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Some results on Shannon wavelet packets

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Abstract. In this article, Shannon wavelet packets are studied together with their properties and focused the properties of packets through inner product.

Keywords: Shannon wavelets, Shannon wavelet packets, Inner products

1. Introduction

The bases of wavelet have poor frequency localization when sufficiently large value of j. Especially, in the signal processing, it is more convenient to have orthonormal bases with better frequency localization. This will be managed by the wavelet packets [1] which are obtained from wavelets associated with multi resolution analysis. Suppose ψ is an orthonormal wavelet generated by an multi resolution analysis then we use $W_j, \ \forall j \in Z$ to denote the closure in $L_2(R)$ which represent linear space generated by the set of wavelet basis $\{\psi_{i,j}: i, j \in Z\}$. Since $W_j, \ \forall j \in Z$ is the direct sum of the two subspaces. Therefore an element which is localized in frequency domain can be expressed as linear sum of elements of direct sum of two subspaces with better frequency localization than the original ones. With respect to that the frequency localization is further reduced by breaking up the new subspaces. Coifman, Meyer and Wickerhauser [2] proposed the decomposition of wavelet spaces $W_j, \ \forall j \in Z$. Their idea was to "split" these spaces into two orthogonal subspaces using the splitting rule as $V_{i+1} = V_i \oplus W_i$. They called as resulting basis functions (constructed from $\phi(x)$ and $\psi(x)$) for the resulting spaces wavelet packet functions. Take all the wavelet packet functions form an orthonormal basis for $L_2(R)$.

Here, we study the shannon wavelet packets before that lets have a look on Shannonwavelets. Shannonwavelets [3] have more advantages than other families of wavelets, they are as follows. In fact, Shannon wavelets

1. are analytically defined;
2. are infinitely differentiable;
3. are sharply bounded in the frequency domain, thus allowing a decomposition of frequencies in narrow bands;

Shannon wavelets are the real part of the so-called harmonic wavelets [4, 5]. Also, some literatures on Shannon wavelets methods in solving differential equations given in [6, 7, 8, 9] and Time–frequency localisation of Shannon wavelet packets [10]. Fortunately, no one discussed the properties of Shannon wavelet packets. So, we thought of to discuss properties of Shannon wavelet and Shannon wavelet packets.

The organization of the rest of the paper is as follows. In section 2, Preliminaries of Shannon Wavelets and Shannon wavelet packets are discussed. Results based on Shannon Wavelets and Shannon wavelet packets are presented in section 3. Finally conclusions are drawn in section 4.

2. Preliminaries of Shannon wavelets and Shannon wavelet packets

Shannon Wavelets

Let Shannon scaling and wavelet functions defined as [3]:

$$\phi(x) = \frac{\sin(\pi x)}{\pi x}$$

and

$$\psi(x) = \frac{\sin\pi(x-\frac{1}{2}) - \sin 2\pi(x-\frac{1}{2})}{\pi(x-\frac{1}{2})}.$$

After doing translation and dilation we obtain,

$$\phi_k^n(x) = 2^{\frac{n}{2}} \frac{\sin\pi(2^n x - k)}{\pi(2^n x - k)} \quad (2.1)$$

and

$$\psi_k^n(x) = 2^{\frac{n}{2}} \frac{\sin\pi(2^n x - k - \frac{1}{2}) - \sin 2\pi(2^n x - k - \frac{1}{2})}{\pi(2^n x - k - \frac{1}{2})}. \quad (2.2)$$

Shannon scaling and wavelet functions satisfies the following properties:

$$< \phi_k^0(x), \phi_h^0(x) >= \delta_{kh}$$

$$< \phi_k^0(x), \psi_h^m(x) >= 0 \quad m \geq 0.$$

$$< \psi_k^n(x), \psi_h^m(x) >= \delta_{nm}\delta_{kh}.$$

Definition 2.1 (Multiresolution Analysis) Let V_j, $\forall j \in Z$, be the sequence of subspaces of $L_2(R)$. Then $\{V_j\}$, $\forall j \in Z$ is a multiresolution analysis (MRA) of $L_2(R)$ if

1. $V_n \subset V_{n+1}$
2. $\overline{\cup_{j\in Z} V_j} = L_2(R)$
3. $\cap_{j\in Z} V_j = \{0\}$
4. $f(x) \in V_0 \Leftrightarrow f(2^j x) = V_j$

and there exists a function $\phi(x) \in V_0$, with $\int_R \phi(x)dx \neq 0$, called scaling function, such that $\{\phi(x-k)\}_{k\in Z}$ is an orthonormal basis for V_0.

