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The perturbed compound Poisson-Geometric risk model with constant interest and a threshold dividend strategy

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Abstract: In this paper, the perturbed compound Poisson-Geometric risk model with constant interest and a threshold dividend strategy are considered. Firstly, the integro-differential equations with boundary conditions for the Gerber-Shiu function is discussed. Then the equation satisfying the ruin probability studied when the claim size is exponential function. Finally, Integro-differential equations with certain boundary for the moment-generation function of the present value of total dividends until ruin is derived.

Keywords: Constant interest; Threshold dividend strategy; Gerber-Shiu discounted penalty function; Integro-differential equation.

1. Introduction

Insurance company is a financial institution that operates risk business, whose operating condition is uncertain. For this reason, scholars have proposed some indicators to describe the operating conditions. The so-called bankruptcy of a company in the mathematical model refers to the probability of negative earnings in a certain period. If some factors affecting the surplus are taken into consideration, the insurance company may obtain a profit. The study of the insolvency theory can be traced back to the doctoral thesis published by Filip Lundbery in 1903[1], who proposed a class of random processes for the first time, that is, the Poisson process, and then Harald Cramer improved the results of Filip Lundberg and developed a strict stochastic process theory. Therefore, the improved model is called a CLASSICAL risk one. Thus, the surplus of an insurance company at time t can be given as

$$U(t) = u + ct - \sum_{t=1}^{N(t)} X_i \, , t \geq 0.$$

There are many conclusions in the classical model [2]. With the deep research of this model, many scholars have improved the classic risk model from different aspects. For example, the interference term or the random factor is added to the model [3, 4]. As we all know, the risk event is equivalent to the actual claim one in the classic risk model. The process which describes the number of claims is a homogeneous Poisson process. In fact, there is a deviation between the number of risk events and the actual claims. Therefore, a kind of composite Poisson-Geometric process is introduced, which is called PG Process [5-6]. The surplus of a Poisson-Geometric risk model at time t can be described as

$$U(t) = u + ct - \sum_{t=1}^{N(t)} X_i + \sigma W(t) \, , t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial surplus; $c > 0$is the constant rate of premium; $\{X_i, i = 1,2, \cdots\}$ is a sequence of independent income size random variables with a common distribution function F (x) which satisfies F (0) = 0 and has density function f (x). $\{N(t) \geq 0, t \geq 0\}$is the Poisson-Geometric income-number process; $\{W(t), t \geq 0\}$ is a standard Brownian motion; σ > 0 is a constant representing diffusion volatility parameter.$\{X_i, i = 1,2, \cdots\}$, $\{N(t) \geq 0, t \geq 0\}$ and $\{W(t), t \geq 0\}$ are mutually independent.

Suppose that the insurer could receive interest form its surplus of (1.1) at a constant force of interest $r > 0$, then the surplus of the insurer at time t is

$$U(t) = ue^{rt} + c\int_0^t e^{r(t-s)} ds - \sum_{t=1}^{N(t)} e^{r(t-S_i)} X_i + \sigma \int_0^t e^{r(t-s)} \, dW(s), t \geq 0, \tag{1.2}$$

where S_i is the inter-time of the ith claim. Then (1.2) also can be rewritten as

$$U(t) = ue^{rt} + \frac{c}{r}(e^{rt} - 1) - \sum_{t=1}^{N(t)} e^{r(t-S_i)} X_i + \sigma \int_0^t e^{r(t-s)} \, dW(s), t \geq 0. \tag{1.3}$$

Gerber and Landry considered its expected discounted value of a penalty that is due at ruin [7]. With the development of the industry, the issue of dividend strategies has received remarkable attention since De Finetti first proposed the so-called barrier strategy to reflect the surplus cash flowed in an insurance portfolio [8]. Classical risk model with constant interest and a threshold dividend strategy were discussed [9, 10]. The expected discounted dividends before ruin under threshold-type dividend strategy was analyzed [11]. The constant barrier for the compound Poisson-Geometric risk model was studied in [12].

