+T^{1-(\beta+1-r)\frac{p}{p-1}}+T^{1-\frac{\beta p}{p-1}-\frac{(1-r)p}{p-1+(1-r)p}}).$$

When $t=0$, we have $b(0)=1$, by the condition (2.5), we can get that $f(1)\ge 0$, $f(0)>0$, so

$$CT^{-\beta}f(0)\leqslant C(T^{1-(\alpha+\beta)\frac{p}{p-1}}+T^{1-\beta\frac{p}{p-1}}+T^{1-(\beta+1-r)\frac{p}{p-1}}+T^{1-\frac{\beta p}{p-1}-\frac{(1-r)p}{p-1+(1-r)p}}).$$

That is

$$f(0)<C(T^{1-(\alpha+\beta)\frac{p}{p-1}+\beta}+T^{1-\beta\frac{p}{p-1}+\beta}+T^{1-(\beta+1-r)\frac{p}{p-1}+\beta}+T^{1-\beta\frac{p}{p-1}-\frac{(1-r)p}{p-1+(1-r)p}+\beta}),$$

Let all exponent of T be negative is to guarantee

$$1-(\alpha+\beta)\frac{p}{p-1}+\beta<0,\quad 1-\beta\frac{p}{p-1}-\frac{(1-r)p}{p-1+(1-r)p}+\beta<0$$

since (2.6), we get $f(0)=0$ by taking $T\to\infty$, which contradicts with $f(0)>0$. Hence, there no-global weak solution for (1.1).

Case 2 For $r=0$, then the function $b(t)\equiv 1$. Repeating the estimates from the previous subsection we arrive at（$I_4=0$）

$$\frac{1}{2}\int_0^T f^p\varphi_2 dt+CT^{1-\alpha-\beta}f(0)\leqslant C(T^{1-(\alpha+\beta)\frac{p}{p-1}}+T^{1-\beta\frac{p}{p-1}}+T^{1-(\beta+1)\frac{p}{p-1}}).$$

So,

$$f(0)<C(T^{1-(\alpha+\beta)\frac{p}{p-1}+\beta}+T^{1-\beta\frac{p}{p-1}+\beta}+T^{1-(\beta+1)\frac{p}{p-1}+\beta}),$$

Since $\frac{p}{p-1}>\frac{1+\beta}{\beta}$, we get $f(0)=0$ by taking $T\to\infty$, which contradicts with $f(0)>0$. Hence, there no-global weak solution for (1.1).

Case 3 When $r\in(-1,0)$, the function $b(t)=(1+t)^r$ is decreasing. If we write $b(t)=(1+t)^{-s}$, $s\in(0,1)$, then

$$I_3=T^{1-(\beta+1)\frac{p}{p-1}}\int_0^1(1+\tau T)^{-\frac{sp}{p-1}}(1-\tau)^{-\frac{k}{p-1}}(1-\tau)^{\frac{(k-\beta-1)p}{p-1}}d\tau,$$

and

$$I_4=T^{1-\beta\frac{p}{p-1}}\int_0^1(1+\tau T)^{-\frac{(s+1)p}{p-1}}(1-\tau)^{-\frac{k}{p-1}}(1-\tau)^{\frac{(k-\beta)p}{p-1}}d\tau.$$

We split the integral

$$\int_0^1=\int_0^{T^{-m}}+\int_{T^{-m}}^1$$

into two integrals. Let us choose $m\in(0,1)$. If k is large, then

$$\int_0^{T^{-m}}(1+\tau T)^{-\frac{sp}{p-1}}(1-\tau)^{(k-\beta-1)\frac{p}{p-1}-\frac{k}{p-1}}d\tau\leqslant CT^{-m}.$$

In the second integral we use the integrand the estimate

$$(1+\tau T)^{-\frac{sp}{p-1}}\le(1+T^{1-m})^{-\frac{sp}{p-1}}.$$

If k is large,

$$\int_{T^{-m}}^1(1+\tau T)^{-\frac{sp}{p-1}}(1-\tau)^{\frac{(k-\beta-1)p-k}{p-1}}d\tau\leqslant(1+T^{1-m})^{-\frac{sp}{p-1}}\int_{T^{-m}}^1(1-\tau)^{\frac{(k-\beta-1)p-k}{p-1}}d\tau\leqslant C(1+T^{1-m})^{-\frac{sp}{p-1}}.$$

