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On solving fuzzy matrix games via linear programming approach

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Abstract In this paper, a two person zero- sum matrix game with $L-R$ fuzzy numbers payoff is introduced. Using the fuzzy number comparison introduced by Rouben's method, 1996, the fuzzy payoff is converted into the corresponding crisp payoff. Then, For each player, a linear programming problem is formulated. Also, a solution procedure for solving each problem is proposed. Finally, a numerical example is given for illustration.

Keywords: Matrix games; $L-R$ fuzzy numbers payoff; Optimal fuzzy strategy

1. Introduction

Game theory is concerned with decision making problem where two or more autonomous decision makers have conflicting interests. They are usually referred to as players who act strategically to find out a compromise solution (Kumar (2016)) . Zero- sum games refer to pure conflict. The payoff of one player is the negative of the payoff of the other player. Peski (2008) compared the structure information in zero- sum games.

As known, fuzzy set theory was introduced by Zadeh (1965) to deal with fuzziness. Up to now, fuzzy set theory has been applied to broad fields. Fuzzy set theory introduced by Zadeh (1965) make a model has to be set up using data which is approximately known. Fuzzy numerical data can be represented by means of fuzzy subsets of the real line, known as fuzzy numbers). For the fuzzy set theory development, we may referee to the papers of Kaufmann(1975), and Dubois and Prade(1980), they extended the use of algebraic operations of real numbers to fuzzy numbers by the use a fuzzifaction principle. Fuzzy linear constraints with fuzzy numbers were studied by Dubois and Prade (1980). In real- world problems, uncertainties may be estimated as intervals, Shaocheng 1994 studied two kinds of linear programming problems with fuzzy numbers called: interval numbers and fuzzy number linear programming, respectively. Tanaka et al. 1984 have formulated and proposed a method for solving fuzzy coefficients linear programming. Bellman and Zadeh (1970) introduced the concept of a maximizing decision making problem. Zhao et al. (1992) introduced the complete solution set for the fuzzy linear programming problems using linear and nonlinear membership functions.

Campos (1989) solved fuzzy matrix game using fuzzy linear programming. Cevikel and A hlatcioglu (2010) introduced new concept of solution for multi- objective two person zero- sum games. Xu (1998) discussed two- person zero- sum game with grey number payoff matrix. Dhingra et al. (1995) introduced a new optimization method to herein as cooperative fuzzy games and also solved the multiple objective optimization problems based on a proposed computational technique. An innovative fuzzy logic approach to analyze $n-$ person cooperative games is proposed by Espin et al.(2007). Xu and Yao (2010) studied rough payoff matrix games. Ein- Dor and Kantar (2001) and Takahashi (2008) discussed two- person zero- sum games with random payoffs. Applications of game theory may be found in economics, engineering, biology, and in many other fields. Three major classes of games are matrix games, continuous static games, and differential games. In continuous static games, the decision possibilities need not be discrete, and the decisions and costs are related in a continuous rather than a discrete manner. The game is static in the sense that no time history is involved in the relationship between costs and decisions. Elshafei (2007) introduced

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an interactive approach for solving Nash Cooperative Continuous Static Games, and also determined the stability set of the first kind corresponding to the obtained compromise solution. Khalifa and ZeinEldein (2015) introduced an interactive approach for solving cooperative continuous static games with fuzzy parameters in the objective function coefficients. Navidi et al. (2014) presented a new game theoretic- based approach for multi response optimization problem. Osman et al. (2015) introduced a new procedure for continuous time open loop stackelberg differential game. Roy et al. (2000) solved linear multiobjective programming based on cooperative game approach.

The remainder of the paper is as: In section 2., some preliminaries need in the paper are presented. In section 3, a two person zero- sum matrix game with fuzzy payoff is defined. In section 4, a solution procedure for solving the problems is introduced. A numerical example is given for illustration in section5. Finally some concluding remarks are reported in section 6.

