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# Existence of periodic solutions for second order delay differential equations with a singularity of repulsive type

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**Abstract:** In this paper, the problem of existence of periodic solution is studied for the second order delay differential equation with a singularity of repulsive type

$$x''(t)+f(x(t))x'(t)+\varphi(t)x(t-\tau_1)-g(x(t-\tau_2))=h(t),$$

where $\tau_1$ and $\tau_2$ are constants, $g(x)$ is singular at $x=0$, $\varphi$ and $h$ are $T-$periodic functions. By using a continuation theorem of coincidence degree theory, a new result on the existence of positive periodic solutions is obtained. The interesting is that the sign of function$\varphi(t)$ is allowed to change fort $\in[0,T]$.

**Keywords:** Liénard equation; Continuation theorem; Singularity; Periodic solution.

## 1. Introduction

The aim of this paper is to search for positive $T-$periodic solutions for second order delay differential equation with a singularity in the following form

$$x''(t)+f(x(t))x'(t)+\varphi(t)x(t-\tau_1)-g(x(t-\tau_2))=h(t), \tag{1.1}$$

where $\tau_1$ and $\tau_2$ are constants, $f:[0,\infty)\to R$ is an arbitrary continuous function, $g\in C((0,+\infty),(0,+\infty))$ and $g(x)$ is singular of repulsive type at $x=0$, i.e.,$g(x)\to+\infty$, as$x\to 0^+$, $\varphi,h:R\to R$ are $T-$periodic with function$h\in L^1([0,T],R)$, $\varphi\in C([0,T],R)$, while the sign of function $\varphi$ being changeable for $t\in[0,T]$.

In recent years, the problem of periodic solutions to the second order singular equation

$$x''(t)+f(x(t))x'(t)+\varphi(t)x(t-\tau_1)-\frac{b(t)}{x^\lambda(t)}=h(t), \tag{1.2}$$

where $f:[0,+\infty)\to R$is an arbitrary continuous function,$\varphi,b,h\in L^1[0,T]$ and $\lambda>0$, has been studied widely. This is due to the fact that the singular term possesses a significant role in many practical situations [1-11]. For example, the singular term in the equations models the restoring force caused by a compressed perfect gas (see [3-6] and the references therein). Lazer and Solimini in the pioneering paper[12] first used the method of topological degree theory, together with the technique of upper and lower solutions, to study the existence of periodic solution to Eq.(1.2) where $f(x)\equiv 0,\varphi(t)\equiv 0,b(t)\equiv 1$. They obtained that if $\lambda\geq 1$,a necessary and sufficient condition for existence of a positive periodic solution to Eq.(1.2) is that$\bar{h}:=\frac{1}{T}\int_0^T h(s)ds<0$. After that, the problem of periodic solutions for singular differential equations like Eq.(1.2) has attracted the attention of many researchers[13-19]. We notice that the condition of $\varphi(t)\geq 0$ for a.e. $t\in[0,T]$ is required in [16-19], since it is crucial for obtain the priori estimates over all the possible periodic solutions to the equations

$$x''(t)+\lambda f(x(t))x'(t)+\lambda\varphi(t)x(t-\tau_1)-\frac{\lambda b(t)}{x^\lambda(t)}=\lambda h(t),\lambda\in(0,1). \tag{1.3}$$

We only find [20,21] where the sign of $\varphi(t)$ is allowed to change. In [20,21], a priori bounds of all the possible periodic solutions to Eq.(1.3) are estimated by using the inequality

$$\int_0^T\frac{u''(t)}{u^\delta(t)}dt\geq 0, \tag{1.4}$$

where $\delta>0$is an arbitrary constant, $u(t)$ is a positive $T-$periodic function with $u\in C^2([0,T],R)$.

Motivated by this, in this paper, we study the existence of positive $T-$periodic solution for the equation (1.1). Since there is a delay $\tau_1$ in (1.1), generally, the inequality like (1.4) for

$$\delta=1$$

$$\int_0^T\frac{u''(t)}{u(t-\tau_1)}dt\geq 0.$$

may not hold. This means that the work to estimate a priori bounds of all the possible periodic solutions to the equations

$$x''(t)+\lambda f(x(t))x'(t)+\lambda\varphi(t)x(t-\tau_1)-\lambda g(x(t-\tau_2))=\lambda h(t),\lambda\in(0,1).$$

is more difficult than the corresponding ones associated to (1.3).