In this paper, we consider the modification of the surplus process by a threshold strategy with a threshold level $b(b > 0)$. when U (b) is below b, the surplus decreases at the original rate c_1; when U (b) is above b, the surplus decreases at a different rate $c_2(c_2 < c_1)$ and dividends are paid at rate $c_1 - c_2$. Incorporating the threshold strategy into (1.3) yields the surplus process $U_b(t)$, $t \geq 0$ which can be expressed by

$$dU_b(t) = \begin{cases} c_1\mathrm{dt} + \mathrm{rU}_b(t)\mathrm{dt} - \mathrm{dS}(t) + \sigma\mathrm{dW}(t) & \text{if } \mathrm{U}_b(t) < \mathrm{b} \\ c_2\mathrm{dt} + \mathrm{rU}_b(t)\mathrm{dt} - \mathrm{dS}(t) + \sigma\mathrm{dW}(t) & \text{if } \mathrm{U}_b(t) \geq b' \end{cases} \tag{1.4}$$

where $S(t) = \sum_{i=1}^{N(t)} X_i$, $U_b(0) = u$.

Let $T_b = \inf\{t \geq 0, U_b(t) \leq 0\}$ be the time of ruin, the Gerber-Shiu discounted penalty function is

$$\varphi(u,b) = E\left[e^{-\delta T_b} w(U_b(T-), |U_b(T)|) I(T_b < \infty) | U_b(t) = u\right],$$

where δ is a nonnegative parameter, $w(x,y)$ is a nonnegative bounded measurable function of $(0,\infty) \times (0,\infty)$, and I(A) is an indicator function.

Let D(t) denote the cumulative amound of dividends paid out up to time t and β be the force of interest, then the present value of all dividends until T_b is

$$D_{u,b} = \int_0^{T_b(t)} e^{-\beta t} dD(t),$$

where $T_b(t) = \inf\{t \geq 0: U_b(t) \leq 0\}$ is the time of ruin. An alternative expression for $D_{u,b}$ is

$$D_{u,b} = (c_1 - c_2) \int_0^{T_b(t)} e^{-\beta t} I(U_b(t) > b)\, dt.$$

In the sequel, we are interested in the following moment generating function

$$M(u,y,b) = E[e^{yD_{u,b}}]$$

and the nth moment function

$$V_n(u,b) = E\left[D_{u,b}^n\right], for\ V_0(u,b) = 1.$$

2 Gerber-Shiu discounted penalty function

Firstly, the compound Poisson-Geometric process and its properties are introduced as Definition 2.1 and Definition 2.2.

Definition 2.1[5] If the probability generating function of random variable ζ is

$$G(t) = exp\frac{\lambda(t-1)}{1-\rho t},$$

then ζ has the compound Poisson-Geometric distribution as PG(λ, ρ).

Definition 2.2[5] N(t) $(t \geq 0)$ is called a compound Poisson-Geometric process with parameter $\lambda > 0$ and $0 \leq \rho < 1$, if the conditions

(1) N $(0) = 0$,

(2) N(t) $(t \geq 0)$ has has stationary and independent increments,

(3) For $t > 0$, N (t) has the PG(λ, ρ) distribution, and

$$EN(t) = \frac{t}{1-\rho}, VarN(t) = \frac{(1+\rho)}{(1-\rho)^2}$$

are satisfied.

Remark 2.1 In definition 2.2, ρ is defined as deviation parameter and describes the difference between risk event numbers and claim event numbers. When $\rho = 0$, the compound Poisson-Geometric process is the Poisson process. So the Poisson-Geometric process is a generalization of the Poisson process.

Lemma 2.1 N (t) is a process of a compound Poisson-Geometric process with parameter λ, ρ. Let $\alpha = \frac{\lambda(1-\rho)}{\rho}$, if $\rho = 0, \alpha = \lambda$, then we can have that when $t \to 0$,

$$\mathrm{P}(N(t) = 0) = e^{-\lambda t} = 1 - \lambda + o(t),$$

$$\mathrm{P}(N(t) = k) = \alpha\rho^k t + A_k(t)o(t), k = 1,2, ...,$$

can be obtained, where $A_k(t) = \rho^k + (k-1)[\rho(1+\alpha t)^{k-2}]$, $o(t)$ is independent of k, and $\sum_{k=0}^{\infty} A_k(t)$ is uniformly convergence.

