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A Quasi-Radial Basis Function Method for European Option Pricing

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Abstract: In this paper, we propose a meshless method for option pricing which uses the radial basis quasi-interpolation method to solve the Black-Scholes equation. The quasi-interpolation operator is used to force the first and second derivatives of stock prices in spatial direction and the forward difference method is used in time direction. Its convergence of order $O\left(\Delta t+h^{\frac{2}{3}}\right)$ in l_∞ −norm is also derived in the paper. The advantage of this method is that it can fit the scattered data well, which makes it be a good approximation method for the option prices that fluctuate randomly. The feasibility of the proposed method is verified by numerical examples of uniform points and scattered points. The results show that this method has a good fitting effect on option prices.

Keywords: Quasi-interpolation; Meshless method; Option pricing; Black-Scholes equation.

1. Introduction

Black-Scholes model is an important milestone in modern finance. It was developed by Black and Scholes in 1973 [1]. Their pricing theory has a very close connection with the actual operation of the financial market and has been directly applied to the practice of financial transactions, which greatly promotes the rapid development of the global derivative financial market.

With the deepening of option pricing theory, on the basis of Black-Scholes model, scholars put forward Merton model [2], Heston model [3] and Bates model [4] to improve the theory of option pricing model. The numerical methods of option pricing mainly include Monte Carlo method [5], binary tree method [**错误!未找到引用源。**], finite difference method [7-10], radial basis function method [**错误!未找到引用源。**] and so on. Boyle [5] proposed the Monte Carlo simulation method for option pricing for the first time. He used the idea of Monte Carlo simulation to simulate the movement trajectory of stock price through random path and obtain option price through risk-free interest rate discount, but the effect was not very good. Brenna and Schwartz [7] applied the finite difference method to option pricing for the first time, and proposed an implicit difference method to price American options in the jump process. Cox and Ross et. al [6] obtained the numerical solution of American option price by using the binary tree method. This method is simple and intuitive, but the convergence speed is very slow. Andreasen and Andreasen [13] proposed an unconditionally stable alternative direction implicit operator splitting method with second-order accuracy for the European option problem, and applied fast Fourier transform to solve the algebraic system generated discretely, which greatly reduced the calculation time. Zhao and Davison [8] proposed a compact difference scheme for option pricing, which achieved higher accuracy. Cen and Le [9] presented robust finite difference methods for European and American option pricing. Kwon and Lee [10] proposed a time discrete-time method for implicit Crank-Nicolson with jump-diffusion model, in which the integral operator in the model was approximated by numerical quadrilles. Guo and Wang [14] proposed an unconditionally stable time splitting method for the nonlinear Black-Sholes equation, and proved the stability, positivity and convergence of the system. Patel and Mehta [11] proposed a high-order compacted difference scheme for solving the Black-Scholes equation based on polynomial interpolation, and compared the errors in Grank-Nicolson scheme, forward difference scheme and backward difference scheme in the time direction. They

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proposed that under a larger time step, the three compact schemes converge to the same precision. Company and Egorova et. al [12] studied the high-dimensional American option pricing problem based on the local radial basis function method and demonstrated the reliability of the method. More articles on option pricing refer to [15-错误!未找到引用源。].

Radial basis function (RBF) interpolation is a real meshless computing method, which does not need to generate regular meshes like the finite difference method. Based on this, we can simulate the solution of Black-Scholes equation at randomly generated points, which can get a better approximation of the actual option price. Three different multiquadric quasi-interpolations were constructed according to whether they are linear or depend on the derivative at the extreme value by Beaston and Powell [18], denoted as $\boldsymbol{L_A}$, $\boldsymbol{L_B}$, $\boldsymbol{L_C}$. Then Wu and Schaback [19] proposed the quasi-interpolation operator $\boldsymbol{L_D}$ which was proved to be llinear and doesn't depend on the derivative at the extreme value. Wu and Chen [20] constructed a special operator $\boldsymbol{L_D}$ on this basis. More details on radial basis quasi-interpolation methods can be found in [21-24]. In this paper, the radial basis quasi-interpolation method is used to solve the Black-Scholes equation, which overcomes the disadvantage of the large number of conditions caused by the radial basis function interpolation, and also has a good fitting effect for solving partial differential equations.