## 2. Preliminary lemmas

Throughout this paper, let $C_T=\{x\in C(R,R): x(t+T)=x(t)$ for all $t\in R\}$with the norm defined by$|x|_\infty=\max_{t\in[0,T]}|x(t)|$. For any $T$ −periodic solution $y(t)$ with$y\in L^1([0,T],R)$,$y_+(t)$ and$y_-(t)$ is denoted by $max\{y(t),0\}$ and $-min\{y(t),0\}$ respectively, and $\overline{y}=\frac{1}{T}\int_0^T y(s)ds$. Clearly, $y(t)=y_+(t)-y_-(t)$ for all$t\in R$,and $\overline{y}=\overline{y_+}-\overline{y_-}$.

The following Lemma is the consequence of Theorem 3.1 in [22].

**Lemma 2.1.** Assume that there exist positive constants $M_0,M_1$ and $M_2$ with $0<M_0<M_1$, such that the following conditions hold.

1.For each $\lambda\in(0,1]$, each possible positive $T$ −periodic solution $x$to the equation

$$u''(t)+\lambda f(u(t))u'(t)+\lambda\varphi(t)u(t-\tau_1)-\lambda g(u(t-\tau_2))=\lambda h(t),$$

satisfies the inequalities$M_0<x(t)<M_1$and$|x'(t)|<M_2$for all$t\in[0,T]$.

2.Each possible solution $c$ to the equation

$$g(c)-c\overline{\varphi}+\overline{h}=0,$$

satisfies the inequality$M_0<c<M_1$.

3.It holds

$$(g(M_0)-\overline{\varphi}M_0+\overline{h})(g(M_1)-\overline{\varphi}M_1+\overline{h})<0,$$

Then Eq.(1.1) has at least one $T$ −periodic solution $u$such that $M_0<u(t)<M_1$ for all$t\in[0,T]$.

**Lemma 2. 2.**[19] Let $x$ be a continuous $T$ −periodic continuous differential function. Then, for any $\tau\in(0,T]$,

$$\left(\int_0^T|x(s)|^2ds\right)^{\frac{1}{2}}\leq\frac{T}{\pi}\left(\int_0^T|x'(s)|^2ds\right)^{\frac{1}{2}}+\sqrt{T}|x(\tau)|.$$

In order to study the existence of positive periodic solutions to Eq.(1.1), we list the following assumptions.

$[H_1]$ The function $\varphi(t)$ satisfies the following conditions

$$\int_0^T\varphi_+(s)ds>0,\sigma:=\frac{\int_0^T\varphi_-(s)ds}{\int_0^T\varphi_+(s)ds}\in[0,1)\text{ and }\quad\sigma_1:\frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T\varphi_+(t)dt\right)^{\frac{1}{2}}\in(0,1);$$

$[H_2]$ there are constants$M>0$and$A>0$such that$g(x)\in(0,A)$ for all$x>M$;

$[H_3]$$\int_0^1 g(s)ds=+\infty$;

$[H_4]$ $\lim_{x\to0^+}g(x)=+\infty$.

**Remark 2.1.** It is noted that assumption$[H_4]$ can not be deduced from assumption $[H_3]$. For example, let$g(x)\frac{1}{x}|\sin\frac{1}{x}|$for all$x\in(0,+\infty)$,then assumption $[H_3]$ is satisfied. But, assumption $[H_4]$ does not hold.

**Remark 2.2.** If assumptions$[H_1]$-$[H_2]$and$[H_4]$ hold, then there are constants$D_1$and$D_2$with$0<D_1<D_2$ such that

$$g(x)-\overline{\varphi}x+\overline{h}>0\text{for all}x\in(0,D_1)$$

and

$$g(x)-\overline{\varphi}x+\overline{h}<0\text{ for all}x\in(D_2,+\infty)$$

Now, we suppose that assumptions$[H_1]$and$[H_2]$ hold, and embed Eq.(1.1) into the following equations family with a parameter$\lambda\in(0,1)$

$$x''(t)+\lambda f(x(t))x'(t)+\lambda\varphi(t)x(t-\tau_1)-\lambda g(x(t-\tau_2))=\lambda h(t),\lambda\in(0,1]. \tag{2.1}$$

