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An explicit finite difference scheme for sine-Gordon equation in two dimensions

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Abstract: In this paper, we aim to construct an explicit finite difference scheme for solving the two-dimensional sine–Gordon equation. By using Taylor expansion, we prove that the local truncation error of the scheme is of $O(h^2+\tau^2)$ with grid size h and time step τ. Numerical results are reported to test the theoretical analysis.

Keywords: Nonlinear sine-Gordon equation; Finite difference method.

1. Introduction

The sine-Gordon equation is an important mathematical model extensively applied in a wild variety of physical fields, and it is often used to describe and simulate physical phenomena, such as nonlinear waves, propagation of fluxons and dislocation of metals [1, 3]. Moreover, it has attracted much attention in investigating solitons and condensed matter physics and the interaction of solitons in collisions plasma [2, 5, 7, 9].

Many scholars have already done a lot of research on the sine-Gordon equation. In [7], Josephson gave the sine-Gordon equation describing the coupled superconducting junction for the first time. And zhou in [13] gave the dimensional estimation of attractors with periodic boundary conditions. Guo [5] et al conducted a numerical study and simulated the collision and reflection of solitons.

Simultaneously, numerical methods of studying this equation have important theoretical significance and practical value. There has already been quite a lot of literature discussing how developing numerical methods to solve the SG equation. Dehghan et al used the collocation method based on the radial basis function to obtain the numerical solutions to the one-dimensional nonlinear sine-Gordon equation [4]. Liang [8] considered the generalized nonlinear SG equations with periodic initial value problems. He established the Fourier quasi-spectrum explicit scheme. In [12], two fully implicit finite difference schemes were constructed. Besides, a split cosine scheme [10], and the multilevel augmentation method [6] have also been applied to solve this equation. [11] studied the interpolated coefficient finite element method and proved the existence of the solution. Furthermore, when the grid size approached zero, the subsequences of numerical solutions obtained on different grids converged to the exact solution in the L_2 norm.

In this paper, using the finite difference method, we construct an explicit scheme that is convenient for computer calculation in section2, and in section 3, two numerical examples are reported to confirm the theoretical analysis.

2. An explicit finite difference scheme

In this paper, we consider the two-dimensional sine-Gordon equation

$$\boldsymbol{\partial_{tt}u - \Delta u + \sin(u) = 0,\ (x,y,t) \in \Omega \times (0,T]} \quad (1)$$

with boundary and initial condition

$$u(x,y,t) = 0,\ (x,y,t) \in \partial\Omega \times (0,T], \quad (2)$$

$$u(x,y,0) = \varphi(x,y),\ \partial_t u(x,y,0) = \psi(x,y),\ (x,y) \in \overline{\Omega}, \quad (3)$$

where $\Omega = (a,b) \times (c,d) \subset R^2$ is a finite computational domain with boundary $\partial\Omega$, $\varphi = \varphi(x,y)$ and $\psi = \psi(x,y)$ are two given functions. Choosing a positive integer N, we denote time step $\tau = T/N$, denote the set of all time grid points by $\Omega_\tau = \{t_n = n\tau | n = 0,1,2,\cdots,N\}$, and denote the space of all grid functions defined on Ω_τ by $S_\tau = \{ w_\tau = w^n | w^n = w_\tau(t_n), t_n \in \Omega_\tau\}$. For a grid function $U \in S_\tau$, we introduce the following average operator and difference quotient operators,

$$\delta_t^+ U^n = \frac{1}{\tau}(U^{n+1} - U^n), \quad \delta_t^2 U^n = \frac{1}{\tau^2}(U^{n+1} - 2U^n + U^{n-1}) \quad (4)$$

Taking two positive integers J and K, we denote space grid size $h_1 = (b-a)/J, h_2 = (d-c)/K$ and $h = \max\{h_1, h_2\}$, denote the set of all space grid points by $\Omega_h = \{(x_j, y_k) | x_j = a + jh_1, y_k = c + kh_2, j = 0,1,2,\cdots,J; k = 0,1,2,\cdots,K\}$, and denote the space of all grid functions defined on Ω_h with homogeneous boundary conditions by $S_h = \{ w_h = w_{j,k} | w_{j,k} = w_h(x_j, y_k) \text{ for } (x_j, y_k) \in \Omega_h, \text{and } w_{j,k} = 0 \text{ for } (x_j, y_k) \in \Omega_h \cap \partial\Omega\}$.