The optimal choice for *m* is given by the condition (for large T)

$$T^{-m}=T^{-(1-m)\frac{sp}{p-1}},$$

that is,

$$m = \frac{sp}{p-1+sp}.$$

So we can estimate I_3 to get

$$I_3 \le CT^{1-\frac{(\beta+1)p}{p-1}-\frac{sp}{p-1+sp}},$$

By same method, we can estimate I_4 to get

$$I_4 \le CT^{1-\frac{\beta p}{p-1}-\frac{(1+s)p}{p-1+(1+s)p}}.$$

Then we have

$$CT^{-\beta} f(0) \leqslant C(T^{1-(\alpha+\beta)\frac{p}{p-1}} + T^{1-\beta\frac{p}{p-1}} + T^{1-\frac{(\beta+1)p}{p-1}-\frac{sp}{p-1+sp}} + T^{1-\frac{\beta p}{p-1}-\frac{(1+s)p}{p-1+(1+s)p}}).$$

So,

$$f(0) < C(T^{1-(\alpha+\beta)\frac{p}{p-1}+\beta} + T^{1-\beta\frac{p}{p-1}+\beta} + T^{1-\frac{(\beta+1)p}{p-1}-\frac{sp}{p-1+sp}+\beta} + T^{1-\frac{\beta p}{p-1}-\frac{(1+s)p}{p-1+(1+s)p}+\beta}).$$

Let all exponents of T be negative is to guarantee

$$1-\beta\frac{p}{p-1}+\beta<0,$$

since (2.7), we get $f(0)=0$ by taking $T\to\infty$, which contradicts with $f(0)>0$.

Hence, there no-global weak solution for (1.1).

Next, we give the proof of Theorem 2.3.

Proof of Theorem 2.3 Let $\Phi \in C_0^\infty(R^n)$ such that $\Phi(x)=1$ for $|x|\le 1$, $\Phi(x)=0$ for $|x|>2$ and $0\le\Phi(x)\le 1$. We defined $\varphi_1(x)=(\Phi(T^{-\frac{\alpha}{2}}))^{\frac{2p}{p-1}}$ and taking $\varphi_2(t)=(1-\frac{t}{T})^k$, $k>(\alpha+\beta)\frac{p}{p-1}-1$. Assume that u is a weak solution of (1.1), and define $\varphi_T(t)={}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))$, then we have

$$\begin{aligned}
&\int_{R^n}\int_0^T I_{0|t}^{\beta}(|u|^p)\,{}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx \\
&+\int_{R^n}\int_0^T (u_1(x)t+u_0(x))\,{}^C D_{t|T}^{\alpha}\,{}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx + b(0)\int_{R^n} u_0(x)\varphi_T(0,x)dx \\
&=\int_{R^n}\int_0^T u(t,x)\,{}^C D_{t|T}^{\alpha}\,{}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx - \int_{R^n}\int_0^T u(t,x)\Delta({}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t)))dtdx \\
&-\int_{R^n}\int_0^T u(t,x)b(t)\partial_t\,{}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx - \int_{R^n}\int_0^T u(t,x)b'(t)\,{}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx.
\end{aligned} \tag{3.5}$$

1. The estimate for the left-hand of (3.5).

Including the test function, we get after using the formula of integration by parts

$$\int_{R^n}\int_0^T I_{0|t}^{\beta}(|u|^p)^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx = \int_{R^n}\int_0^T {}^C D_{0|t}^{\beta} I_{0|t}^{\beta}(|u|^p)(\varphi_1(x)\varphi_2(t))dtdx = \int_{R^n}\int_0^T |u|^p \varphi_1(x)\varphi_2(t)dtdx. \quad (3.6)$$

For the second term of the left-hand side of (3.5)

$$\int_{R^n}\int_0^T (u_1(x)t+u_0(x))^C D_{t|T}^{\alpha}\, {}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx = \int_{R^n}\int_0^T (u_1(x)t+u_0(x))^C D_{t|T}^{\alpha+\beta}(\varphi_1(x)\varphi_2(t))dtdx = \int_{R^n}\int_0^T (u_1(x)t+u_0(x))\varphi_1(x)^C D_{t|T}^{\alpha+\beta}\varphi_2(t)dtdx. \quad (3.7)$$

For the third term, after using (2.2), we get

$$b(0)\int_{R^n} u_0(x)\varphi_T(0,x)dx = b(0)\int_{R^n} u_0(x)^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(0))dx = b(0)C_1T^{-\beta}\int_{R^n} u_0(x)\varphi_1(x)dx. \quad (3.8)$$

2.The estimates for the right-hand of (3.5).