Let

$$\Omega=\{x\in C_T: x''(t)+\lambda f(x(t))x'(t)+\lambda\varphi(t)x(t-\tau_1)-\lambda g(x(t-\tau_2))=\lambda h(t),\lambda\in(0,1],\ x(t)>0,\forall t\in[0,T]\},$$

and

$$M_0 = \frac{T^{\frac{1}{2}}}{1-\sigma}\left[\frac{T^{\frac{3}{4}}(\overline{h_-})^{\frac{1}{2}}}{(1-\sigma_1)(1-\sigma)^{\frac{1}{2}}} + \left(\frac{A_0}{1-\sigma_1}\right)^{\frac{1}{2}}\right]^2 + \frac{max\{A+\bar{h},0\}}{(1-\sigma)\overline{\varphi_+}}, \tag{2.2}$$

where

$$A_0 = T^{\frac{1}{2}}(\overline{\varphi_+})^{\frac{1}{2}}\frac{A+|\bar{h}|}{(1-\sigma)\overline{\varphi_+}} + \left(\frac{T^{\frac{1}{2}}\overline{h_-}\,max\{A+\bar{h},0\}}{(1-\sigma)\overline{\varphi_+}}\right)^{\frac{1}{2}},$$

$A$is a constant determined by assumption $[H_2]$ Clearly,$M_0$and$A_0$are all independent of$(\lambda, x) \in (0,1] \times \Omega$, and there is a positive integer $k_0$such that

$$k_0 M \geq M_0, \tag{2.3}$$

where$M$is a constant determined by assumption$[H_2]$.

**Lemma 2.3.** Assume that assumptions$[H_1]$-$[H_2]$ hold, then there is an integer$k^* > k_0$ such that for each function$u \in \Omega$, there is a point$t_0 \in [0,T]$ satisfying

$$u(t_0) \leq k^* M.$$

**Proof:** If the conclusion does not hold, then for each$k > k_0$, there is a function$u_k \in \Omega$satisfying

$$u_k(t) > kM \text{for all} t \in [0,T]. \tag{2.4}$$

From the definition of$\Omega$, we see

$$u_k'' + \lambda f(u_k)u_k' + \lambda\varphi(t)u_k(t-\tau_1) - \lambda g(u_k(t-\tau_2)) = \lambda h(t), \lambda \in (0,1], \tag{2.5}$$

and by using assumption$[H_2]$,

$$0 < g(u_k(t)) < A, \text{ for all} \tag{2.6}$$

By integrating (2.5) over the interval [0,T], we have

$$\int_0^T \varphi(t)u_k(t-\tau_1)dt = \int_0^T g(u_k(t-\tau_2))dt + \int_0^T h(t)dt,$$

i.e.,

$$\int_0^T \varphi_+(t)u_k(t-\tau_1)dt = \int_0^T \varphi_-(t)u_k(t-\tau_1)dt + \int_0^T g(u_k(t-\tau_2))dt + \int_0^T h(t)dt,$$

Since$\varphi_+(t) \geq 0$and$\varphi_-(t) \geq 0$for all$t \in [0,T]$, it follows from the integral mean value theorem that there are two points$\xi, \eta \in R$such that

$$u_k(\xi)T\overline{\varphi_+} = T\overline{\varphi_-}u_k(\eta) + \int_0^T g(u_k(t))dt + T\bar{h}$$

$$\leq T\overline{\varphi_-}|u_k|_\infty + \int_0^T g(u_k(t))dt + T\bar{h},$$

which together with (2.6) yields

$$u_k(\xi)T\overline{\varphi_+} < T\overline{\varphi_-}|u_k|_\infty + TA + T\bar{h},$$

i.e.,

$$u_k(\xi) < \sigma|u_k|_\infty + \frac{A+\bar{h}}{\overline{\varphi_+}}. \tag{2.7}$$

In view of the inequality

$$|u_k|_\infty \leq u_k(\xi) + T^{\frac{1}{2}}\left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}},$$

it follows from (2.7) and the condition of$\sigma \in [0,1)$, which is determined in assumption $[H_1]$, that

$$|u_k|_\infty \leq \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} + \frac{max\{A+\bar{h},0\}}{(1-\sigma)\overline{\varphi_+}}. \tag{2.8}$$