For a grid function $W \in S_h$, we introduce the following difference quotient operators,

$$\delta_x^2 W_{j,k} = \frac{1}{h_1^2}(W_{j+1,k} - 2W_{j,k} + W_{j-1,k}), \quad (5)$$

$$\delta_y^2 W_{j,k} = \frac{1}{h_2^2}(W_{j,k+1} - 2W_{j,k} + W_{j,k-1}), \quad (6)$$

$$\Delta_h W_{j,k} = \delta_x^2 W_{j,k} + \delta_y^2 W_{j,k}. \quad (7)$$

For simplicity, we introduce three index sets as follows:

$$I_h^0 = \{(j,k) | j = 0,1,2,\cdots,J;\ k = 0,1,2,\cdots,K\},$$

$$I_h = \{(j,k) | j = 1,2,\cdots,J-1;\ k = 1,2,\cdots,K-1\},$$

$$\Gamma_h = \{(j,k) | (j,k) \in I_h^0, but\ (j,k) \notin I_h.$$

We consider the 2D nonlinear SG equation at (x_j, y_k, t_n),

$$u_{tt}(x_j, y_k, t_n) - \Delta u(x_j, y_k, t_n) + \sin u(x_j, y_k, t_n) = 0. \quad (8)$$

Denote numerical solution of $u_{j,k}^n$ by $U_{j,k}^n$, appling second-order centered finite difference operators to temporal and spatial derivatives, i.e.

$$\frac{\partial^2 u}{\partial t^2}(x_j, y_k, t_n) = \delta_t^2 U_{j,k}^n - \frac{\tau^2}{12}\partial_t^4 u(x_j, y_k, \xi_1), t_{n-1} < \xi_1 < t_{n+1}, \quad (9)$$

$$\frac{\partial^2 u}{\partial x^2}(x_j, y_k, t_n) = \delta_x^2 U_{j,k}^n - \frac{h_1^2}{12}\partial_x^4 u(\eta_x, y_k, t_n), x_{j-1} < \eta_x < x_{j+1}, \quad (10)$$

$$\frac{\partial^2 u}{\partial y^2}(x_j, y_k, t_n) = \delta_y^2 U_{j,k}^n - \frac{h_2^2}{12}\partial_y^4 u(x_j, \eta_y, t_n), y_{k-1} < \eta_y < y_{k+1}. \quad (11)$$

So we rewrite the equation as:

$$\delta_t^2 U_{j,k}^n - (\delta_x^2 + \delta_y^2) U_{j,k}^n + \sin(U_{j,k}^n) = R_{j,k}^n,\ (j,k) \in I_h, 0 \le n \le N-1, \quad (12)$$

we can get from (9)-(11) the local truncation error $|R_{j,k}^n| \le c(\tau^2 + h_1^2 + h_2^2)$.

Replace numerical solution by exact solution，and then ignore the small items, we get the finite difference scheme as follows:

$$\begin{aligned}
&\delta_t^2 u_{j,k}^n - \Delta_h u_{j,k}^n + \sin(u_{j,k}^n) = 0, (j,k) \in I_h, 0 \le n \le N-1, \\
&u_{j,k}^0 = \varphi_{j,k},\ \delta_t^+ u_{j,k}^0 = \psi_{j,k}, (j,k) \in I_h, \\
&u_{j,k}^n = 0, (j,k) \in \Gamma_h, 0 \le n \le N.
\end{aligned} \tag{13}$$

With von Neumann analysis method, it can be known that the stability condition of this finite difference scheme is $\frac{\tau^2}{h_1^2} + \frac{\tau^2}{h_2^2} < \frac{1}{2}$.