For the first term,

$$\int_{R^n}\int_0^T u(t,x)^C D_{t|T}^{\alpha}\, {}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx = \int_{R^n}\int_0^T u(t,x)\varphi_1(x)^C D_{t|T}^{\alpha+\beta}\varphi_2(t)dtdx. \quad (3.9)$$

By a simple calculation, we have, for $i=1,2,\dots,n.$

$$\frac{\partial^2}{\partial x_i^2}\phi_T(x) = \frac{2p(p+1)}{(p-1)^2}T^{-\alpha}(\Phi(T^{-\frac{\alpha}{2}}x))^{\frac{2}{p-1}}\Phi_{x_i}^2(T^{-\frac{\alpha}{2}}x) + \frac{2p}{p-1}T^{-\alpha}(\Phi(T^{-\frac{\alpha}{2}}x))^{\frac{p+1}{p-1}}\Phi_{x_ix_i(T^{-\frac{\alpha}{2}}x)},$$

Observing that $|\Phi|\le 1$ and $\Phi\in C_0^{\infty}(R^n)$, one see that

$$|\Delta\varphi_1| \le CT^{-\alpha}(\Phi(T^{-\frac{\alpha}{2}}x))^{\frac{2}{p-1}} = CT^{-\alpha}\varphi_1^{\frac{1}{p}},$$

for some positive constant C independent of T. Combining the above estimates and observing $0\le\varphi_1\le 1$, we derive

$$\int_{R^n}\int_0^T u(t,x)\Delta({}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t)))dtdx = \int_{R^n}\int_0^T u(t,x)\Delta\varphi_1(x)^C D_{t|T}^{\beta}\varphi_2(t)dtdx. \quad (3.10)$$

For the third term,

$$\int_{R^n}\int_0^T u(t,x)b(t)\partial_t{}^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx = -\int_{R^n}\int_0^T u(t,x)b(t)\varphi_1(x)^C D_{t|T}^{\beta+1}\varphi_2(t)dtdx. \quad (3.11)$$

For the fourth term, we have

$$\int_{R^n}\int_0^T u(t,x)b'(t)^C D_{t|T}^{\beta}(\varphi_1(x)\varphi_2(t))dtdx = \int_{R^n}\int_0^T u(t,x)b'(t)\varphi_1(x)^C D_{t|T}^{\beta}\varphi_2(t)dtdx. \quad (3.12)$$

We deduce from (3.6) to (3.12) to get

$$\int_{R^n}\int_0^T |u|^p \varphi_1(x)\varphi_2(t)dtdx + \int_{R^n}\int_0^T (u_1(x)t + u_0(x))\varphi_1(x)^C D_{t|T}^{\alpha+\beta}\varphi_2(t)dtdx + b(0)C_1T^{-\beta}\int_{R^n} u_0(x)\varphi_1(x)dx$$
$$= \int_{R^n}\int_0^T u(t,x)\varphi_1(x)^C D_{t|T}^{\alpha+\beta}\varphi_2(t)dtdx + \int_{R^n}\int_0^T u(t,x)b(t)\varphi_1(x)^C D_{t|T}^{\beta+1}\varphi_2(t)dtdx$$
$$-\int_{R^n}\int_0^T u(t,x)\Delta\varphi_1(x)^C D_{t|T}^{\beta}\varphi_2(t)dtdx - \int_{R^n}\int_0^T u(t,x)b'(t)\varphi_1(x)^C D_{t|T}^{\beta}\varphi_2(t)dtdx.$$