On the other hand, by multiplying (2.5) with $u_k(t)$, and integrating it over the interval [0,T], we obtain

$$\int_0^T |u_k'(t)|^2 dt = -\lambda\int_0^T g(u_k(t-\tau_2))u_k(t)dt + \lambda\int_0^T \varphi(t)u_k(t-\tau_1)u_k(t)dt - \lambda\int_0^T h(t)u_k(t)dt,$$

which together with the fact of $g(x) > 0$ for all $x > 0$ gives

$$\int_0^T |u_k'(t)|^2 dt < \lambda\int_0^T \varphi_+(t)u_k(t-\tau_1)u_k(t)dt + \lambda\int_0^T h_-(t)u_k(t)dt,$$

$$\leq T\overline{\varphi_+}|u_k|_\infty^2 + T\overline{h_-}|u_k|_\infty,$$

i.e.,

$$\left(\int_0^T |u_k'(t)|^2 dt\right)^{\frac{1}{2}} < (T\overline{\varphi_+})^{\frac{1}{2}}|u_k|_\infty + \left(T\overline{h_-}\right)^{\frac{1}{2}}|u_k|_\infty^{\frac{1}{2}}. \tag{2.9}$$

Substituting (2.8) into the above formula,

$$\left(\int_0^T |u_k'(t)|^2 dt\right)^{\frac{1}{2}} < (T\overline{\varphi_+})^{\frac{1}{2}}\left[\frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} + \frac{max\{A+\overline{h},0\}}{(1-\sigma)\overline{\varphi_+}}\right] +$$

$$\left(T\overline{h_-}\right)^{\frac{1}{2}}\left[\frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} + \frac{max\{A+\overline{h},0\}}{(1-\sigma)\overline{\varphi_+}}\right]^{\frac{1}{2}}$$

$$= \sigma_1\left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} + \frac{T^{\frac{3}{4}}\left(\overline{h_-}\right)^{\frac{1}{2}}}{(1-\sigma)^{\frac{1}{2}}}\left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{4}} + A_0,$$

where$\sigma_1 = \frac{T^{\frac{1}{2}}}{(1-\sigma)}\left(\int_0^T \varphi_+(t)dt\right)^{\frac{1}{2}} \in (0,1)$, which is determined by assumption $[H_1]$, and

$$A_0 = (T\overline{\varphi_+})^{\frac{1}{2}}\frac{max\{A+\overline{h},0\}}{(1-\sigma)\overline{\varphi_+}} + \left(\frac{T\overline{h}\,max\{A+\overline{h},0\}}{(1-\sigma)\overline{\varphi_+}}\right)^{\frac{1}{2}},$$

and then

$$\left(\int_0^T |u_k'(t)|^2 dt\right)^{\frac{1}{2}} < \frac{T^{\frac{3}{4}}\left(\overline{h_-}\right)^{\frac{1}{2}}}{(1-\sigma)(1-\sigma)^{\frac{1}{2}}}\left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{4}} + \frac{A_0}{1-\sigma_1},$$

which results in

$$\left(\int_0^T |u_k'(t)|^2 dt\right)^{\frac{1}{4}} < \frac{T^{\frac{3}{4}}\left(\overline{h_-}\right)^{\frac{1}{2}}}{(1-\sigma)(1-\sigma)^{\frac{1}{2}}} + \left(\frac{A_0}{1-\sigma_1}\right)^{\frac{1}{2}}. \tag{2.10}$$

Substituting (2.10) into (2.8), we have

$$|u_k|_\infty < M_0,$$

where $M_0$ is determined by (2.2). This is

$$u_k(t) < M_0 \text{for all} t \in [0,T]. \tag{2.11}$$

By the definition of$k_0$, we see from (2.3) that (2.11) contradicts to (2.4). This contradiction implies that the conclusion of Lemma 2.3is true.