2. Preliminaries

In order discuss our problem conveniently, we introduce fuzzy numbers and some of the results of applying fuzzy arithmetic on them and also comparison of fuzzy numbers by Roubens' method (Kauffmann and Gupta(1988); Fttemps and Roubens (1996)).

Definition1. A fuzzy number $\tilde{a}$ is a mapping defined as:

$\mu_{\tilde{a}} : R \to [0, 1]$, with the following:

(i) $\mu_{\tilde{a}}(x)$ is an upper semi- continuous membership function;

(ii) $\tilde{a}$ is a convex fuzzy set, i. e., $\mu_{\tilde{a}}(\lambda x + (1-\lambda) y) \geq \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)\}$, for all $x, y \in R, 0 \leq \lambda \leq 1$;

(iii) $\tilde{a}$ is normal, i. e., $\exists x_0 \in R$ for which $\mu_{\tilde{a}}(x_0) = 1$;

(iv) Supp $(\tilde{a}) = \{x \in R : \mu_{\tilde{a}}(x) > 0\}$ is the support of the $\tilde{a}$, and its closure cl (supp $(\tilde{a})$) is compact set.

Definition2. The $\alpha -$ level set of the fuzzy number $\tilde{a}$, , denoted by $(\tilde{a})_\alpha$ and is defined as the ordinary set:

$$(\tilde{a})_\alpha = \begin{cases} \{x \in R : \mu_{\tilde{a}}(x) \geq \alpha, \ 0 < \alpha \leq 1 \\ cl(\sup p(\tilde{a})), \qquad \alpha = 0 \end{cases}$$

A function, usually denoted by " L " or " R ", is a reference function of a fuzzy number if and only if

1. $L(x) = L(-x)$,
2. $L(0) = 1$,
3. L is nonincreasing on $[0, -\infty[$.

A convenient representation of fuzzy numbers in the $L - R$ flat fuzzy number which is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} L((A^- - x)\eta), & if\ x \leq A^-, \eta > 0, \\ R((x - A^+)\beta), & if\ x \geq A^+,\ \beta > 0, \\ 1, & eleswhere \end{cases}$$

where, $A^- < A^+, [A^-, A^+]$ is the core of $\tilde{A}$, $\mu_{\tilde{A}}(x) = 1; \forall x \in [A^-, A^+], A^-, A^+$ are the lower and upper modal values of $\tilde{A}$, and $\eta > 0, \beta > 0$ are the left- hand and right- hand spreads (Roubens (1991)).

Remark1. A flat fuzzy number is denoted by $\tilde{A} = (A^-, A^+, \eta, \beta)_{LR}$

Among the various type of $L-R$ fuzzy numbers, trapezoidal fuzzy numbers, denoted by $\tilde{A}=(A^-,A^+,\eta,\beta)$, are the greatest importance (Roubens (1991)).

Let $\tilde{p}=(p^-,p^+,\eta,\beta)$, and $\tilde{q}=(q^-,q^+,\gamma,\delta)$ both trapezoidal fuzzy numbers, the formulas for the addition, subtraction, and scalar multiplication are as follow:

- **Addition:** $\tilde{p}\oplus\tilde{q}=(p^-+q^-,p^++q^+,\eta+\gamma,\beta+\delta).$
- **Subtraction:** $\tilde{p}(-)\tilde{q}=(p^--q^+,p^+-q^-,\eta+\delta,\beta+\gamma).$
- **Scalar multiplication:**

$$x>0,\quad x\in R: x\otimes\tilde{p}=(xp^-,xp^+,x\eta,x\beta),$$

$$x<0,\quad x\in R: x\otimes\tilde{p}=(xp^+,xp^-,-x\beta,-x\eta).$$

The main concept of comparison of fuzzy numbers is based on the compensation of areas determined by the membership functions(Baldwin and Guild (1979); Nakamura (1986)).

