ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 16, No. 2, 2021, *pp.*122-125

An explicit finite difference scheme for solving the space fractional nonlinear Schrödinger equation

Dongsheng Tang
School of Mathematics and Statistics, Nanjing University of Information Science & Technology,
Nanjing, 210044, China
Email:dongshengtang@nuist.edu.cn
(Received August 01, 2021, accepted October 09, 2021)

Abstract: This paper uses the finite difference method to numerically solve the space fractional nonlinear Schrodinger equation. First, we give some properties of the fractional Laplacian Δ_h^α. Then we construct a numerical scheme which satisfies the mass conservation law without proof and the scheme's order is $O(\tau^2+h^2)$ in the discrete L^∞ norm. Moreover, The scheme conserves the mass conservation and is unconditionally stable about the initial values. Finally, this article gives a numerical example to verify the relevant properties of the scheme.

Keywords: Partial Differential Equations; Finite difference method; Numerical solutions

1. Introduction

We consider the space fractional nonlinear Schrödinger equation (SFNLS) equation

$$i\frac{\partial u(x,t)}{\partial t}-(-\Delta)^{\frac{\alpha}{2}}u(x,t)+\beta|u(x,t)|^2u(x,t)=0,\quad x\in(a,b),\quad t\in(0,T],\tag{1.1}$$

with boundary condition

$$u(a,t)=u(b,t)=0,\quad t\in(0,T],\tag{1.2}$$

and initial condition

$$u(x,0)=u_0(x),\quad x\in[a,b],\tag{1.3}$$

where $1<\alpha\le 2, i=\sqrt{-1}$ is the complex unit, $u=u(x,t)$ is the unknown complex-valued function, $u_0=u_0(x)$ is a given smooth complex-valued function, and β is a given non-zero real number. When $\alpha=2$, the SFNLS equation is reduced into the standard cubic nonlinear Schrödinger equation. Here the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ is defined as a pseudo-differential operator with $|\xi|^\alpha$ in Fourier space[8], i.e.,

$$-(-\Delta)^{\frac{\alpha}{2}}u(x,t)=-\mathcal{F}^{-1}\left(|\xi|^\alpha\hat{u}(\xi,t)\right),\tag{1.4}$$

where $\mathcal{F}$ is the standard Fourier transform and $\hat{u}(\xi,t)=\mathcal{F}[u(x,t)]$.

2. Construction of Scheme

In this section, we will construct a finite difference scheme for the system (1.1)-(1.3), which preserves the total mass given in **Theorem 2.1**. For given positive integers J,N, we set the time step $\tau=\frac{T}{N}$ and grid size $h=\frac{b-a}{J}$. Denote the space and time discrete node sets by $\Omega_h=\{x_j|x_j=a+jh,0\le j\le J\}$ and $\Omega_\tau=\{t_n|t_n=n\tau,0\le n\le N\}$. Then the space and time grid point set is defined by $\Omega_h^\tau=\Omega_h\times\Omega_\tau$. Denote $\mathcal{S}_h=\{u|u=(u_0,u_1,u_2,\cdots,u_J),u_0=u_J=0\}$ as a grid function space. For any grid function $w^n\in\mathcal{S}_h$ for $n=1,2,\cdots,N-1$, we introduce the following notations

$$\delta_{\hat{t}}w_j^n:=\frac{w_j^{n+1}-w_j^{n-1}}{2\tau},\ \delta_t w_j^n:=\frac{w_j^{n+1}-w_j^n}{\tau},\ w_j^{n+\frac{1}{2}}:=\frac{w^{n+1}+w_j^n}{2},\ w_j^{[n]}:=\frac{w_j^{n+1}+w_j^{n-1}}{2}.\tag{2.1}$$

for any grid functions $w,v\in\mathcal{S}_h$, we define the discrete inner product and the two norms over $\mathcal{S}_h$ as

$$(v,w) := h\sum_{j=1}^{J-1} v_j\,\overline{w_j},\quad |v| := \sqrt{(v,v)},\quad |v|_\infty := \sup_{1\le j\le J-1}|v_j|. \tag{2.2}$$

Lemma2.1[11] For a function $\phi \in C^5(R)\cap L^1(R)$, we have

$$\frac{\partial^\alpha \phi(x)}{\partial |x|^\alpha} = -\frac{1}{h^\alpha}\sum_{k=-\infty}^{+\infty} c_k^{(\alpha)}\phi(x-kh) + O(h^2),\quad \forall 1<\alpha\le 2, \tag{2.3}$$

where $c_k^{(\alpha)} := \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k+1\right)}$.