The remaining work of this paper is as follows. In Section 2, multiquadric (MQ) quasi-interpolation operator and the Black-Scholes equation are introduced. In Section 3, the numerical scheme for solving European options with MQ quasi-interpolation operator and the convergence of the scheme are given. The results of numerical experiments are presented in Section 4. The conclusions are given in section 5.

2. Preliminarie

2.1 The Black-Scholes equation

In this paper, we consider the Black-Scholes equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+\left(r-\frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial S}-rV=0,\ \ (S,t)\epsilon[0,+\infty)\times[0,T], \tag{2.1}$$

$$V(S,T)=\begin{cases}\max(S-K,0) & (for\ call),\\ \max(K-S,0) & (for\ put),\end{cases} \tag{2.2}$$

where V is the value of European option with the strike K and expiry date T, S is the price of stock, r is the risk-free rate, σ is the volatility. By using the algebraic transformation $\tau=T-t$, $x=lnS$ and $u(x,\tau)=V(x,t)$, we can get the new form of the Black-Scholes equation

$$\frac{\partial u}{\partial \tau}-\frac{1}{2}\sigma^2\frac{\partial^2u}{\partial x^2}-\left(r-\frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}+ru=0,\qquad (x,\tau)\epsilon(-\infty,+\infty)\times(0,T], \tag{2.3}$$

$$V(S,0)=\begin{cases}\max(e^x-K,0) & (for\ call),\\ \max(K-e^x,0) & (for\ put).\end{cases} \tag{2.4}$$

The solution $V(S,t)$ of the Black-Scholes equation (2.3) with the initial value (2.4) is

$$u(S,\tau)=\begin{cases}SN(d_1)-Ke^{-r\tau}N(d_2) & (for\ \ call),\\ Ke^{-r\tau}N(-d_2)-SN(-d_1) & (for\ put),\end{cases}$$

where

$$d_1=\frac{ln\frac{S}{K}+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},\ \ d_2=d_1-\sigma\sqrt{\tau}.$$

2.2 MQ quasi-interpolation

In this paper, we choose the MQ quasi-interpolation operator proposed by Wu and Chen [20] to approximate f.

Given the points (x_j,f_j), $\{f=f(x_j),\ j=0,1,\dots,m,\ x_0<x_1<\cdots<x_m\}$, the MQ quasi-interpolation is defined as

$$f^*(x)=\sum_{j=0}^{m}f_j\Psi_j(x), \tag{2.5}$$

where

$$\Psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})},$$

$$\phi_j(x) = \sqrt{(x - x_j)^2 + c^2}, \qquad 0 \le j \le m-1, c \in \mathbb{R}.$$

And when the following definitions are met, i.e.,

$$\begin{cases} \phi_{\boldsymbol{m}}(x) = \phi_0(x) - 2x + x_m + x_0, \\ \phi_{-1}(x) = \phi_0(x) + x_0 - x_{-1}, \\ \phi_{m+1}(x) = \phi_{\boldsymbol{m}}(x) + x_{m+1} - x_m, \end{cases}$$

the MQ quasi-interpolation can be rewritten as

$$f^*(x) = \frac{f_0 + f_m}{2} + \frac{1}{2}\sum_{j=0}^{m-1} \frac{\phi_j(x) - \phi_{j+1}(x)}{x_{j+1} - x_j}\Big(f_{j+1}(x) - f_j(x)\Big). \tag{2.6}$$