Let $\tilde{p},\tilde{q}$ be fuzzy and numbers and $S_L(\tilde{p},\tilde{q}),S_R(\tilde{p},\tilde{q})$ be the areas determined by their membership functions according to

$$S_L(\tilde{p},\tilde{q})=\int_{I(\tilde{p},\tilde{q})}(\inf\tilde{p}_\alpha-\inf\tilde{q}_\alpha)\,d\alpha,$$

and

$$S_R(\tilde{p},\tilde{q})=\int_{S(\tilde{p},\tilde{q})}(\sup\tilde{p}_\alpha-\sup\tilde{q}_\alpha)\,d\alpha,$$

where $I(\tilde{p},\tilde{q})=\{\alpha:\inf\tilde{p}_\alpha\geq\inf\tilde{q}_\alpha\}\subset[\theta,1],\theta>0$, and $S(\tilde{p},\tilde{q})=\{\alpha:\sup\tilde{p}_\alpha\geq\sup\tilde{q}_\alpha\}\subset[\theta,1],\theta>0$.

The degree to which $\tilde{p}\geq\tilde{q}$ is defined (Roubens (1991)) as

$$C(\tilde{p},\tilde{q})=S_L(\tilde{p},\tilde{q})-S_L(\tilde{q},\tilde{p})+S_R(\tilde{p},\tilde{q})-S_R(\tilde{q},\tilde{p}).$$

Here, let us consider that $\tilde{p}\geq\tilde{q}$ when $C(\tilde{p},\tilde{q})\geq 0.$

Proposition 1. (Roubens(1991)). Let $\tilde{p}$, and $\tilde{q}$ be $L-R$ fuzzy numbers with parameters $(p^-,p^+,\eta,\beta),(q^-,q^+,\gamma,\delta)$ and reference functions $(L_{\tilde{p}},R_{\tilde{p}}),(L_{\tilde{q}},R_{\tilde{q}})$, where all reference functions are invertible. Then $\tilde{p}\geq\tilde{q}$ if and only if

$$\sup\tilde{p}_{\alpha_{\tilde{p},R}}+\inf\tilde{p}_{\alpha_{\tilde{p},L}}\geq\sup\tilde{q}_{\alpha_{\tilde{q},R}}+\inf\tilde{q}_{\alpha_{\tilde{q},L}}.$$

If $k=\tilde{p}\otimes\tilde{q}$. Then

$$\alpha_{k.R}=R_k\left(\int_0^1 R_k^{-1}(\alpha)\,d\alpha\right),\quad \alpha_{k.L}=L_k\left(\int_0^1 L_k^{-1}(\alpha)\,d\alpha\right).$$

Remark 2. $\tilde{p}\geq\tilde{q}$ if and only if

$$p^-+p^++\frac{1}{2}(\beta-\eta)\geq q^-+q^++\frac{1}{2}(\delta-\gamma). \tag{1}$$

Notation. The associated real number p corresponding to $\tilde{p}=(p^-,p^+,\eta,\beta)_{LR}$ is $\hat{p}=p^-+p^++\frac{1}{2}(\beta-\eta).$

Let $F(R)$ be the set of all trapezoidal fuzzy numbers.

3. Problem formulation and solution concepts

Consider the following fuzzy matrix game

$$\text{Player } I \quad \overset{\text{Player } II}{\begin{pmatrix} \tilde{p}_{11} & \tilde{p}_{12}\cdots & \tilde{p}_{1j}\ldots & \tilde{p}_{1m} \\ .. & .. & .. & .. \\ \tilde{p}_{i1} & \tilde{p}_{i2}\cdots & \tilde{p}_{ij}\cdots & \tilde{p}_{im} \\ .. & .. & .. & .. \\ \tilde{p}_{n1} & \tilde{p}_{n2} & \tilde{p}_{nj}\cdots & \tilde{p}_{nm} \end{pmatrix}} \tag{2}$$