Then

$$\begin{aligned}
&\int_{R^n}\int_0^T |u|^p \varphi_1(x)\varphi_2(t)dtdx + \int_{R^n}\int_0^T u_1 t\varphi_1(x)^C D_{t|T}^{\alpha+\beta}\varphi_2(t)dtdx \\
&+\int_{R^n}\int_0^T u_0\varphi_1(x)^C D_{t|T}^{\alpha+\beta}\varphi_2(t)dtdx + b(0)C_1T^{-\beta}\int_{R^n} u_0(x)\varphi_1(x)dx \\
&\le \int_{R^n}\int_0^T |\ u(t,x)|\ \ \varphi_1(x)|^C D_{t|T}^{\alpha+\beta}\varphi_2(t)|\ \ dtdx + \int_{R^n}\int_0^T |\ u(t,x)|\ \ b(t)\varphi_1(x)|^C D_{t|T}^{\beta+1}\varphi_2(t)|\ \ dtdx \\
&+\int_{R^n}\int_0^T |\ u(t,x)|\ \ \Delta\varphi_1(x)|^C D_{t|T}^{\beta}\varphi_2(t)|\ \ dtdx + \int_{R^n}\int_0^T |\ u(t,x)|\ \ |\ b'(t)|\ \ \varphi_1(x)|^C D_{t|T}^{\beta}\varphi_2(t)|\ \ dtdx.
\end{aligned} \tag{3.13}$$

Now, by the assumption and $T > 0, C_1 > 0$, the integrals over the data are assumed to be positive for large k and $b(0) > 0,$ we may conclude from (3.13) the following estimate:

$$\begin{aligned}
&\int_{R^n}\int_0^T |u|^p \varphi_1(x)\varphi_2(t)dtdx \\
&\le \int_{R^n}\int_0^T |\ u(t,x)|\ \ \varphi_1(x)|^C D_{t|T}^{\alpha+\beta}\varphi_2(t)|\ \ dtdx + \int_{R^n}\int_0^T |\ u(t,x)|\ \ b(t)\varphi_1(x)|^C D_{t|T}^{\beta+1}\varphi_2(t)|\ \ dtdx \\
&+\int_{R^n}\int_0^T |\ u(t,x)|\ \ \Delta\varphi_1(x)|^C D_{t|T}^{\beta}\varphi_2(t)|\ \ dtdx + \int_{R^n}\int_0^T |\ u(t,x)|\ \ |\ b'(t)|\ \ \varphi_1(x)|^C D_{t|T}^{\beta}\varphi_2(t)|\ \ dtdx.
\end{aligned} \tag{3.14}$$

In order to estimate the terms of the right-hand of (3.14), let $\psi(t,x) = \varphi_1(x)\varphi_2(t)$. We apply the ε-Young's inequality, then we find the estimate for the first term of

$$\begin{aligned}
&\int_{R^n}\int_0^T |u|\varphi_1(x)\left|{}^C D_{t|T}^{\alpha+\beta}\varphi_2(t)\right|dtdx = \int_{R^n}\int_0^T |u|\psi^{\frac{1}{p}}(t,x)\psi^{-\frac{1}{p}}(t,x)\varphi_1(x)\left|{}^C D_{t|T}^{\alpha+\beta}\varphi_2(t)\right|dtdx \\
&\leqslant \varepsilon\int_{R^n}\int_0^T |u|^p \psi(t,x)dtdx + C(\varepsilon)\int_{R^n}\int_0^T \psi^{-\frac{1}{p-1}}(t,x)\varphi_1^{\frac{p}{p-1}}(x)\left|{}^C D_{t|T}^{\alpha+\beta}\varphi_2(t)\right|^{\frac{p}{p-1}} dtdx,
\end{aligned} \tag{3.15}$$

Similarly, for the second term

$$\begin{aligned}
&\int_{R^n}\int_0^T |u| b(t)\varphi_1(x)\left|{}^C D_{t|T}^{\beta+1}\varphi_2(t)\right|dtdx \\
&\leqslant \varepsilon\int_{R^n}\int_0^T |u|^p \psi(t,x)dtdx + C(\varepsilon)\int_{R^n}\int_0^T \psi^{-\frac{1}{p-1}}(t,x)b(t)^{\frac{p}{p-1}}\varphi_1^{\frac{p}{p-1}}(x)\left|{}^C D_{t|T}^{\beta+1}\varphi_2(t)\right|^{\frac{p}{p-1}} dtdx,
\end{aligned} \tag{3.16}$$