So, on $[x_0, x_m]$, we can get

$$(f^*(x))' = \frac{1}{2}\sum_{j=0}^{m-1} \frac{\phi_j(x)' - \phi_{j+1}(x)'}{x_{j+1} - x_j}(f_{j+1}(x) - f_j(x)), \tag{2.7}$$

$$(f^*(x))'' = \frac{1}{2}\sum_{j=0}^{m-1} \frac{\phi_j(x)'' - \phi_{j+1}(x)''}{x_{j+1} - x_j}\Big(f_{j+1}(x) - f_j(x)\Big). \tag{2.8}$$

Theorem2.1 [24] As for $f(x) \in C^2\ [x_0, x_m]$, when $h \to 0$, there exist constants K_0, K_1, K_2, K_3 independent of h, c, such that the error of the quasi-interpolation operator satisfies

$$\|f'(x) - (f^*(x))'\|_\infty \le O(h), \tag{2.9}$$

$$\|f''(x) - (f^*(x))''\|_\infty \le O(h^{\frac{2}{3}}). \tag{2.10}$$

3. MQ quasi-interpolation for solving Black-Scholes equation

3.1 MQ quasi-interpolation numerical scheme

In this paper, we use the quasi-interpolation method to solve the Black-Scholes equation. First, we spread the space by using the MQ quasi-interpolation to approximate the first and second derivatives of stock prices, while in time with step Δt, we get

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{1}{2}\sigma^2 (u_{xx})_j^n - (r - \frac{1}{2}\sigma^2)(u_x)_j^n + r u_j^n = 0, \tag{3.1}$$

where u_j^n is the approximation of $u(x,\tau)$ at point (x_j, τ_n) by quasi-interpolation and $(u_x)_j^n$, $(u_{xx})_j^n$ is defined in (2.7) and (2.8). Here we call it the MQ scheme.

In order to compare with the finite difference method, we first write the numerical scheme of the finite difference method (FDM) [25]

$$\frac{(u_F)_j^{n+1} - (u_F)_j^n}{\Delta t} = \frac{1}{2}\sigma^2 \frac{(u_F)_{j+1}^n - 2(u_F)_j^n + (u_F)_{j-1}^n}{h^2} + (r - \frac{1}{2}\sigma^2)\frac{(u_F)_{j+1}^n - (u_F)_j^n}{h} - r(u_F)_j^n, \tag{3.2}$$

where $(u_F)_j^n$ is the approximation of $u(x,\tau)$ at point (x_j, τ_n) by the finite difference method.

3.2 Convergence analysis

Theorem3.1 The solution of the MQ scheme we proposed converges to the exact solution of the Black-Scholes equation (2.3), and under the condition of $\frac{\Delta t}{h}|r - \frac{\sigma^2}{2}| < 1$, the convergence order is $O(h^{\frac{2}{3}} + \Delta t)$.

Proof: (3.1) - (3.2) lead to

$$u_j^{n+1} - (u_F)_j^{n+1} = (1 - r\Delta t)(u_j^n - (u_F)_j^n) + (r - \frac{1}{2}\sigma^2)\Delta t[(u_x)_j^n - \frac{(u_F)_{j+1}^n - (u_F)_j^n}{h}]$$

$$+\ \frac{1}{2}\sigma^2\Delta t[(u_{xx})_j^n - \frac{(u_F)_{j+1}^n - 2(u_F)_j^n + (u_F)_{j-1}^n}{h^2}].$$

Introduce some notations

$$A = (u_x)_j^n - \frac{(u_F)_{j+1}^n - (u_F)_j^n}{h},$$

$$B = (u_{xx})_j^n - \frac{(u_F)_{j+1}^n - 2(u_F)_j^n + (u_F)_{j-1}^n}{h^2}.$$

Then,

$$|A| \leq \left|(u_x)_j^n - (u_x)(x_j,\tau_n)\right| + \left|(u_x)(x_j,\tau_n) - \frac{(u_F)_{j+1}^n - (u_F)_j^n}{h}\right| \leq O(h) + O(h) = O(h). \quad (3.3)$$

