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THE GROWTH OF GENERALIZED ITERATED ENTIRE FUNCTIONS –I

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Abstract. In this paper, we study the generalized iteration of entire functions and investigate the growth of iterated entire functions of finite iterated order. Here we prove some results on the growth of iterated entire functions of finite iterated order. The results improve and generalize some earlier results.

Keywords: entire functions, growth, iteration.

1. Introduction, Definitions and Notation

In order to study the growth properties of generalized iterated entire functions, it is very much necessary to mention some relevant notations and definitions. For standard notations and definitions we refer to [4].

Notation 1.1. [10] Let and for positive integer *m*, $\log^{[0]} x = x$, $\exp^{[0]} = x$ and

$$\log^{[m]} x = \log(\log^{[m-1]} x), \quad \exp^{[m]} x = \exp(\exp^{[m-1]} x).$$

Definition 1.2. The order ρ_f and lower order λ_f of a meromorphic function *f* is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If *f(z)* is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Definition 1.3. The hyper order ρ_f and hyper lower order λ_f of a meromorphic function *f* is defined as

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

and

$$\bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

If *f(z)* is entire then

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$

and

$$\bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 1.4. [6] A function A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function *f* if

(i) $\lambda_f(r)$ is nonnegative and continuous for $r \ge r_0$, say;

(ii) $\lambda_f(r)$ is differentiable for $r \ge r_0$ except possibly at isolated points at which $\lambda'_f(r-0)$ and $\lambda'_f(r+0)$ exist;

(iii) $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$;

(iv) $\lim_{r\to\infty} r\lambda'_f(r)\log r = 0$; and

(v) $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1$.

Definition 1.5. [6] Let $f(z)$ and $g(z)$ are two entire functions defined in the open complex plane and $\alpha \in (0,1]$. Then the generalized iterations of *f* with respect to *g* is defined as follows:

$$f_{1,g}(z) = (1-\alpha)z + \alpha f(z)$$

$$f_{2,g}(z) = (1-\alpha)g_{1,f}(z) + \alpha f\left(g_{1,f}(z)\right)$$

$$f_{3,g}(z) = (1-\alpha)g_{2,f}(z) + \alpha f\left(g_{2,f}(z)\right)$$

....

....

$$f_{n,g}(z) = (1-\alpha)g_{n-1,f}(z) + \alpha f\left(g_{n-1,f}(z)\right)$$

and so

$$g_{1,f}(z) = (1-\alpha)z + \alpha g(z)$$
$$g_{2,f}(z) = (1-\alpha)f_{1,g}(z) + \alpha g\left(f_{1,g}(z)\right)$$
$$g_{3,f}(z) = (1-\alpha)f_{2,g}(z) + \alpha g\left(f_{2,g}(z)\right)$$
....
....
$$g_{n,f}(z) = (1-\alpha)f_{n-1,g}(z) + \alpha g\left(f_{n-1,g}(z)\right).$$

Clearly all $f_{n,g}(z)$ and $g_{n,f}(z)$ are entire functions.

For two non-constant entire functions *f* and *g*, it is well known that

$$\log M(r, f(g)) \le \log M(M(r,g), f). \qquad \ldots\ldots \quad (1)$$

Let *f(z)* and *g(z)* be two transcendental entire functions defined in the open complex plane **C**. J. Clunie [3] proved that

$$\lim_{r\to\infty} \frac{T(r, fog)}{T(r, f)} = \infty \text{ and } \lim_{r\to\infty} \frac{T(r, fog)}{T(r, g)} = \infty.$$

In 1985, Singh [11] proved some comparative growth properties of $\log T(r, fog)$ and $T(r, f)$. After this Lahiri [6] proved some results on the comparative growth of $\log T(r, fog)$ and $T(r, g)$.

Recently Lahiri and Datta [7] made a close investigation on the comparative growth properties of $\log T(r, fog)$ and $T(r, g)$. They also proved some results on the comparative growth properties of $\log\log T(r, fog)$ and $T(r, f^k)$.

In 2011, Banerjee and Dutta [2] proved some results on comparative growth of iterated entire functions which improve some earlier results.

In this paper, we study the growth of generalized iterated entire functions and prove some results which generalize and improve some earlier results.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [4] Let *f(z)* be an entire function. For $0 \le r < R < \infty$, we have

$$T(r, f) \le \log^+ M(r, f) \le \frac{R+r}{R-r} T(R, f).$$

Lemma 2.2. [4] Let *f(z)* and *g(z)* be two transcendental entire functions, then

$$\lim_{r\to\infty} \frac{T(r, f)}{T(r, g(f))} = 0.$$

Lemma 2.3. [8] Let *f(z)* and *g(z)* be two entire functions. If $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$, for any $\varepsilon > 0$, then

$$T(r, f(g)) < (1+\varepsilon)T(M(r,g), f).$$

In particular if *g(0)=0*, then $T(r, f(g)) \le T(M(r,g), f)$ for all *r >0*.