# 3. Main results

**Theorem 3.1.** Assume that $[H_1]$-[ $H_4$] hold. Then Eq.(1.1) has at least one positive $T$ −periodic solution.

**Proof.** Firstly, we will show that there exist$M_1, M_2$with $M_1 > k^*M$ and $M_2 > 0$such that each positive $T$ −periodic solution $u(t)$ of Eq.(2.1) satisfies the inequalities

$$u(t) < M_1, |u'(t)| < M_2, \text{ for all } t \in [0,T]. \tag{3.1}$$

In fact, if $u$is an arbitrary positive $T$ −periodic solution of Eq.(2.1), then

$$u''(t) + \lambda f(u(t))u'(t) + \lambda\varphi(t)u(t-\tau_1) - \lambda g(u(t-\tau_2)) = \lambda h(t), \lambda \in (0,1]. \tag{3.2}$$

This implies $u \in \Omega$. So by using Lemma 2.2 that there is a point$t_0 \in [0,T]$ such that

$$u(t_0) \le k^*M, \tag{3.3}$$

and then

$$|u_k|_\infty \le u_k(\xi) + T^{\frac{1}{2}}\left(\int_0^T |u_k'(s)|^2 ds\right)^{\frac{1}{2}} \tag{3.4}$$

Integrating (3.2) over the interval [0,T], we have

$$\int_0^T \varphi(t)u(t-\tau_1)dt - \int_0^T g(u(t-\tau_2))dt = \int_0^T h(t)dt. \tag{3.5}$$

By assumption $[H_4]$, we see from (3.5) that there is a point $t_1 \in [0,T]$ such that

$$u(t_1) \ge \gamma, \tag{3.6}$$

where $\gamma < k^*M$is a positive constant, which is independent of$\lambda \in (0,1]$. Similar to the proof of (2.9), we have

$$\left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} < (T\overline{\varphi_+})^{\frac{1}{2}}|u|_\infty + \left(T\overline{h_-}\right)^{\frac{1}{2}}|u|_\infty^{\frac{1}{2}}. \tag{3.7}$$

Substituting (3.4) into (3.7), we have

$$\left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} \le (T\overline{\varphi_+})^{\frac{1}{2}}\left[k^*M + T^{\frac{1}{2}}\left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{2}}\right] +$$

$$\left(T\overline{h_-}\right)^{\frac{1}{2}}\left[k^*M + T^{\frac{1}{2}}\left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\le (T\overline{\varphi_+})^{\frac{1}{2}}\left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{2}} + T^{\frac{3}{4}}\left(\overline{h_-}\right)^{\frac{1}{2}}\left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{4}} +$$

$$k^*M(T\overline{\varphi_+})^{\frac{1}{2}} + \left(Tk^*M\overline{h_-}\right)^{\frac{1}{2}},$$

which results in

$$\left[1 - T(\overline{\varphi_+})^{\frac{1}{2}}\right]\left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}}$$

$$\le T^{\frac{3}{4}}\left(\overline{h_-}\right)^{\frac{1}{2}}\left(\int_0^T |u'(s)|^2 ds\right)^{\frac{1}{4}} + k^*M(T\overline{\varphi_+})^{\frac{1}{2}} + \left(Tk^*M\overline{h_-}\right)^{\frac{1}{2}}. \quad (3.8)$$

Since

$$(T\overline{\varphi_+})^{\frac{1}{2}} = T^{\frac{1}{2}}\left(\int_0^T \varphi_+(s) ds\right)^{\frac{1}{2}} < \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T \varphi_+(s) ds\right)^{\frac{1}{2}},$$

it follows from assumption $[H_2]$ that

$$1 - T(\overline{\varphi_+})^{\frac{1}{2}} > 0,$$

which together (3.8) yields that there is a constant $\rho > 0$, which is independent of$\lambda \in (0,1]$, such that

$$\left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}} < \rho,$$

and then by (3.4), we have

$$u(t) < k^*M + T^{\frac{1}{2}}\rho :\ = M_1, \text{ for all } t \in [0,T]. \quad (3.9)$$

Now, if $u$attains its maximum over [0,T] at $t_2 \in [0,T]$, then $u'(t_2) = 0$ and we deduce from (3.2) that

$$u'(t) = \lambda\int_{t_2}^{t} [-f(u)u' - \varphi(t)u(t-\tau_1) + g(u(t-\tau_2)) + h(t)]dt$$