Then we discuss the Gerber-Shiu discounted penalty function $\varphi(u,b)$ which includes two parts, the expected discounted penalty function $\varphi_s(u,b)$ with the ruin caused by a claim

$$\varphi_s(u,b) = E\left[e^{-\delta T_b}\omega(U_b(T(-),|U_b(T)|)I(T_b < \infty, U_b(T_b) < 0)|U_b(t) = u\right]$$

and the Laplace transform $\varphi_d(u,b)$ of ruin time Tb with the ruin due to the oscillation

$$\varphi_d(u,b) = E\left[e^{-\delta T_b}I(T_b < \infty, U_b(T_b) = 0)|U_b(t) = u\right]. \tag{2.1}$$

Let

$$\varphi_s(u,b) = \begin{cases} \varphi_{s1(u,b)}, & if \quad 0 \le u < b \\ \varphi_{s2(u,b)}, & if \quad u \ge b \end{cases}$$

and

$$\varphi_d(u,b) = \begin{cases} \varphi_{d1(u,b)}, & if \quad 0 \le u < b \\ \varphi_{d2(u,b)}, & if \quad u \ge b \end{cases}$$

Then we can get

$$\varphi(u,b) = \begin{cases} \varphi_{1(u,b)} = \varphi_{s1(u,b)} + \varphi_{d1(u,b)}, & if \quad 0 \le u < b \\ \varphi_{2(u,b)} = \varphi_{s2(u,b)} + \varphi_{d2(u,b)}, & if \quad u \ge b \end{cases}.$$

Furthermore, the main results can be given as following Theorems.

Theorem 2.1 For $0 \le u < b$, we have

$$\frac{\sigma^2}{2}\varphi_{s1}''(u,b) + (ru + c_1)\varphi_{s1}'(u,b) = (\lambda+\delta)\varphi_{s1}(u,b) - \lambda\int_0^u \varphi_{s1}(u-x,b)f_\rho(x)dx$$
$$-\lambda\int_u^\infty \omega(u,u-x)f_\rho(x)dx. \tag{2.2}$$

for $u \ge b$, we have

$$\frac{\sigma^2}{2}\varphi_{s2}''(u,b) + (ru + c_2)\varphi_{s2}'(u,b) = (\lambda+\delta)\varphi_{s2}(u,b) + \lambda\int_0^{u-b} \varphi_{s2}(u-x,b)f_\rho(x)dx$$
$$-\lambda\int_{u-b}^u \varphi_{s1}(u-x,b)f_\rho(x)dx - \lambda\int_u^\infty \omega(u,u-x)f_\rho(x)dx, \tag{2.3}$$

where boundary conditions are

$$\varphi_{s1}(0,b) = 0, \lim_{u\to\infty}\varphi_{s2}(u,b) = 0, \varphi_{s1}(b^-,b) = \varphi_{s2}(b,b).$$

Proof. When $0 \leqslant u < b$, we consider an infinitesimal time $[0,t]$, $v(t) = ue^{rt} + \frac{c_1(e^{rt}-1)}{r} + \sigma\int_0^t e^{r-s}\,dW(s)$. On the occurrence of claim on the time, applying Lemma 2.1 and the total probability formula. we can get

$$\phi_{s1}(u,b) = \left(1-\lambda t + o(t)\right)e^{-\delta t}E[\phi_{s1}(v(t),b)] + \sum_{k=1}^{\infty}\left(\alpha\rho^k t + A_k(t)o(t)\right)e^{-\delta t}$$
$$E\left[\int_0^{v(t)}\varphi_{s1}(v(t)-x,b)dF^{*k}(x) + \int_{v(t)}^\infty \omega(u,x-u)dF^{*k}(x)\right], \tag{2.4}$$

where $F^{*k}(x)$ is k-convolution of $\mathrm{F}(x)$ and $f^{*k}(x)$ is k-convolution of $\mathrm{f}(x)$.