Definition 2.2 (Wavelet Packet Functions) [11] Suppose that $\phi(x)$ generates an orthogonal multiresolution analysis V_j, $\forall j \in Z$ with associated wavelet function $\psi(x)$. The wavelet packet functions are defined by $W^0(x) = \phi(x)$, $W^1(x) = \psi(x)$ and for $n = 2,3,\ldots$

$$W^{2n}(x) = \sqrt{2}\sum_{k\in Z} h_k W^n(2x-k) \quad (2.3)$$

$$W^{2n+1}(x) = \sqrt{2}\sum_{k\in Z} g_k W^n(2x-k) \quad (2.4)$$

Definition 2.3 (Shannon wavelet packets) [10] Clearly $\phi^\pi(x) = \mathrm{sinc}(x)$ is called Shannon scaling function. Now, denote $\phi_1^\pi(x) = \phi^\pi(x)$. The Shannon wavelets packets $\phi_1^\pi(x); n = 1,2,3,\ldots$ associated with the finite sequences h and g are defined recursively as,

$$\phi_{2n}^\pi(x) = 2\sum_{k\in Z} h_k \phi_n^\pi(2x-k) \quad (2.5)$$

$$\phi_{2n+1}^\pi(x) = 2\sum_{k\in Z} g_k \phi_n^\pi(2x-k) \quad (2.6)$$

Proposition 2.4 (Properties of scaling filter) [11] Suppose that $\{V_j\}$ $\forall j \in Z$ is a multiresolution analysis of $L_2(R)$ with scaling function $\phi(x)$. If h is the scaling filter with real valued elements from the dilation equation $\phi(x) = \sqrt{2}\sum_{k\in Z} h_k \phi(2x-k)$ where, $h_k =< \phi(x), \phi_{1,k}(x) >$. Then

- i) $\sum_{k\in Z} h_k = \sqrt{2}$
- ii) $\sum_{k\in Z} h_k h_{k-2l} = \delta_{0,l}$ for all $l \in Z$
- iii) $\sum_{k\in Z} h_k^2 = 1$

3. Results

Theorem 3.1 Let $\phi(x)$ and $\psi(x)$ are Shannon scaling and wavelet functions respectively. Then $\psi(x) = \phi(x-\frac{1}{2}) - 2\phi(2x-1)$.

Proof. Since,

$$\phi(x) = \frac{\sin(\pi x)}{\pi x}$$

Therefore,

$$\phi(x-\tfrac{1}{2}) = \frac{\sin\pi(x-\frac{1}{2})}{\pi(x-\frac{1}{2})}$$

$$\phi(2x-1) = \frac{\sin 2\pi(x-\frac{1}{2})}{2\pi(x-\frac{1}{2})}$$

$$\phi(x-\tfrac{1}{2}) - 2\phi(2x-1) = \frac{\sin\pi(x-\frac{1}{2})}{\pi(x-\frac{1}{2})} - 2\frac{\sin 2\pi(x-\frac{1}{2})}{2\pi(x-\frac{1}{2})}$$

$$\phi(x-\tfrac{1}{2})-2\phi(2x-1)=\frac{\sin\pi(x-\frac{1}{2})-\sin 2\pi(x-\frac{1}{2})}{\pi(x-\frac{1}{2})}$$

$\phi(x-\frac{1}{2})-2\phi(2x-1)=\psi(x)$ is called Shannon wavelet function.

Theorem 3.2 Let $\phi(x)$ and $\psi(x)$ are Shannon scaling and wavelet functions respectively. Then

$$\phi(2x)=\psi(2x+\tfrac{1}{2})+2\phi(4x)$$

$$\phi(2x-1)=\tfrac{1}{2}(\phi(x-\tfrac{1}{2})-\psi(x))$$

Lemma 3.3 Let $\phi(x)$ and $\psi(x)$ are Shannon scaling and wavelet functions respectively. Then

$$\phi(2^j x)=\psi(2^j x+\tfrac{1}{2})+2\phi(2^{j+1}x)$$

$$\phi(2^j x-1)=\tfrac{1}{2}(\phi(2^{j-1}x-\tfrac{1}{2})-\psi(2^{j-1}x))$$

This lemma follows by replacing x by $2^{j-1}x$ in above theorem. This lemma can be used to decompose $\phi(2^j x-k)$ in to its W_k components.