the third term

$$\begin{aligned}
&\int_{R^n}\int_0^T |u|\Delta\varphi_1(x)\left|{}^C D_{t|T}^{\beta}\varphi_2(t)\right|dtdx \\
&\leqslant \varepsilon\int_{R^n}\int_0^T |u|^p \psi(t,x)dtdx + C(\varepsilon)\int_{R^n}\int_0^T \psi^{-\frac{1}{p-1}}(t,x)\left|\Delta\varphi_1(x)\right|^{\frac{p}{p-1}}\left|{}^C D_{t|T}^{\beta}\varphi_2(t)\right|^{\frac{p}{p-1}} dtdx,
\end{aligned} \tag{3.17}$$

and the fourth term

$$\int_{R^n}\int_0^T |u||b'(t)|\varphi_1(x)\left|{}^C D_{t|T}^{\beta}\varphi_2(t)\right|dtdx$$
$$\leqslant \varepsilon\int_{R^n}\int_0^T |u|^p \psi(t,x)dtdx + C(\varepsilon)\int_{R^n}\int_0^T \psi^{-\frac{1}{p-1}}(t,x)|b'(t)|^{\frac{p}{p-1}}\varphi_1^{\frac{p}{p-1}}(x)\left|{}^C D_{t|T}^{\alpha}\varphi_2(t)\right|^{\frac{p}{p-1}}dtdx. \tag{3.18}$$

Taking into account the estimates (3.15)-(3.18), we may conclude from (3.14), for $\varepsilon > 0$ small enough and $\psi(t,x) = \varphi_1(x)\varphi_2(t)$, the estimate

$$\begin{aligned}
&\int_{R^n}\int_0^T |u|^p \varphi_1(x)\varphi_2(t)dtdx\\
&\le C(\varepsilon)\int_{R^n}\int_0^T \psi^{-\frac{1}{p-1}}(t,x)\varphi_1^{\frac{p}{p-1}}(x)\left|{}^C D_{t|T}^{\alpha+\beta}\varphi_2(t)\right|^{\frac{p}{p-1}}dtdx\\
&+C(\varepsilon)\int_{R^n}\int_0^T \psi^{-\frac{1}{p-1}}(t,x)b(t)^{\frac{p}{p-1}}\varphi_1^{\frac{p}{p-1}}(x)\left|{}^C D_{t|T}^{\beta+1}\varphi_2(t)\right|^{\frac{p}{p-1}}dtdx\\
&+C(\varepsilon)\int_{R^n}\int_0^T \psi^{-\frac{1}{p-1}}(t,x)|b'(t)|^{\frac{p}{p-1}}\varphi_1^{\frac{p}{p-1}}(x)\left|{}^C D_{t|T}^{\beta}\varphi_2(t)\right|dtdx\\
&+C(\varepsilon)\int_{R^n}\int_0^T \psi^{-\frac{1}{p-1}}(t,x)|\Delta\varphi_1(x)|^{\frac{p}{p-1}}\left|{}^C D_{t|T}^{\beta}\varphi_2(t)\right|^{\frac{p}{p-1}}dtdx\\
&\leqslant C(\varepsilon)\Big(\int_{\Omega_{T^{\frac{\alpha}{2}}}}\varphi_1(x)dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}}\left|{}^C D_{t|T}^{\beta+\alpha}\varphi_2(t)\right|^{\frac{p}{p-1}}dt\\
&+\int_0^T b(t)^{\frac{p}{p-1}}\varphi_2(t)^{-\frac{1}{p-1}}\left|{}^c D_{t|T}^{\beta+1}\varphi_2(t)\right|^{\frac{p}{p-1}}dt + \int_0^T \varphi_2(t)^{\frac{-1}{p-1}}b'(t)^{\frac{p}{p-1}}\left|{}^C D_{t|T\varphi_2(t)}^{\beta}\right|^{\frac{p}{p-1}}dt\Big)\\
&\lesssim C(\varepsilon)\Big(\int_{\Omega_{T^{\frac{\alpha}{2}}}}\varphi_1(x)dx\Big)\Big(I_1+I_2+I_3\Big)+C(\varepsilon)I_4.
\end{aligned} \tag{3.19}$$

Obviously, $\int_{R^n}\varphi_1(x)dx = \int_{\Omega_{T^{\frac{\alpha}{2}}}}\varphi_1(x)dx$, where $\Omega_{T^{\frac{\alpha}{2}}} = \{x\in R^n : |x| \le 2T^{\frac{\alpha}{2}}\}$.