Similary, we can get

$$|\ B\ | \leq O\left(h^{\frac{2}{3}}\right) + O(h^2) = O\left(h^{\frac{2}{3}}\right), \quad (3.4)$$

then we can get

$$\left\|\tilde{R}_j^{n+1}\right\| \leq \|(1+r\Delta t)\tilde{R}_j^n\|_\infty + \Delta t O(h^{\frac{2}{3}}) \leq \|e^{rn\Delta t}(\tilde{R}^0 + \sum_{n=0}^{M}\Delta t O(h^{\frac{2}{3}}))\|_\infty \leq O(h^{\frac{2}{3}}), \quad (3.5)$$

where $\tilde{R}_j^n = u_j^n - (u_F)_j^n$.

This means that the solution u_j^n of the MQ scheme we proposed converges to the difference scheme solution $(u_F)_j^n$. Because the finite difference scheme converges to the exact solution of the equation, and the order of convergence $O(\Delta t + h)$ [25]. Therefore, we can obtain that, under the condition of satisfying the stability of the difference scheme, the solution of the MQ scheme converges to the exact solution, and the convergence order is $O\left(\Delta t + h^{\frac{2}{3}}\right)$.

4. Numerical experiments

In order to demonstrate the accuracy of the MQ scheme for option pricing, we take the European put option as an example under uniform points and scattered points for comparison .

TABLE 1 : Parameters of numerical example.

Name	Value
Asset price value	$S\epsilon[e^{-3.5}, e^{4.5}]$
Space-step size	$h = 8/1280$
Time-step size	$\Delta t = 5 \times 10^{-4}$
Expiration date	$T = 0.5$
Exercise price	$K = 10$
Risk free interest rate	$r = 0.5$
Volatility	$\sigma = 0.2$
Support radius	$c = 0.1h^{\frac{1}{3}}$

Example 4.1 At first, we consider approximating option prices on the uniform grids. The settings of relevant parameters are shown in the TABLE 1. In TABLE 2, we give the $L_\infty - error$ and mean square error (RMSE), which are defined as

$$L_\infty - error = \max_{0\leq j\leq M} |\ u_{exact}(j) - u_{apper}(j)\ |,$$

$$RMSE = \sqrt{\frac{1}{M}\sum_{J=0}^{J}(u_{exact}(j) - u_{apper}(j))^2}.$$

The numerical results obtained by the MQ scheme are shown in Fig. 1. Fig. 2 shows the approximate errors of option price by the MQ scheme and the finite difference method at $\tau = 200\Delta t$.

From the TABLE 2 and Figs. 1-2, we can observe that we get accurate results by using the MQ scheme. Fig. 2 shows that the solution of the MQ scheme is close to the finite difference numerical solution at $\tau = 200\Delta t$, and the effect is slightly better than the finite difference method.

TABLE 2: Comparison between the exact solution and the numerical solution.

S	Exact	MQ	error
1	8.7531	8.7530	1.0000e-04
2	7.7519	7.7518	1.0000e-04
3	6.7489	6.7488	1.0000e-04
4	5.7483	5.7481	2.0000e-04
5	4.7348	4.7375	2.7000e-03
6	3.7412	3.7410	2.0000e-04
7	2.7285	2.7286	1.0000e-04
8	1.7856	1.7863	7.0000e-04
9	0.9709	0.9713	4.0000e-04
10	0.4274	0.4275	1.0000e-04
RMSE		1.6860e-04	
$L_\infty - error$		5.3358e-03	

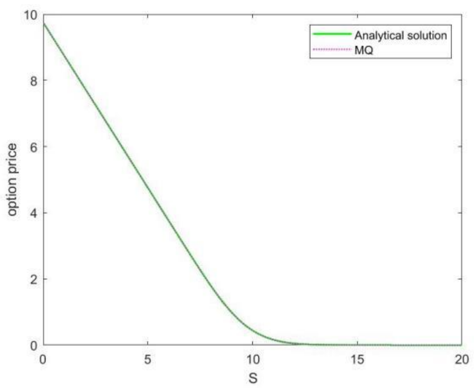


Fig.1 Comparison between the exact solution and the numerical solution of the MQ scheme.