3. Numerical experiments

In this section, we aim to show the performance of the explicit finite difference scheme, numerical tests can confirm the theoretical findings which obtained in the previous.

Example 1: Considering the following issue

$$\begin{aligned}
&u_{tt} - \Delta u + \sin u = 0, (\boldsymbol{x}, t) \in \Omega \times (0,1], \\
&u(x,y,0) = \sin(\pi x) \cdot \sin(\pi y), \\
&u_t(x,y,0) = -\sin(\pi x) \cdot \sin(\pi y).
\end{aligned}$$

In this test, we choose $\Omega = (0,2) \times (0,2)$. And we fix temporal step $\tau = 1 \times 10^{-3}$, $h_1 = h_2 = 0.02$, we use the solution obtained by the Fourier spectral method as the numerical exact solution u_e of this experiment. Fig.1 shows the surfaces of $u_{j,k}^n$ at different times respectively and it is a time evolution process of the numerical solution. The overall simulation results are very satisfactory.

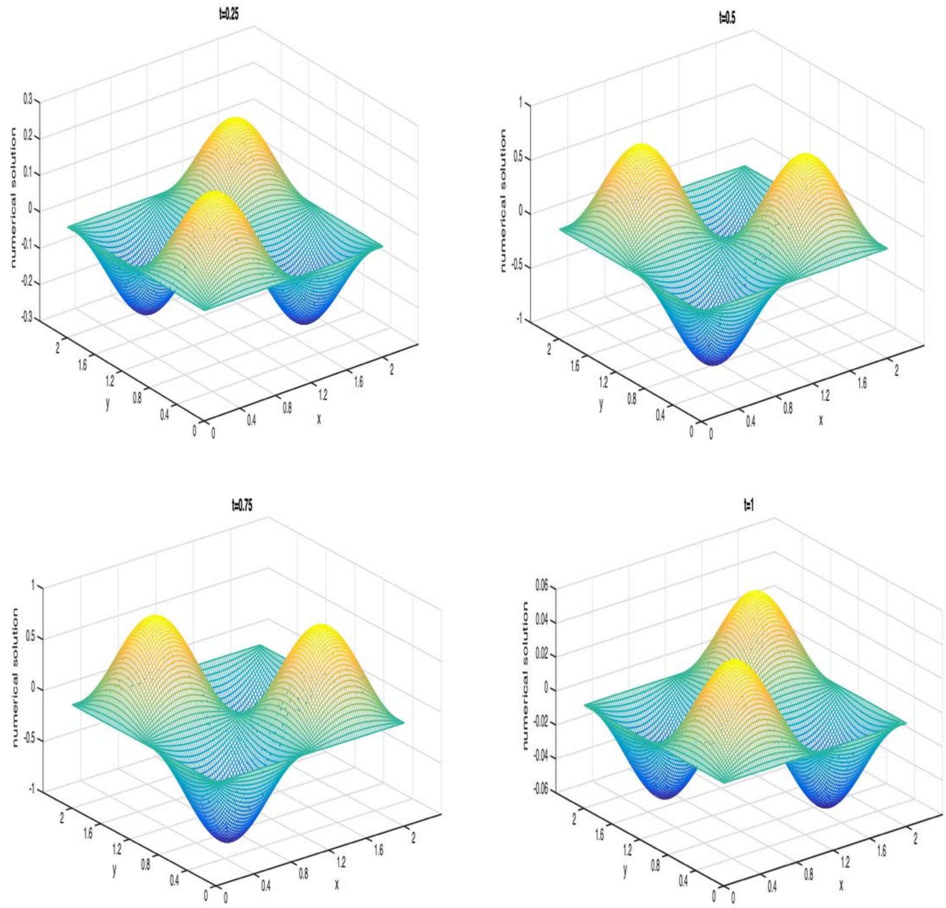

Figure 1: Numerical results at different times for example 1

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