First, let $x = T^{-\frac{\alpha}{2}}y$, we have

$$\int_{\Omega_{T^{\frac{\alpha}{2}}}}\varphi_1(x)dx = T^{\frac{n\alpha}{2}}\int_{|y|\leqslant 2}(\Phi(y))^{\frac{2p}{p-1}}dy = CT^{\frac{n\alpha}{2}} \tag{3.20}$$

with some $C > 0$.

Then, we have to estimate the integrals I_1, I_2, I_3 and I_4. By Fubin's theorem, we got for I_1 the estimate

$$I_1 = \int_0^T \varphi_2(t)^{-\frac{1}{p-1}}\left|{}^c D_{t|T}^{\beta+\alpha}\varphi_2(t)\right|^{\frac{p}{p-1}}dt = CT^{1-(\alpha+\beta)\frac{p}{p-1}}. \tag{3.21}$$

Before we estimate I_2 and I_3, we deal with I_4 first.

$$I_4 = (\int_{\Omega_{T^{\frac{\alpha}{2}}}} \varphi_1^{-\frac{1}{p-1}}(x)\left|\Delta\varphi_1(x)\right|^{\frac{p}{p-1}} dx)(\int_0^T \varphi_2^{\frac{-1}{p-1}}(t)\left|{}^C D_{t|T}^{\beta}\varphi_2(t)\right|^{\frac{p}{p-1}} dt) = I_{41}I_{42}. \tag{3.22}$$

Observing that $\left|\Phi(x)\right| \leqslant 1$ and $\Phi(x) \in C_0^{\infty}(R^n)$, so

$$\left|\Delta\varphi_1(x)\right|^{\frac{p}{p-1}} \leqslant CT^{-\alpha}(\Phi(T^{-\frac{\alpha}{2}}x)^{\frac{1}{p}})^{\frac{p}{p-1}} = CT^{-\frac{\alpha p}{p-1}}\varphi_1^{\frac{1}{p-1}}.$$

Then

$$I_{41} = \int_{\Omega_{T^{\frac{\alpha}{2}}}} \varphi_1^{-\frac{1}{p-1}}(x)\left|\Delta\varphi_1(x)\right|^{\frac{p}{p-1}} dx \le CT^{-\frac{\alpha p}{p-1}+\frac{n\alpha}{2}}. \tag{3.23}$$

Moreover, we have the relations

$$I_{42} = \int_0^T \varphi_2^{\frac{-1}{p-1}}(t)\left|{}^c D_{t|T}^{\beta}\varphi_2(t)\right|^{\frac{p}{p-1}} dt = \int_0^1 (1-\tau)^{\frac{-k}{p-1}}(T^{-\beta}(1-\tau)^{k-\beta})^{\frac{p}{p-1}} T d\tau \leqslant CT^{1-\frac{\beta p}{p-1}}. \tag{3.24}$$

After replacing (3.23) and (3.24) into (3.22) we find

$$I_4 \le CT^{-\frac{\beta p}{p-1}-\frac{\alpha p}{p-1}+\frac{n\alpha}{2}+1}, \tag{3.25}$$

with some constant $C > 0$.

Now we estimate I_2 and I_3. Similar to the treatment of I_3 and I_4 in Theorem 2, the calculation process is omitted here for convenience, we can get the following result:

(1) When $r \in (0,1)$, the function $b = b(t) = (1+t)^r$ is strictly increasing, then

$$I_2 \lesssim T^{-(\beta+1-r)\frac{p}{p-1}+1}, I_3 \lesssim T^{-\frac{\beta p}{p-1}-\frac{(1-r)p}{p-1+(1-r)p}+1}, \tag{3.26}$$

At this point, including all the estimates we may conclude

$$\int_{\Omega_{T^{\frac{\alpha}{2}}}} \int_0^T |u|^p \psi(t,x) dt dx \leqslant C\Big(T^{-\frac{(\alpha+\beta)p}{p-1}+\frac{n\alpha}{2}+1} + T^{-\frac{(\beta+1-r)p}{p-1}+\frac{n\alpha}{2}+1} + T^{-\frac{\beta p}{p-1}-\frac{(1-r)p}{p-1+(1-r)p}+\frac{n\alpha}{2}+1} + T^{-\frac{(\alpha+\beta)p}{p-1}+\frac{n\alpha}{2}+1}\Big). \tag{3.27}$$