Players I, and II have mixed strategies denoted by M_{S_I}, and $M_{S_{ii}}$, respectively and are defined as

$$M_{S_I} = \left\{ x \in R^{n:} : x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}, \text{ and} \tag{3}$$

$$M_{S_{ii}} = \left\{ y \in R^{m:} : y_j \geq 0, \sum_{j=1}^{m} y_j = 1 \right\} \tag{4}$$

The fuzzy mathematical expectation for player I is

$$\tilde{Z} = \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{p}_{ij}\, x_i y_j,$$

and for player II is

$$\tilde{Z} = \sum_{j=1}^{m} \sum_{i=1}^{n} \tilde{p}_{ij}\, x_i y_j$$

Remark 3. It is clear that the two mathematical expectation are the same since the sums are finite.

Definition3. In one two- person zero- sum game, player I 's mixed strategy $x^{\bullet}$ player II 's mixed strategy $y^{\bullet}$ are said to be optimal fuzzy strategies if $x^T \tilde{P}\, y^{\bullet} \leq x^{\bullet T} \tilde{P}\, y^{\bullet} \leq x^{\bullet T} \tilde{P}\, y$ for any mixed strategies x and y.

Remark 4. The optimal fuzzy strategy of player I is the strategy which maximizes Z irrespective of II ' s strategy. Also, the optimal fuzzy strategy of player II is the strategy which minimizes Z irrespective of I ' s strategy.

Let us consider the game with fuzzy payoff matrix (2), and the mixed strategies of players I, and II defined in (3) and (4), respectively. If E is the fuzzy optimum value of the game of a player II, then the linear programming model for player II becomes

$$\begin{aligned} &\min \mathrm{E} \\ &\text{Subject to} \\ &\sum_{j=1}^{m} \tilde{p}_{ij} y_j \leq E;\; y_j \geq \tilde{0},\; j = 1,2,\ldots,m. \end{aligned} \tag{5}$$

Putting $y_j^{'} = y_j / \mathrm{E}$, . Then problem(5) becomes

$$\begin{aligned} &\max_j \left(\sum_{j=1}^{m} y_j^{'} \right) \\ &\text{Subject to} \\ &\sum_{j=1}^{m} \tilde{p}_{ij} y_j^{'} \leq \tilde{1}; \\ &y_j^{'} \geq \tilde{0}, \forall j \end{aligned} \tag{6}$$

Similarly, the linear programming model for player I is as

$$\max \Phi$$

Subject to (7)

$$\sum_{i=1}^{n} \tilde{p}_{ij} x_i \geq \Phi;$$

$$x_i \geq \tilde{0}, i = 1,2,...,n.$$

Putting $x_i^{'} = x_i / \Phi, i = 1,2,...,n$. Then problem(7) becomes

$$\min_i \left(\sum_{i=1}^{n} x_i^{'} \right)$$

Subject to (8)

$$\sum_{i=1}^{n} \tilde{p}_{ij} x_i^{'} \geq \tilde{1};$$

$$x_i^{'} \geq \tilde{0}; \forall i.$$

Where, $\tilde{p}_{ij} = (p_{ij}^{-}, p_{ij}^{+}, \alpha_{ij}, \beta_{ij})_{LR} \in F(R)$.

4. Solution procedure

In this section, a solution procedure for solving the problem under study is introduced as in the following steps:

Step1: Translate the payoff matrix (2) into the corresponding problem (6) into problem(8)

Step2: Based on the operations of $L-R$ fuzzy numbers, problem(6), and problem(8) are converted into the corresponding crisp models as:

Model 1: $$\max_j \left(\sum_{j=1}^{m} y_j^{'} \right)$$

Subject to

$$\sum_{j=1}^{m} p_{ij} y_j^{'} \leq 1;$$

$$y_j^{'} \geq 0, \forall j$$

And,

Model 2: $$\min_i \left(\sum_{i=1}^{n} x_i^{'} \right)$$

Subject to

$$\sum_{i=1}^{n} p_{ij} x_i^{'} \geq 1;$$

$$x_i^{'} \geq 0; \forall i.$$

Step3: Apply the simplex method or any software (Lingo) to solve the Model1,and Model2 to obtain the optimal strategies for players II, and I , respectively.