For all $t \in [t_2, t_2+T]$. Thus, if $F' = f$, then

$$|u'(t)| \le \lambda|F(u(t)) - F(u(t_2))| + \lambda\int_{t_2}^{t_2+T} |\varphi(s)|u(s-\tau_1))ds + \lambda\int_{t_2}^{t_2+T} g(u(t-\tau_2))dt +$$

$$\lambda\int_{t_2}^{t_2+T} |h(s)|ds$$

$$\le 2\lambda \max_{0\le u\le M_1}|F(u)| + \lambda\int_0^T g(u(s))ds + \lambda T\overline{|\varphi|}\,|u|_\infty + \lambda\overline{|h|}. \quad (3.10)$$

From (3.5), we see that

$$\int_0^T g(u(s))ds = \int_0^T \varphi(t)u(t-\tau_1)dt - T\bar{h}$$

$$\le T\overline{\varphi_+}|u|_\infty + T\overline{h_-}.$$

It follows from (3.10) that

$$|u'(t)| \le 2\lambda\left(\max_{0\le u\le M_1}|F(u)| + \lambda T\overline{|\varphi|}|u|_\infty + \overline{|h|}\right)$$

$$< 2\lambda\left(\max_{0\le u\le M_1}|F(u)| + M_1T\overline{|\varphi|} + T\overline{|h|}\right)$$

$$:= \lambda M_2, \text{ for all } t \in [0,T], \quad (3.11)$$

and then

$$|u'(t)| < M_2, \text{ for all } t \in [0,T]. \quad (3.12)$$

(3.10) and (3.12) imply that (3.1) holds.

Below, we will show that then there exists a constant $\gamma_0 \in (0,\gamma)$, such that each positive $T$ −periodic solution of Eq.(2.1) satisfies

$$u(t) > \gamma_0, \text{for all } t \in [0,T]. \quad (3.13)$$

Suppose that $u(t)$ is an arbitrary positive $T$ −periodic solution of Eq.(2.1), then

$$u''(t) + \lambda f(u)u' + \lambda\varphi(t)u(t-\tau_1) - \lambda g(u(t-\tau_2)) = \lambda h(t), \lambda \in (0,1]. \quad (3.14)$$

Let $t_1$ be determined in (3.6). Multiply (3.14) by $u'(t)$ and integrating it over the interval $[t_1+\tau_1, t+\tau_1]$ (or $[t+\tau_1, t_1+\tau_1]$) , we get

$$\frac{|u'(t+\tau_1)|^2}{2} - \frac{|u'(t_1+\tau_1)|^2}{2} + \lambda\int_{t_1+\tau_1}^{t+\tau_1} f(u(s))(u'(s))^2 ds$$

$$= \lambda\int_{t_1+\tau_1}^{t+\tau_1} g(u(s-\tau_2))(u'(s))ds - \lambda\int_{t_1+\tau_1}^{t+\tau_1}\varphi(s)u(s-\tau_1)(u'(s))ds\,\lambda\int_{t_1+\tau_1}^{t+\tau_1} h(t)u'(s)ds$$

$$= \lambda\int_{t_1+\tau_1}^{t+\tau_1} g(u(s-\tau_2))(u'(s-\tau_2))ds + \lambda\int_{t_1+\tau_1}^{t+\tau_1} g(u(s-\tau_2))[u'(s) - u'(s-\tau_2)]ds -$$

$$\lambda\int_{t_1+\tau_1}^{t+\tau_1}\varphi(s)u(s-\tau_1)(u'(s))ds + \lambda\int_{t_1+\tau_1}^{t+\tau_1} h(t)u'(s)ds$$

$$= \lambda\int_{t_1+\tau_1}^{t+\tau_1} g(u(s))u'(s)ds + \lambda\int_{t_1+\tau_1}^{t+\tau_1} g(u(s-\tau_2))[u'(s) - u'(s-\tau_2)]ds -$$

$$\lambda\int_{t_1+\tau_1}^{t+\tau_1}\varphi(s)u(s-\tau_1)(u'(s))ds + \lambda\int_{t_1+\tau_1}^{t+\tau_1} h(t)u'(s)ds,$$

which yields the estimate

$$\lambda\left|\int_{u(t)}^{u(t_1)} g(s)ds\right| \le \frac{|u'(t+\tau_1)|^2}{2} + \frac{|u'(t_1+\tau_1)|^2}{2} + \lambda\int_0^T |f(u)|(u')^2 dt +$$