To have that all exponents of T on the right-hand side are negative it is sufficient to guarantee

$$-\frac{\beta p}{p-1} - \frac{(1-r)p}{p-1+(1-r)p} + \frac{n\alpha}{2} + 1 < 0.$$

Consequently,

$$\frac{p}{p-1} > \frac{n\alpha(1-r)-2\beta}{4\beta(1-r)} + \sqrt{(\frac{2\beta+(1-r)n\alpha}{4\beta(1-r)})^2 + \frac{1}{\beta(1-r)}},$$

that is condition (2.9) with $\beta \in (0,1)$, $\alpha \in (1,2)$. Then the estimate (3.27) will be write

$$\int_{\Omega_{T^{\frac{\alpha}{2}}}} \int_0^T |u|^p \psi(t,x)dtdx \lesssim T^{-\delta}, \tag{3.28}$$

where $\delta = \delta(p,n,\gamma,r) > 0$. Then after passing to the limit $T \to \infty$ in (3.27) and using the dominated convergence theorem and the fact that

$$\lim_{T\to\infty} \psi(t,x) = 1 \text{ for all } (t,x) \in (0,T) \times R^n,$$

It follows

$$\int_0^\infty \int_{R^n} |u|^p \, dtdx = 0.$$

This gives immediately $u \equiv 0$ and this is a contradiction to (2.8).

(2) For $r = 0$, the function $b \equiv 1$. Repeating the estimate from the previous subsection we arrive at $(I_3 = 0)$

$$\int_{\Omega_{T^{\frac{\alpha}{2}}}} \int_0^T |u|^p \psi(t,x)dtdx \leqslant C(T^{-\frac{(\alpha+\beta)p}{p-1}+\frac{n\alpha}{2}+1} + T^{-\frac{(1+\beta)p}{p-1}+\frac{n\alpha}{2}+1} + T^{-\frac{(\alpha+\beta)p}{p-1}+\frac{n\alpha}{2}+1}).$$

As above we derive the condition

$$\frac{p}{p-1} > \frac{n\alpha+2}{2(\beta+1)},$$

that is condition (2.10) with $\beta = \in (0,1)$, $\alpha \in (1,2)$. In the same way we conclude $u \equiv 0$ and this a contradiction to (2.8).

(3) When $r \in (0,-1)$, then the function $b = b(t) = (1+t)^r$ is decreasing. Then

$$I_2 \lesssim T^{-(\beta+1)\frac{p}{p-1}-\frac{sp}{p-1+sp}+1}, \; I_3 \lesssim T^{-\beta\frac{p}{p-1}-\frac{(s+1)p}{(2+s)p-1}+1}.$$

Next, we can conclude the following estimate:

$$\int_{\Omega_{T^{\frac{\alpha}{2}}}} \int_0^T |u|^p \psi(t,x)dtdx \leqslant C\Big(T^{-(\alpha+\beta)\frac{p}{p-1}+\frac{n\alpha}{2}+1} + T^{-(\beta+1)\frac{p}{p-1}-\frac{sp}{p-1+sp}+\frac{n\alpha}{2}+1} + T^{-\beta\frac{p}{p-1}-\frac{(1+s)p}{(2+s)p-1}+\frac{n\alpha}{2}+1} + T^{-(\alpha+\beta)\frac{p}{p-1}+\frac{n\alpha}{2}+1}\Big). \tag{3.29}$$

To have all exponents of T on the right-hand side are negative it is sufficient to guarantee

$$-(\alpha+\beta)\frac{p}{p-1}+\frac{n\alpha}{2}+1<0 \text{ and } -\beta\frac{p}{p-1}-\frac{(1+s)p}{(2+s)p-1}+\frac{n\alpha}{2}+1<0.$$

The first inequality implies the relation

$$\frac{p}{p-1}>\frac{n\alpha+2}{2(\alpha+\beta)}.$$

From the second inequality we verify

$$\frac{p}{p-1}>\frac{(s+1)n\alpha-2\beta}{4\beta(s+1)}+\sqrt{(\frac{2\beta+(s+1)n\alpha}{4\beta(s+1)})^2+\frac{1}{\beta(s+1)}}.$$