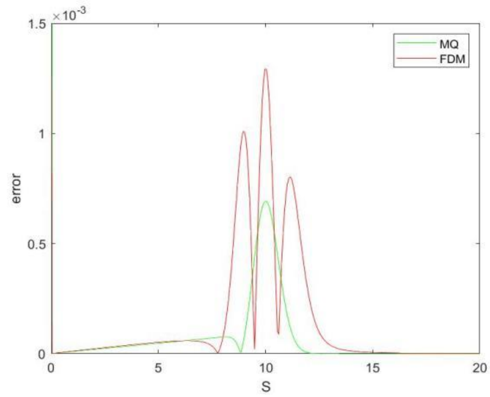
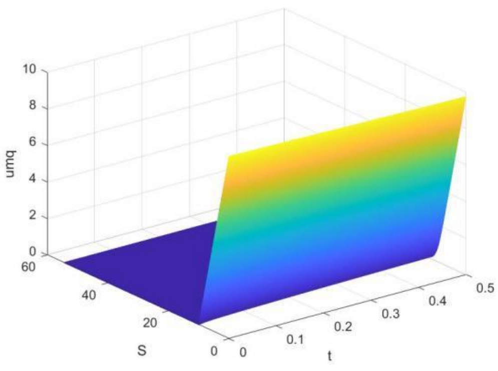
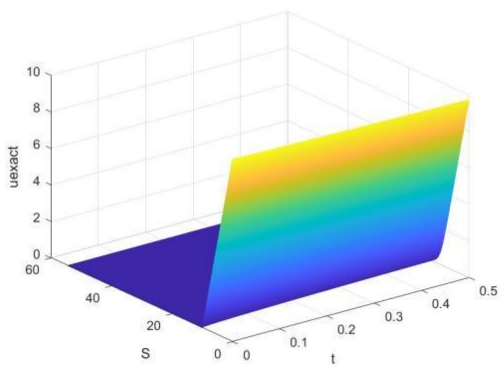


Fig. 2 Errors of the MQ scheme and the FDM scheme.





(a) The solution of the MQ scheme. (b) Exact solution.

Fig. 3 Comparison between the exact solution and the numerical solution.

Example 4.2 For scattered datas, we choose $x_j = jh - 0.3sin(2\pi h), h = max \mid x_{j+1} - x_j \mid, J = 200, j = -\frac{J}{2}, \dots, \frac{J}{2}$, and the shape parameter of quasi-interpolation is $c = 0.1h^{\frac{1}{3}}$. Fig. 3 shows the numerical and exact solution, and Fig. 4 shows the approximate errors of the MQ scheme.

From the Figs. 3-4, we can obtain that the solution of the MQ scheme also has good agreement with exact solution and we can get small errors in the option price under scattered points.

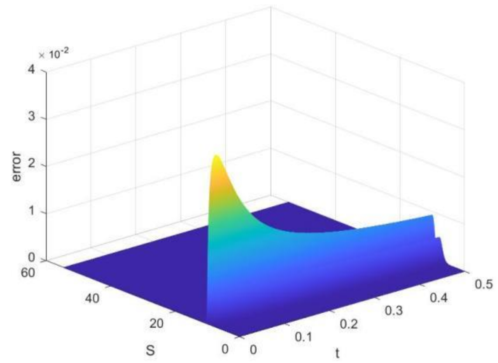


Fig. 4 Errors of the MQ scheme under scattered points.

5. Conclusions

MQ quasi-interpolation method not only provides understanding of interpolation formula, but also provides the derivative of the interpolation formula, which makes it close to the change of the asset price and does not need additional interpolation technology. Based on the above, this method is easy to calculate. From the results, we can obtain that the MQ scheme proposed here can well simulate the option price whether it is under uniform points or scattered points.

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