Both sides of (2.3) is multiplied by $e^{\delta t}$ and it can be gotten that

$$e^{\delta t}\varphi_{s1}(u,b) = E\left(\varphi_{s1}(v(t),b)\right) - \lambda t E[\varphi_{s1}(v(t),b)] + \sum_{k=1}^{\infty}\left(\alpha\rho^k t + A_k(t)o(t)\right)$$
$$E\left[\int_0^{v(t)}\varphi_{s1}(v(t)-x,b)\,dF^{*k}(x) + \int_{v(t)}^\infty \omega(u,x-u)\,dF^{*k}(x)\right]. \tag{2.5}$$

By Taylor's expansion, we can get

$$\varphi_{s1}(v(t),b) = \varphi_{s1}(u,b) + (ru+c_1)t\varphi_{s1}'(u,b) + \tfrac{1}{2}t\varphi_{s1}''(u,b) + o(t).$$

Substituting the above expression and $e^{\delta t} = 1 + \delta t + o(t)$ into (2.4), we can get

$$(1-\lambda t)\left[\varphi_{s1}(u,b) + (ru+c_1)t\varphi_{s1}'(u,b) + \frac{\sigma^2}{2}\varphi_{s1}''(u,b)t\right] = \left(1+\delta t + o(t)\right)\varphi_{s1}(u,b)$$
$$-\sum_{k=1}^{\infty}\left(\alpha\rho^k t - A_k(t)o(t)\right)E\left[\int_0^{v(t)}\varphi_{s1}(v(t)-x,b)\,dF^{*k}(x) - \int_{v(t)}^\infty \omega(u,x-u)\,dF^{*k}(x)\right]. \tag{2.6}$$

Dividing both side of (2.5) by t and letting $t \to 0$, we have

$$(\lambda+\delta)\varphi_{s1}(u,b) = (ru+c_1)t\varphi'_{s1}(u,b) + \frac{\sigma^2}{2}\varphi''_{s1}(u,b) + \sum_{k=1}^{\infty}\alpha\rho^k \left[\int_0^{v(t)}\varphi_{s1}(v(t)-x,b)\,dF^{*k}(x) + \int_{v(t)}^{\infty} d\,F^{*k}(x)\right]. \quad (2.7)$$

Substitute $\alpha = \frac{\lambda(1-\rho)}{\rho}$ into (2.6) and let $f_\rho(x) = \sum_{k=1}^{\infty}(1-\rho)\rho^{k-1}f^{*k}(x)$, where $f^{*k}(x)$ is density function of $F^{*k}(x)$, and we can obtain(2.1) via using Lemma 2.1. When $u \geq b$, similarly, we also consider an infinitesimal tome [0, t], $v(t) = ue^{rt} + \frac{c_2(e^{rt}-1)}{r} + \sigma\int_0^t e^{r-s}dw(s)$. (2.2) can be gotten. The proof of Theorem 2.1 is completed.

Theorem 2.2 For $0 \leq u < b$, we can reach

$$\frac{\sigma^2}{2}\varphi''_{d1}(u,b) + (ru+c_1)\varphi'_{d1}(u,b) = (\lambda+\delta)\varphi_{d1}(u,b) - \lambda\int_0^u u_{d1}\,(u-x,b)f_\rho(x)dx. \quad (2.8)$$

when $u \geq b$, we can get

$$\frac{\sigma^2}{2}\varphi''_{d2}(u,b) + (ru+c_2)\varphi'_{d2}(u,b) = (\lambda+\delta)\varphi_{d2}(u,b) - \lambda\int_0^{u-b} u_{d2}\,(u-x,b)f_\rho(x)dx - \lambda\int_{u-b}^{b} u_{d1}\,(u-x,b)f_\rho(x)dx \quad (2.9)$$

with boundary conditions

$$\varphi_{s1}(0,b) = 0, \lim_{u\to\propto}\varphi_{s2}(u,b) = 0, \varphi_{s1}(b^-,b) = \varphi_{s2}(b,b).$$

Theorem 2.3 For $0 \leq u < b$,

$$\frac{\sigma^2}{2}\varphi''_1(u,b) + (ru+c_1)\varphi'_1(u,b) = (\lambda+\delta)\varphi_1(u,b) - \lambda\int_0^u \varphi_1\,(u-x,b)f_\rho(x)dx - \lambda\int_u^{\infty}\omega(u,u-x)f_\rho(x)dx \quad (2.10)$$

can be followed. When $u \geq b$,

$$\frac{\sigma^2}{2}\varphi''_2(u,b) + (ru+c_2)\varphi'_2(u,b) = (\lambda+\delta)\varphi_2(u,b) - \lambda\int_0^{u-b}\varphi_2\,(u-x,b)f_\rho(x)dx - \lambda\int_{u-b}^{u}\varphi_1\,(u-x,b)f_\rho(x)dx - \lambda\int_u^{\infty}\omega\,(u,u-x)f_\rho(x)dx \quad (2.11)$$

with boundary conditions

$$\varphi_1(0,b) = 0\ , \lim_{u\to\infty}\varphi_2 = 0\ ,\ \varphi_1(b^-,b) = \varphi_{s2}(b,b)$$

can be achieved.