Lemma 2.4. [9] Let $f(z)$ and $g(z)$ be two entire functions. Then we have

$$T(r, fog) \geq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

Lemma 2.5. [5] Let *f* be an entire function. Then for *k>2,*

$$\liminf_{r\to\infty} \frac{\log^{[k-1]} M(r, f)}{\log^{[k-2]} T(r, f)} = 1.$$

Lemma 2.6. [7] Let *f* be a meromorphic function. Then for $\delta(>0)$ the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is an increasing function of *r*.

Lemma 2.7. Let $f(z)$ and $g(z)$ be two entire functions such that $0<\lambda_f \leq \rho_f < \infty$ and $0<\lambda_g \leq \rho_g < \infty$ respectively, then for any $\varepsilon(0<\varepsilon<\min\{\lambda_f, \lambda_g\})$

$$\log^{[n-1]} T\left(r, f_{n,g}\right) \leq \begin{cases} (\rho_f+\varepsilon)(1+0(1))\log M\left(r, g\right)+O(1) & \textit{when n is even,} \\ (\rho_g+\varepsilon)(1+0(1))\log M\left(r, f\right)+O(1) & \textit{when n is odd} \end{cases}$$

and

$$\log^{[n-1]} T\left(r, f_{n,g}\right) \geq \begin{cases} (\lambda_f-\varepsilon)(1+0(1))\log M\left(\frac{r}{4^{n-1}}, g\right)+O(1) & \textit{when n is even,} \\ (\lambda_g-\varepsilon)(1+0(1))\log M\left(\frac{r}{4^{n-1}}, f\right)+O(1) & \textit{when n is odd}, \end{cases}$$

for all sufficiently large values of *r*.

Proof. We get from Lemma 2.2, Lemma 2.3 and (1) for $\varepsilon > 0$ and for all large values of *r,*

$$\begin{aligned} T\left(r, f_{n,g}\right) &\leq T\left(r, g_{n-1,f}\right) + T\left(r, f(g_{n-1,f})\right) + O(1) \\ &\leq (1+0(1))T\left(r, f(g_{n-1,f})\right) \\ &\leq 2T(M\left(r, g_{n-1,f}\right), f) \end{aligned}$$

i.e., $\log T(r, f_{n,g}) \leq \log T(M(r, g_{n-1,f}))+O(1)$

$$\leq (\rho_f+\varepsilon)\log M\left(r, g_{n-1,f}\right)+O(1), \quad \text{using } \textit{Definition } 1.1.$$

$$So,\ l\,\text{og}^{[2]}T(r,f_{n,g})\le\log^{[2]}M(r,g_{n-1,f})+O(1)$$
$$\le\log\{\log M(r,f_{n-2,g})+\log M(r,g(f_{n-2,g}))+O(1)\}+O(1)$$
$$\le\log\{\log M(M(r,f_{n-2,g}),g)+\log M(M(r,f_{n-2,g}),g)+O(1)\}+O(1)$$
$$\le\log\log M(M(r,f_{n-2,g}),g)+O(1)$$
$$\le(\rho_g+\varepsilon)\log M(r,f_{n-2,g})+O(1).$$

....

....

$$Therefore,\ \log^{[n-1]}T(r,f_{n,g})\le(\rho_f+\varepsilon)\log M(r,g_{1,f})+O(1)$$
$$\le(\rho_f+\varepsilon)\{\log M(r,z)+\log M(r,g)+O(1)\}+O(1)$$
$$\le(\rho_f+\varepsilon)(1+0(1))\log M(r,g)+O(1)\ when\ n\ is\ even.$$

Similarly,

$$\log^{[n-1]}T(r,f_{n,g})\le(\rho_g+\varepsilon)(1+0(1))\log M(r,f)+O(1)\quad when\ n\ is\ odd.$$

Again for $\varepsilon(0<\varepsilon<\min\{\lambda_f,\lambda_g\})$ we get from Lemma 2.2 and Lemma 2.4 for all large values of r,

$$T\left(r,fn,g\right)\ge T\left(r,f(g_{n-1,f})\right)-T\left(r,g_{n-1,f}\right)+O(1)$$
$$\ge(1+0(1))T\left(r,f(g_{n-1,f})\right)$$
$$\ge(1+0(1))\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1,f}\right)+O(1),f\right)$$
$$\ge(1+0(1))\frac{1}{3}\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1,f}\right)+O(1)\right)^{\lambda_f-\varepsilon}$$
$$\ge(1+0(1))\frac{1}{3}\left(\frac{1}{9}M\left(\frac{r}{4},g_{n-1,f}\right)\right)^{\lambda_f-\varepsilon},$$

$$that\ is,\ \log T\left(r,fn,g\right)\ge(\lambda_f-\varepsilon)\log M\left(\frac{r}{4},g_{n-1,f}\right)+O(1)$$
$$\ge(\lambda_f-\varepsilon)T\left(\frac{r}{4},g_{n-1,f}\right)+O(1)$$
$$\ge(\lambda_f-\varepsilon)\left[T\left(\frac{r}{4},g(f_{n-2,g})\right)-T\left(\frac{r}{4},f_{n-2,g}\right)+O(1)\right]+O(1)$$
$$\ge(\lambda_f-\varepsilon)(1+0(1))T\left(\frac{r}{4},g(f_{n-2,g})\right)+O(1)$$