5. Numerical example

Consider the following $L-R$ fuzzy numbers payoff matrix game as

$$\tilde{P} = \left(\tilde{p}_{ij}\right)_{3\times3} = Player\,I \overset{Player\,II}{\begin{pmatrix} (4,5,3,1) & (2,4,1,1\ ) & (4,5,1,3) & (5,7,1,1) \\ (10,12,5,5) & (9,13,1,5) & (7,10,2,4) & (10,11,3,1) \\ (0,2,1,1) & (2,3,3,1) & (17,21,9,9) & (6,7,1,3) \end{pmatrix}}$$

Referring to the previous notation the above payoff matrix can be reduced to the corresponding associated ordinary payoff as:

$$P = \left(p_{ij}\right)_{3\times 3} = Player\,I \overset{Player\,II}{\begin{pmatrix} 8 & 6 & 10 & 12 \\ 22 & 12 & 18 & 20 \\ 2 & 4 & 38 & 14 \end{pmatrix}}$$

For simplicity, let us take 2 as common factor . The optimal strategies will be the same and the game value is just multiply by 2. The new matrix game is

$$P = \left(p_{ij}\right)_{3\times 3} = Player\,I \overset{Player\,II}{\begin{pmatrix} 4 & 3 & 5 & 6 \\ 11 & 6 & 9 & 10 \\ 1 & 2 & 19 & 7 \end{pmatrix}}$$

According to Model1, we have

$$\max\left(y_1^1 + y_2^{'} + y_3^{'} + y_4^{'}\right)$$

Subject to

$$\begin{aligned} &4y_1^1 + 3y_2^{'} + 5y_3^{'} + 6y_4^{'} \le 1, \\ &11y_1^1 + 6y_2^{'} + 9y_3^{'} + 10y_4^{'} \le 1, \\ &y_1^1 + 2y_2^{'} + 19y_3^{'} + 7y_4^{'} \le 1, \\ &y_1^1, y_2^{'}, y_3^{'}, y_4^{'} \ge 0. \end{aligned}$$

Using the simplex method (Phase2), we get

Table1. The optimal strategy of player I

Variables	Optimal strategy	Game value
$y_1^{'}$	0	2
$y_2^{'}$	0	
$y_3^{'}$	0.231	
$y_4^{'}$	0.769	

Referring to Model2, we have

$$\min\left(x_1^1 + x_2^{'} + x_3^{'}\right)$$

Subject to

$$\begin{aligned} &4x_1^1 + 11x_2^{'} + x_3^{'} \ge 1, \\ &3x_1^1 + 6x_2^{'} + 2x_3^{'} \ge 1, \\ &5x_1^1 + 9x_2^{'} + 19x_3^{'} \ge 1, \\ &6x_1^1 + 10x_2^{'} + 7x_3^{'} \ge 1; x_1^1, x_2^{'}, x_3^{'} \ge 0. \end{aligned}$$

It is clear from the duality theorem that the optimal strategy of player I is $\left(x_1^1, x_2^{'}, x_3^{'}\right) = \left(0, 0.923, 0.077\right)$, which are the coefficients slack variables in the final table of the simplex method .

Thus,

Table2. The optimal fuzzy strategy

Player I	Player II	Game value
$x_1^{'} = 0$ $x_2^{'} = 0.923$ $x_3^{'} = 0.077$	$y_1^{'} = 0$ $y_2^{'} = 0$ $y_3^{'} = 0.231$ $y_4^{'} = 0.769$	$(9.248018, 10.727805, 2.775083, 1.900361)$

6. Concluding Remarks

In this paper, a two person zero- sum matrix game with $L-R$ fuzzy numbers payoff has been introduced. For each player, a linear programming problem has been formulated. A solution method for solving each problem with fuzziness in relations has been proposed.

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