Theorem 3.4 Let $\phi_n^\pi(x)$ is defined as in the definition 2.3 $\forall j,k\in Z$ then

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-j),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=\tfrac{1}{2}\delta_{j,k}.$$

Proof. We know that, suppose that $f(x)$ and $g(x)$ are functions in $L_2(R)$ $\forall k,l,m\in Z$. Then

$$<f(x-k),g(x-l)>=<f(x),g(x-(l-k))> \quad (3.1)$$

$$<f(2^m x-k),g(2^m x-l)>=2^{-m}<f(x),g(x-(l-k))> \quad (3.2)$$

Therfore it is sufficient to show that, $<\phi_n^\pi(x),\phi_n^\pi(x-k)>=\delta_{0,k}$. We can proceed by mathematical induction on n.

Case I: Consider $n=1$ then we have,

$$<\phi_1^\pi(x),\phi_1^\pi(x-k)>=<\phi^\pi(x),\phi^\pi(x-k)>=\delta_{0,k}$$

Since, $\{\phi(x-k)\}_{k\in Z}$ is an orthonormal set of functions. Hence,

$$<\tfrac{1}{\sqrt{2}}\phi_1^\pi(x),\tfrac{1}{\sqrt{2}}\phi_1^\pi(x-k)>=\tfrac{1}{2}\delta_{0,k}$$

Case II: Now assume that,

$$<\tfrac{1}{\sqrt{2}}\phi_l^\pi(x),\tfrac{1}{\sqrt{2}}\phi_l^\pi(x-k)>=\tfrac{1}{2}\delta_{0,k} \quad \text{for } 1<l<n.$$

Case III: a) If n is even and put $p=2k+j$. Consider

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=<\tfrac{2}{\sqrt{2}}\textstyle\sum_{i\in Z}h_i\phi_{\frac{n}{2}}^\pi(2x-i),\tfrac{2}{\sqrt{2}}\sum_{j\in Z}h_j\phi_{\frac{n}{2}}^\pi 2(x-k)-j>$$

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=2\textstyle\sum_{i\in Z}\sum_{j\in Z}h_ih_j<\phi_{\frac{n}{2}}^\pi(2x-i),\phi_{\frac{n}{2}}^\pi(2(x-k)-j)>$$

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=2\textstyle\sum_{i\in Z}\sum_{j\in Z}h_ih_{p-2k}<\phi_{\frac{n}{2}}^\pi(2x-i),\phi_{\frac{n}{2}}^\pi(2x-p)>$$

From 3.4 (at m=1) we get,

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=\tfrac{2}{2}\textstyle\sum_{i\in Z}\sum_{j\in Z}h_ih_{p-2k}<\phi_{\frac{n}{2}}^\pi(x),\phi_{\frac{n}{2}}^\pi(x-(p-i))>$$

We know that, $\delta_{0,p-i}=\delta_{p,i}$ and by case II we obtain,

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=\tfrac{1}{2}\textstyle\sum_{i\in Z}\sum_{j\in Z}h_ih_{p-2k}\delta_{p,i}=\tfrac{1}{2}\sum_{i\in Z}h_ih_{i-2k}.$$

by proposition 2.4 (ii)

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=\tfrac{1}{2}\delta_{0,k}.$$

b) If n is odd and put $p=2k+j$. Consider

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=<\tfrac{2}{\sqrt{2}}\textstyle\sum_{i\in Z}g_i\phi_{\frac{n}{2}}^\pi(2x-i),\tfrac{2}{\sqrt{2}}\sum_{j\in Z}g_j\phi_{\frac{n}{2}}^\pi 2(x-k)-j>$$

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=2\textstyle\sum_{i\in Z}\sum_{j\in Z}g_ig_j<\phi_{\frac{n}{2}}^\pi(2x-i),\phi_{\frac{n}{2}}^\pi(2(x-k)-j)>$$

$$<\tfrac{1}{\sqrt{2}}\phi_n^\pi(x),\tfrac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=2\textstyle\sum_{i\in Z}\sum_{p\in Z}g_ig_{p-2k}<\phi_{\frac{n}{2}}^\pi(2x-i),\phi_{\frac{n}{2}}^\pi(2x-p)>$$