According to the initial value of the original equation and the above lemma, we have

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = -\frac{1}{h^\alpha}\sum_{k=-(b-x)/h}^{-(a-x)/h} c_k^{(\alpha)} u(x-kh,t) + O(h^2), \tag{2.4}$$

and

$$-(-\Delta)^{\frac{\alpha}{2}} u_j^n = -\frac{1}{h^\alpha}\sum_{k=-J+j}^{j} c_k^{(\alpha)} u_{j-k}^n + O(h^2) = -\frac{1}{h^\alpha}\sum_{k=1}^{J-1} c_{j-k}^{(\alpha)} u_k^n + O(h^2). \tag{2.5}$$

For the sake of brevity, we denote

$$\Delta_h^\alpha u_j^n = \frac{1}{h^\alpha}\sum_{k=1}^{J-1} c_{j-k}^{(\alpha)} u_k^n,\quad 1\le j\le J-1,\quad 0\le n\le N, \tag{2.6}$$

and denote matrix $\boldsymbol{C}$ as

$$C = \begin{pmatrix} c_0^{(\alpha)} & c_{-1}^{(\alpha)} & \cdots & c_{-J+2}^{(\alpha)} \\ c_1^{(\alpha)} & c_0^{(\alpha)} & \cdots & c_{-J+3}^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{J-2}^{(\alpha)} & c_{J-3}^{(\alpha)} & \cdots & c_0^{(\alpha)} \end{pmatrix} \tag{2.7}$$

The eigenvalues of the real positive Toeplitz matrix $\boldsymbol{C}$ is denoted by λ_j for $1\le j \le J-1$ and we have [12]

$$0<\lambda_j<2c_0^{(\alpha)},\quad j=1,2,\cdots,J-1. \tag{2.8}$$

The scheme of the equation (1.1)-(1.3) constructed in this paper is

$$i\delta_{\hat{t}} U_j^n - \Delta_h^\alpha U_j^n + \beta\left|U_j^n\right|^2 U_j^{[n]} = 0,\quad 1\le j\le J-1,\ \ 1\le n\le N-1, \tag{2.9}$$

$$i\delta_t U_j^{\frac{1}{2}} - \Delta_h^\alpha U_j^{\frac{1}{2}} + \beta\left|\hat{U}_j^{\frac{1}{2}}\right|^2 U_j^{\frac{1}{2}} = 0,\ 1\le j\le J-1, \tag{2.10}$$

$$U_j^0 = u_0(x_j),\quad 1\le j\le J-1, \tag{2.11}$$

$$U_0^n = U_J^n = 0,\quad 1\le n\le N, \tag{2.12}$$

where $U_j^{\frac{1}{2}} = u(x_j,0)+\frac{\tau}{2}u_t(x_j,0)$, $1\le j\le J-1$.

Theorem 2.1 *The above scheme* (2.9)-(2.12) *satisfies the following mass conservation*

$$M^n := \frac{1}{2}(|U^n|^2+|U^{n+1}|^2) \equiv M^0,\quad 0\le n\le N-1. \tag{2.13}$$

3. Numerical Experiment

In this section, we use our scheme to compute one numerical example to show our theoretical results.

Example 3.1 *We consider the following SFNLS equation*

$$i\frac{\partial u(x,t)}{\partial t} - (-\Delta)^{\frac{\alpha}{2}} u(x,t) + \beta|u(x,t)|^2 u(x,t) = 0, \quad a < x < b, \quad 0 < t \leq T, \tag{3.1}$$

with homogeneous Dirichlet boundary conditions and the following initial condition

$$u(x,0) = \text{sech}(x) \cdot \exp(2ix), \quad a \leq x \leq b. \tag{3.2}$$

For this problem, we take $\alpha = 2, \beta = 2$, and the exact solution is given by

$$u(x,t) = sech(x - 4t) \cdot exp\big(i(2x - 3t)\big). \tag{3.3}$$

In this example, we set the interval $[a,b] = [-20,20]$, M^n is the discrete total mass at $t_n = n\tau$.

Table 1: Convergence test for $1 < \alpha \leq 2$ with $\tau = 0.04h$.

α	$error$	$h = 0.2$	$h = 0.1$	$order$
1.2	$\|\|e^N\|\|_\infty$	1.6945e-01	4.4717e-02	1.92
1.4	$\|\|e^N\|\|_\infty$	2.4276e-01	5.0584e-02	2.26
1.8	$\|\|e^N\|\|_\infty$	1.8285e-01	4.0860e-02	2.16
2.0	$\|\|e^N\|\|_\infty$	1.4267e-01	3.4243e-02	2.06

Table 1 gives the errors together with the corresponding orders of numerical solutions in the maximum norm at $T = 1$, and shows the convergence order in time and space direction for $1 < \alpha \leq 2$. For the exact solution of the equation when $1 < \alpha < 2$, this paper uses the numerical accurate solution to replace it, i.e., we take $h = 0.025, \tau = 1e - 04$ to get a 'numerical eaxct' solution.