3 Closed form expression for ruin probability

In this section, we give the closed form expression for ruin probability $\phi(u)$ if claim size X_i has exponential distribution with parameter θ, that is

$$f(x) = \theta e^{-\theta x}(x>0,\theta>0).$$

Then, $f^{*k}(x)$ is the Gamma distribution with parameters (k,θ), that is

$$f^{*k}(x) = \frac{\theta^k}{(k-1)!}x^{k-1}e^{-\theta x}.$$

Thus, it can be known that

$$f_\rho(x) = \theta(1-\rho)e^{-(1-\rho)\theta x}.$$

Let

$$\varphi(u,b) = \begin{cases}\varphi_1(u,b), & if\ 0 \leq u < b\\ \varphi_2(u,b), & if\ \ u \geq b\end{cases}, \quad (3.1)$$

set $\delta = 0$ and $w(x,y) = 1$ in (2.1) and (2.2), and we have that when $0 \leq u < b$,

$$\frac{\sigma^2}{2}\phi''_1(u,b) + (ru+c_1)\phi'_1(u,b) = (\lambda+\delta)\phi_1(u,b) - \lambda\int_0^u \phi_1(u-x,b)\,f_\rho(x)dx - \lambda\int_u^{\infty} w(u,u-x)\,f_p(x)d_{x.} \quad (3.2)$$

Derivate on both sides of the equation (3.2), and we can get

$$\frac{\sigma^2}{2}\Phi_1{}'''(u,b) + (ru+c_1)\Phi_1{}''(u,b) = (\lambda+\delta)\Phi_1{}'(u,b) - \lambda\int_0^u \Phi_1{}'(u,b)f_\rho\,(0) + \lambda f_\rho(u). \quad (3.3)$$

(3.2) multiplied by $(1-\rho)\theta$ and plus (3.3), we can get

$$\frac{\sigma^2}{2}\Phi_1'''(u,b)+\left[(1-\rho)\theta\frac{\sigma^2}{2}+ru+c_1\right]\Phi_1''(u,b)-[(1-\rho)\theta(ru+c_1)-\lambda]\Phi_1'(u,b)=0. \quad (3.4)$$

Similarly, when $u \geq b$,

$$\frac{\sigma^2}{2}\Phi_2'''(u,b)+\left[(1-\rho)\theta\frac{\sigma^2}{2}+ru+c_2\right]\Phi_2''(u,b)-[(1-\rho)\theta(ru+c_2)-\lambda]\Phi_2'(u,b)=0 \quad (3.5)$$

with boundary conditions

$$\phi_1(u,b)=0\ ,\ \lim_{u\to\infty}\phi_2(u,b)=0 \quad (3.6)$$

$$\phi_1(b^-,b)=\phi_2(b,b) \quad (3.7)$$

$$\frac{\sigma^2}{2}\Phi_1''(b^-,b)+(ru+c_1)\Phi_1'(b^-,b)=\frac{\sigma^2}{2}\Phi_2''(b,b)+(ru+c_2)\Phi_2'(b,b) \quad (3.8)$$

can be obtained.

We define

$$K_1=\frac{c_1}{r}-\frac{(1-\rho)\theta\sigma^2}{2r},$$

$$K_2=\frac{c_2}{r}-\frac{(1-\rho)\theta\sigma^2}{2r},$$

$$M(a,b,x)=\frac{\Gamma(\mathrm{b})}{\Gamma(\mathrm{b-a})\Gamma(\mathrm{a})}\int_0^1 e^{xt}t^{a-1}(1-t)^{b-a-1}dt, b>a>0,$$

$$U(a,b,x)=\frac{1}{\Gamma(\mathrm{a})}\int_0^\infty e^{-xt}t^{a-1}(1-t)^{b-a-1}dt, a>0,$$

where $M(a,b,x)$ is the standard confluent hypergeometric function and $\mathrm{U}(a,b,x)$ indicates its second form [13], suppose for $i=1,2$,