From 3.4 (at m=1) we get,

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x),\frac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=\frac{2}{2}\sum_{i\in Z}\sum_{p\in Z} g_i g_{p-2k} <\phi_{\frac{n}{2}}^\pi(x),\phi_{\frac{n}{2}}^\pi(x-(p-i))>$$

We know that, $\delta_{0,p-i}=\delta_{p,i}$ and by case II we obtain,

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x),\frac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=\frac{1}{2}\sum_{i\in Z}\sum_{p\in Z} g_i g_{p-2k}\delta_{p,i}=\frac{1}{2}\sum_{i\in Z} g_i g_{i-2k}.$$

by proposition 2.4 (ii)

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x),\frac{1}{\sqrt{2}}\phi_n^\pi(x-k)>=\frac{1}{2}\delta_{0,k}.$$

Theorem 3.5 Let $\phi_n^\pi(x)$ is defined as in the definition 2.3 $\forall j,k\in Z$ and $\forall n,m\in N$ then $<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=\frac{1}{2}\delta_{n,m}\delta_{j,k}$.

Proof. We know that, suppose that $f(x)$ and $g(x)$ are functions in $L_2(R)$ $\forall k,l,m\in Z$. Then

$$<f(x-k),g(x-l)>=<f(x),g(x-(l-k))> \quad (3.3)$$

$$<f(2^m x-k),g(2^m x-l)>=2^{-m}<f(x),g(x-(l-k))> \quad (3.4)$$

Therfore it is sufficient to show that, $<\phi_n^\pi(x),\phi_m^\pi(x-k)>=\delta_{n,m}\delta_{0,k}$. If $n=m$ then this theorem reduces to theorem 3.4. So, we prove this by mathematical induction at $m\neq n$.

Case I: If m and n are sufficiently small values, assume that $m=1$ and $n=2$. Then,

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x),\frac{1}{\sqrt{2}}\phi_m^\pi(x)>=\frac{1}{2}\delta_{n,m}.$$

Therefore,

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=\frac{1}{2}\delta_{n,m}\delta_{j,k}$$

Case II: Assume the result is true for all $1<l<n, 1<k<m$.

Case III: Let for all $n,m\in N$ and $p=2k+j$

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=<\frac{2}{\sqrt{2}}\sum_{i\in Z} h_i\phi_{\frac{n}{2}}^\pi(2x-i),\frac{2}{\sqrt{2}}\sum_{j\in Z} h_j\phi_{\frac{m}{2}}^\pi 2(x-k)-j>$$

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=2<\sum_{i\in Z} h_i\phi_{\frac{n}{2}}^\pi(2x-i),\sum_{j\in Z} h_j\phi_{\frac{m}{2}}^\pi 2(x-k)-j>$$

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=2\sum_{i\in Z}\sum_{j\in Z} h_i h_j <\phi_{\frac{n}{2}}^\pi(2x-i),\phi_{\frac{m}{2}}^\pi(2x-(2k+j))>$$

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=2\sum_{i\in Z}\sum_{p\in Z} h_i h_{p-2k} <\phi_{\frac{n}{2}}^\pi(2x-i),\phi_{\frac{m}{2}}^\pi(2x-p)>$$

From 3.4 (at m=1) we get,

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=\frac{2}{2}\sum_{i\in Z}\sum_{p\in Z} h_i h_{p-2k} <\phi_{\frac{n}{2}}^\pi(x),\phi_{\frac{m}{2}}^\pi(x-(p-i))>$$

We know that, $\delta_{0,p-i}=\delta_{p,i}$ and by case II we obtain,

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=\frac{1}{2}\delta_{n,m}\sum_{i\in Z}\sum_{p\in Z} h_i h_{p-2k}=\frac{1}{2}\delta_{n,m}\sum_{i\in Z} h_i h_{i-2k}$$

by proposition 2.4 (ii).

$$<\frac{1}{\sqrt{2}}\phi_n^\pi(x-j),\frac{1}{\sqrt{2}}\phi_m^\pi(x-k)>=\frac{1}{2}\delta_{n,m}\delta_{0,k}$$

Corollary 3.6 Shannon wavelets functions $\{\psi_n^\pi(x)\}$, $\forall n\in Z$ are linearly independent.

4. Conclusions

In this article, we expressed Shannon scaling functions genarally by sum of scaling and wavelet function. Also, discussed the properties of Shannon packets under inner products.

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