Cause of $s=-r$, we derived the condition

$$\frac{p}{p-1}>\inf\max\left\{\frac{n\alpha+2}{2(\alpha+\beta)};\frac{(1-r)n\alpha-2\beta}{4\beta(1-r)}+\sqrt{(\frac{2\beta+(1-r)n\alpha}{4\beta(1-r)})^2+\frac{1}{\beta(1-r)}}\right\}, \quad (3.30)$$

that is condition (2.11) with $\beta\in(0,1)$ and $\alpha\in(1,2)$. Then find the estimate

$$\int_{\Omega_{T^{\frac{\alpha}{2}}}}\int_0^T|u|^p\psi(t,x)dtdx\lesssim T^{-\delta}, \quad (3.31)$$

where $\delta=\delta(p,n,\gamma,r)>0$. Then after passing to the limit $T\to\infty$ in (3.31) and using the dominated convergence theorem and the fact that

$$\lim_{T\to\infty}\psi(t,x)=1 \text{ for all } (t,x)\in(0,T)\times R^n,$$

It follows

$$\int_0^\infty\int_{R^n}|u|^p\,dtdx=0.$$

This gives immediately $u\equiv 0$ and this is a contradiction to (2.8).

4. References

[1] P. M. de Carvalho-Neto, G. Planas, *Mild solutions to the time fractional Navier- Stokes equations in* R^N, J. Differ. Equ. 259 (2015), pp. 2948-2980.

[2] P. Clment, S. O. Londen, G. Simonett, *Quasilinear evolutionary equations and continuous interpolation spaces*, J. Differ. Equ. 196 (2004), pp. 418-447.

[3] A. Z. Fino, M. Kirane, *Qualitative properties of solutions to a time-space fractional evolution equation*. Quart. Appl. Math. 70 (2012), pp. 133-157.

[4] T. H. Kaddour, M. Reissig. *Blow-up results for effectively damped wave models with nonlinear memory, J. Commun.* Pur. Appl. Anal. *20* (7-8) (2021), pp. 2687.

[5] A. A. Kilbas, H.M. Sr*ivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations,* Elsevier Science B.V. Amsterdam, 204 (2006).

[6] I. Kim, K. H. Kim, S. Lim, *An* $L^q(L^p)$ *-theory for the time fractional evolution equations with variable coefficients*, Adv. Math. 306 (2017), pp. 123-176.

[7] Y. N. Li, *Regularity of mild Solutions for fractional abstract Cauchy problem with order* $\alpha\in(1,2)$.

Z. *Angew*. Math. Phy. 66, 6 (2015), pp. 3283-3298.

[8] Y. N. Li, H. R. Sun, Z. S. Feng, *Fractional abstract Cauchy problem with order* $\alpha \in (1,2)$. Dynamics of PDE, 13, 2 (2016), pp. 155-177.

[9] Jiahui Liang, *Some properties of the Caputo Fractional Derivatives*. Mathematics in Practice and Theory. 51 (9) (2021), pp. 14.

[10] R. Metzler, J. Klafter, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*. J.Phys. A 37, 31 (2004), pp. 161-208.

[11] K. Sakamoto, M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl. 382 (2011), pp. 426-447.

[12] W.R. Schneider, W. Wyss, *Fractional diffusion and wave equations*. J. Math. Phy. 30, No 1 (1989), pp. 134-144.

[13] R. Zacher, *Maximal regularity of type for abstract parabolic Volterra equations*, J. Evol. Equ. 5 (2005), pp. 79-103.

[14] R. Zacher, *Convergence to equilibrium for second order differential equations with weak damping of memory type*, Adv. Differential Equ. 14 (2009), pp. 749-770.

[15] Q.G. Zhang, Y. Li, *Global well-posedness and blow-up solutions of the Cauchy problem for a time-fractional superdiffusion equation*. J. Evol. Equ. 19, 1 (2019), pp. 271-303.

[16] Q.G. Zhang, H.R. Sun, *The blow-up and global existence of solutions of Cauchy problems for a time fractional diffusion equation*. Topol. Meth. Nonlinear Anal. 46, 1 (2015), pp. 69-92.