$$q_i(x)=e^{-(1-\rho)\theta x+\frac{r(x+k_i)^2}{\sigma^2}}U\left(\frac{\lambda}{2r},\frac{1}{2},\frac{r(x+k_i)^2}{\sigma^2}\right),$$

$$l_i(x)=(x+k_i)e^{-(1-\rho)\theta x+\frac{r(x+k_i)^2}{\sigma^2}}M\left(\frac{1}{2}+\frac{\lambda}{2r},\frac{3}{2},\frac{r(x+k_i)^2}{\sigma^2}\right),$$

$$Q_1(u)=\int_0^u q_1(x)dx\ ,\quad Q_2(u)=\int_u^\infty q_2(x)dx\ ,$$

$$L_1(u)=\int_0^u l_1(x)dx\ ,\quad L_2(u)=\int_u^\infty l_2(x)dx\ ,$$

it can be followed that

$$\Phi_1(u,b)=a_1Q_1(u)+a_2L_1(u)\ ,\quad 0\leq u<b,$$
$$\Phi_2(u,b)=a_3Q_2(u)+a_4L_2(u),\quad u\geq b,$$

the coefficient a1-a4 is determined by the following equations,

$$\begin{cases} a_1Q_1(0)+a_2L_1(0)=0 \\ a_1Q_1(b)+a_2L_1(b)-a_3Q_2(b)-a_4L_2(b)=0 \\ a_1Q_1'(b)+a_2L_1'(b)-a_3Q_2'(b)-a_4L_2'(b)=0 \\ a1\left(\frac{\sigma^2}{2}Q_1''(b)+(rb+c_1)Q_1'(b)\right)+a_2\left(\frac{\sigma^2}{2}L_1''(b)+(rb+c_1)L_1'(b)\right)+ \\ a_3\left(\frac{\sigma^2}{2}Q_2''(b)+(rb+c_2)Q_2'(b)\right)\Bigg)+a_4\left(\frac{\sigma^2}{2}L_2''(b)+(rb+c_2)L_2'(b)\right)=0 \end{cases}. \quad (3.9)$$

We define vector $\vec{A}=(a_1,a_2,a_3,a_4)^T$, vector $\vec{B}=(0,1,0,0)^T$ and

$$\vec{M}=\begin{pmatrix} Q_1(0) & L_1(0) & 0 & 0 \\ Q_1(b) & L_1(b) & -Q_2(b) & -L_2(b) \\ Q_1'(b) & L_1'(b) & -Q_2'(b) & -L_2'(b) \\ P_1(b) & S_1(b) & -P_2(b) & -S_2(b) \end{pmatrix}.$$

When $u = 0$ or b $(i = 1,2)$, we have

$$P_i(u) = \frac{\sigma^2}{2} Q_i''(u) + (ru + c_i)Q_i'(u), S_i(u) = \frac{\sigma^2}{2} L_i''(u) + (ru + c_i)L_i'(u).$$

Solving linear equations $\vec{M}\vec{A} = \vec{B}$, we can get ai. Furthermore, we can obtain $\phi_i(u, b)$.

4 Integro-differential equations for $M(u, y; b)$ and $V_n(u; b)$

In this section, we will discuss the moment generating function $M\ (u, y;\ b)$ when its surplus u is below or above the barrier level b. For this end, we set

$$M(u, y; b) = \begin{cases} M_1(u, y; b) & if\ 0 \le u < b \\ M_2(u, y; b) & if\ \ u \ge b \end{cases}. \tag{4.1}$$

We firstly derive the integro-differential equations satisfied by M (u, y; b).Some relative results can be gotten as Theorem 4.1and Theorem 4.2.

Theorem 4.1 For $0 \le u < b$, we can have

$$\frac{\sigma^2}{2} \frac{\partial^2 M_1}{\partial u^2}\left(u, y; b + (ru + c_1) \frac{\partial M_1}{\partial u}(u, y; b)\right) = \beta y \frac{\partial M_1}{\partial u}(u, y; b) + \lambda M_1(u, y; b)$$
$$+\lambda \int_0^u M_1(u - x, y; b) f_\rho(x) dx + \lambda \bar{F}_\rho(u)\ . \tag{4.2}$$