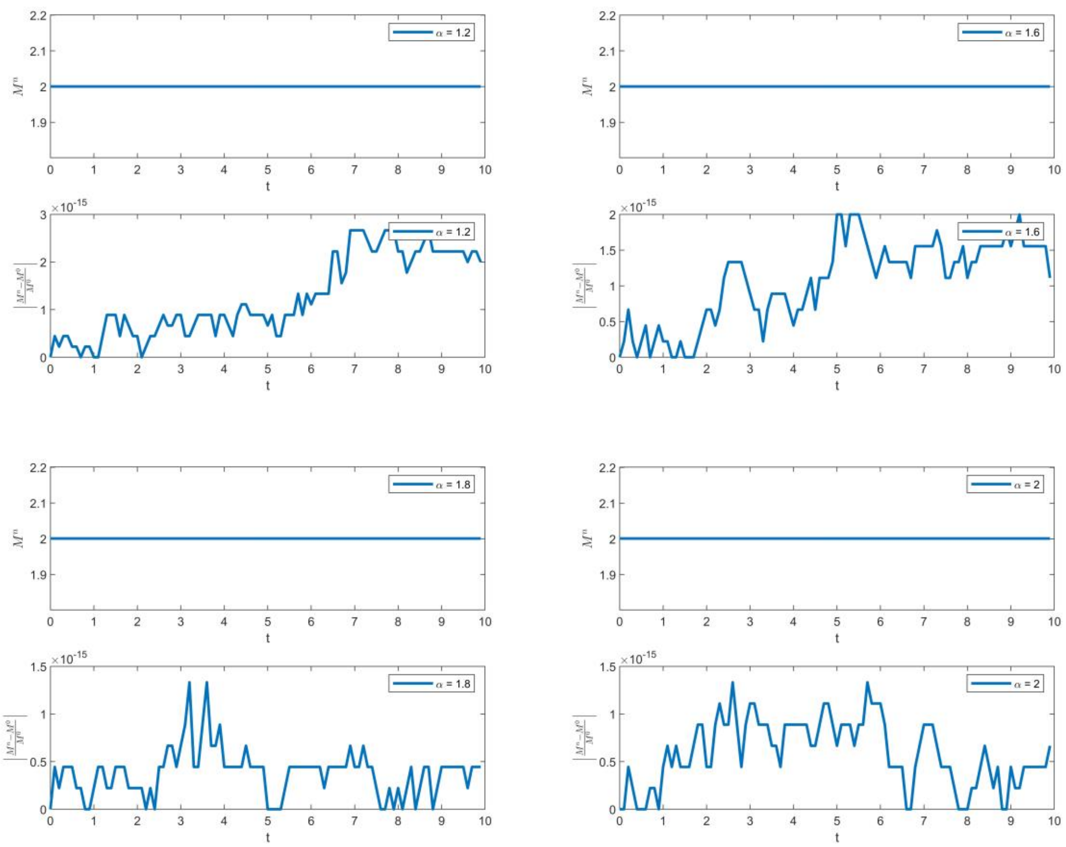

Figure 1: Mass conservation law and its errors for different α by $[a,b] = [-20,20], T = 10, h = 0.1, \tau = 0.1$.

References

[1] A. Sapora, P. Cornetti, A. Carpinteri, Wave propagation in nonlocal elastic continua modelled by a fractional calculus approach, Commun. Nonlinear Sci. Numer. Simulat., 18, 2013, 63-74.

[2] H. Nasrolahpour, A note on fractional electrodynamics, Commun. Nonlinear Sci. Numer. Simulat., 18, 2013, 2589-2593.

[3] V.E. Tarasov, E.C. Aifantis, Non-standard extensions of gradient elasticity: Fractional nonlocality, memory and fractality, Commun. Nonlinear Sci. Numer. Simulat., 22, 2015, 197-227.

[4] J.E. Macías-Díaz, A structure-preserving method for a class of nonlinear dissipative wave equations with Riesz space-fractional derivatives, J. Comput. Phys., 351, 2017, 40-58.

[5] N. Laskin, Fractional quantum mechanics, Phys. Rev. E., 62, 2000, 3135.

[6] N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A., 268, 2000, 298-305.

[7] N. Laskin, Fractional schrödinger equation, Phys. Rev. E., 66, 2002, 056108.

[8] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Part. Diff. Eq., 32, 2007, 1245-1260.

[9] L. Roncal, P.R. Stinga, Fractional Laplacian on the torus, Commun. Contemp. Math., 18, 2016, 1550033.

[10] W. Zeng, A. Xiao, X. Li, Error estimate of Fourier pseudo-spectral method for multidimensional nonlinear complex fractional Ginzburg-Landau equations, Appl. Math. Lett., 93, 2019, 40-45.

[11] M.D. Ortigueira, Riesz potential operators and inverses via fractional centred derivatives, Int. J. Math. Math. Sci., 2006.

[12] C. Çelik, M. Duman, Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative, J. Comput. Phys., 231, 2012, 1743-1750.