$$\lambda\int_0^T |\varphi(t)uu'|\,dt + \lambda\int_0^T |h(t)u'|dt.$$

From (3.10) and (3.11), we get

$$\lambda\left|\int_{u(t)}^{u(t_1)} g(s)ds\right| \le \lambda M_2^2 + \lambda\max_{0\le u\le M_1}|f(u)|TM_2^2 + \lambda M_1M_2T\overline{|\varphi|} + \lambda M_2T\overline{|h|}$$

which gives

$$\left|\int_{u(t)}^{u(t_1)} g(s)ds\right| \le M_3, \text{ for all } t \in [t_1, t_1+T] \quad (3.15)$$

with

$$M_3 = M_2^2 + \max_{0\le u\le M_1}|f(u)|TM_2^2 + M_1M_2T\overline{|\varphi|} + M_2T\overline{|h|}.$$

From $[H_3]$ there exists $\gamma_0 \in (0,\gamma)$ such that

$$\int_\eta^\gamma g_2(u)du > M_3, \text{ for all } \eta \in (0,\gamma_0] \quad (3.16)$$

Therefore, if there is a $t^* \in [t_1, t_1+T]$ such that $u(t^*) \le \gamma_0$, then from (3.16) we get

$$\int_{u(t^*)}^{\gamma} g(s)ds > M_3,$$

and then

$$\int_{u(t^*)}^{u(t_1)} g(s)ds > \int_{u(t^*)}^{\gamma} g(s)ds > M_3,$$

which contradicts (3.15). This contradiction gives that $u(t) > \gamma_0$ for all $t \in [0,T]$. So (3.13) holds.

Let $m_0 \in min\{D_1,\gamma_0\}$ and $m_1 \in (M_1 + D_2, +\infty)$ be two constants, then from(3.1) and (3.12), we see that each possible positive $T$ −periodic solution $u$ satisfies

$$m_0 < u(t) < m_1, |u'(t)| < M_2$$

This implies that condition 1 and condition 2 of Lemma 2.1 are satisfied. Also, we can deduce from Remark 2.2 that

$$g(c) - \overline{\varphi}c + \overline{h} > 0, \text{ for } c \in (0, m_0]$$

and

$$g(c) - \overline{\varphi}c + \overline{h} < 0, \text{ for } c \in [m_1, +\infty)$$

which results in

$$(g(m_0) - \overline{\varphi}m_0 + \overline{h})(g(M_1) - \overline{\varphi}M_1 + \overline{h}) < 0.$$

So condition 3 of Lemma 2.1 holds. By using Lemma 2.1, we see that Eq.(1.1) has at least one positive $T$ −periodic solution. The proof is complete.

**Example 3.1:** Considering the following equation

$$x''(t) + f(x(t))x'(t) + a(1 + 2\sin t)x(t - \tau_1) - \frac{1}{x^2(t-\tau_2)} = \cos t, \tag{3.17}$$

where $f$ is an arbitrary continuous function, $\tau_1, \tau_2 \in [0, +\infty)$ and $a \in (0, +\infty)$ are constants. Corresponding to Eq.(1.1), we have $g(u) = \frac{1}{u^2}$, $\varphi(t) = a(1 + 2\sin t)$ and $h(t) = \cos t$.By simple calculating, we can verify that assumptions $[H_2]$-$[H_4]$ are all satisfied. Furthermore,

$$\int_0^T \varphi_+(t)dt = (\frac{4\pi}{3} + 2\sqrt{3})a, \int_0^T \varphi_-(t)dt = (2\sqrt{3} - \frac{2\pi}{3})a,$$

and then

$$\sigma := \frac{\int_0^T \varphi_-(s)ds}{\int_0^T \varphi_+(s)ds} = \frac{2\sqrt{3} - \frac{2\pi}{3}}{\frac{4\pi}{3} + 2\sqrt{3}} \in (0,1)$$

and

$$\sigma_1 : \frac{T^{\frac{1}{2}}}{1-\sigma}(\int_0^T \varphi_+(t)dt)^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}(\frac{4\pi}{3} + 2\sqrt{3})^{\frac{3}{2}}.$$

If

$$a < \frac{2\pi}{(\frac{4\pi}{3} + 2\sqrt{3})^3},$$

then $\sigma_1 \in (0,1)$, this implies that assumption $[H_1]$ holds. Thus, by using Theorem 3.1, we have that Eq.(3.17) has at least one positive $2\pi$ −periodic solution.

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## 5. References

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