When $u \ge b$, we can get

$$\frac{\sigma^2}{2} \frac{\partial^2 M_2}{\partial u^2}\left(u, y; b + (ru + c_2) \frac{\partial M_2}{\partial u}(u, y; b)\right) = \beta y \frac{\partial M_2}{\partial u}(u, y; b) + \lambda M_2(u, y; b)$$
$$+\lambda \int_0^{u-b} M_2(u - x, y; b) f_\rho(x) dx + \lambda \int_{u-b}^{u} M_1(u - x, y; b) f_\rho(x) dx + \lambda \bar{F}_\rho(u) \tag{4.3}$$

with boundary conditions

$$M_1(0, y; b) = 1, \lim_{u\to\infty} M_2(u, y; b) = e^{\frac{(c_1 - c_2)y}{\beta}},$$
$$M_1(b^-, y; b) = M_2(b, y; b),$$
$$\frac{\partial M_1}{\partial u}(u, y; b)|_{u=b^-} = \frac{\partial M_2}{\partial u}(u, y; b)|_{u=b}. \tag{4.4}$$

Proof. When $0 \le u < b$, let $h(t) = ue^{rt} + \frac{c_1(e^{rt} - 1)}{r} + \sigma \int_0^t e^{(r-s)} dW(s)$, considering the occurrence time$(0,t]$ of the first claim, we have

$$M_1(u, y; b) = \big(1 - \lambda t + o(t)\big) E\big[M_1\big(h(t), ye^{\beta t}, b\big)\big] + \sum_{k=1}^{\infty} \big(\alpha \rho^k t + A_k(t) o(t)\big)$$
$$E\left[\int_0^{h(t)} \varphi_{s1}(v(t) - x, b) dF^*(x) + \int_{v(t)}^{\infty} \omega(u, x - u) dF^*(x)\right]. \tag{4.5}$$

Using Taylor's expansion, we have

$$M_1\big(h(t), ye^{-\beta t}; b\big) = M_1(u, y; b) + (ru + c_1)t \frac{\partial M_1(u, y; b)}{\partial u} - \beta y t \frac{\partial M_1(u, y; b)}{\partial y}$$
$$+ \frac{\sigma^2}{2} t \frac{\partial^2 M_1(u,y;b)}{\partial u^2} + o(t).$$

Substituting the above expression into (4.5), dividing both side by t and letting $t \to 0$, we can get (4.2). When u ≥ b, similarly, we can obtain (4.3).

For simplicity, we denote

$$V_n(u, b) = \begin{cases} V_{n1}(u, b), & if\ 0 \le u < b \\ V_{n2}(u, b), & if\ \quad u \ge b \end{cases}. \tag{4.6}$$

Theorem 4.2 For $0 \le u < b$, we can get

$$\frac{\sigma^2}{2} V_{n1}''(u, b) + (ru + c_1) V_{n1}'(u, b) = (\lambda + n\beta) V_{n1}(u, b) - \lambda \int_0^u V_{n1}(u - x, b) f_\rho(x) dx. \tag{4.7}$$

when $u \ge b$, we have

$$\frac{\sigma^2}{2} V_{n2}''(u, b) + (ru + c_2) V_{n2}'(u, b) = (\lambda + n\beta) V_{n2}(u, b) - \lambda \int_0^u V_{n2}(u - x, b) f_\rho(x) dx$$
$$+\lambda \int_0^u V_{n1}(u - x, b) f_\rho(x) dx \tag{4.8}$$

with boundary conditions

$$V_{n1}(0, b) = 0,$$
$$\lim_{u\to\infty} V_{n2}(u, b) = \left(\frac{c_1 - c_2}{\beta}\right)^n,$$

$$V_{n1}(b^-,b) = V_{n2}(b,b),$$
$$\frac{\partial V_{n1}}{\partial u}(u,b)|_{u=b^-} = \frac{\partial V_{n2}}{\partial u}(u,b)|_{u=b} \ . \qquad (4.9)$$

5 Conclusion

In the context of a perturbed Poisson-Geometic risk model, we established the model with constant interest and a threshold dividend strategy. Based on it, the integro-differential equations with boundary conditions for the Gerber-Shiu function are obtained by Taylor expansion and the property of surplus process. Especially, when claim size has exponential distribution, the closed form expression for ruin probability is provided. Finally, in a similar way, we could derive Integro-differential equations with certain boundary for the moment- generation function of the present value of total dividends until ruin.

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