$$\geq (\lambda_f - \varepsilon)\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2,g}\right) + O(1), g\right) + O(1)$$

$$\geq (\lambda_f - \varepsilon)\frac{1}{3}\left(\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2,g}\right) + O(1)\right)^{\lambda_g - \varepsilon} + O(1)$$

$$\geq (\lambda_f - \varepsilon)\frac{1}{3}\left(\frac{1}{9}M\left(\frac{r}{4^2}, f_{n-2,g}\right)\right)^{\lambda_g - \varepsilon} + O(1),$$

that is, $\log^{[2]} T(r, fn,g) \geq (\lambda_g - \varepsilon)\log M\left(\frac{r}{4^2}, f_{n-2,g}\right) + O(1).$

....

....

Therefore, $\log^{[n-1]} T(r, fn,g) \geq (\lambda_f - \varepsilon)\log M\left(\frac{r}{4^{n-1}}, g_{1,f}\right) + O(1)$

$$\geq (\lambda_f - \varepsilon)(1 + 0(1))\log M\left(\frac{r}{4^{n-1}}, g\right) + O(1) \qquad \text{when } n \text{ is even.}$$

Similarly if *n* is odd then

$$\log^{[n-1]} T(r, fn,g) \geq (\lambda_g - \varepsilon)(1 + 0(1))\log M\left(\frac{r}{4^{n-1}}, f\right) + O(1).$$

This proves the lemma.

3. Theorems

Theorem 3.1. Let *f* and *g* be two non-constant entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$ respectively. Then for k = 0,1,2,3,.......

$$\frac{\bar{\lambda}_g}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r, fn,g)}{\log T(r, g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n+1]} T(r, fn,g)}{\log T(r, g^{(k)})} \leq \frac{\bar{\rho}_g}{\lambda_g}$$

when n is even and

$$\frac{\bar{\lambda}_f}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r, fn,g)}{\log T(r, f^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n+1]} T(r, fn,g)}{\log T(r, f^{(k)})} \leq \frac{\bar{\rho}_f}{\lambda_f}$$

when n is odd, where f^k denote the k-th derivative of *f*.

Proof. First suppose that *n* is even. Then for given $\varepsilon(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$ we get from Lemma 2.7 for all large values of *r*,

$$\log^{[n-1]} T\left(r, f_{n,g}\right) \geq (\lambda_f - \varepsilon)(1+0(1))\log M\left(\frac{r}{4^{n-1}}, g\right) + O(1)$$

$$\geq (\lambda_f - \varepsilon)(1+0(1))T\left(\frac{r}{4^{n-1}}, g\right) + O(1)$$

$$i.e.\ \log^{[n]} T\left(r, f_{n,g}\right) \geq \log T\left(\frac{r}{4^{n-1}}, g\right) + O(1).$$

$$So,\ \log^{[n+1]} T\left(r, f_{n,g}\right) \geq \log^{[2]} T\left(\frac{r}{4^{n-1}}, g\right) + O(1).$$

So for all large values of *r*,

$$\frac{\log^{[n+1]} T\left(r, f_{n,g}\right)}{\log T\left(r, g^{(k)}\right)} \geq \frac{\log^{[2]} T\left(\frac{r}{4^{n-1}}, g\right)}{\log \frac{r}{4^{n-1}}} . \frac{\log \frac{r}{4^{n-1}}}{\log T\left(r, g^{(k)}\right)} + 0(1). \qquad \text{…. } (3.1)$$

Since,

$$\limsup_{r\to\infty} \frac{\log T\left(r, g^{(k)}\right)}{\log r} = \rho_g,$$

so for all large values of *r* and arbitrary $\varepsilon > 0$ we have

$$\log T\left(r, g^{(k)}\right) < (\rho_g + \varepsilon)\log r. \qquad \text{…. } (3.2)$$

Since $\varepsilon > 0$ is arbitrary, so from (3.1) and (3.2) we have

$$\liminf_{r\to\infty} \frac{\log^{[n+1]} T\left(r, f_{n,g}\right)}{\log T\left(r, g^{(k)}\right)} \geq \liminf_{r\to\infty} \frac{\log^{[2]} T\left(\frac{r}{4^{n-1}}, g\right)}{\log \frac{r}{4^{n-1}}} . \left(\frac{\log r - \log 4^{n-1}}{\rho_g \log r}\right)$$

$$\geq \frac{\bar{\lambda}_g}{\rho_g}. \qquad \text{…. } (3.3)$$

Again from Lemma 2.7 for all large values of *r*,

$$\log^{[n-1]} T(r, f_{n,g}) \leq (\rho_f + \varepsilon)(1+0(1))\log M(r,g) + O(1)$$

$$i.e.,\ \frac{\log^{[n+1]} T\left(r, f_{n,g}\right)}{\log T\left(r, g^{(k)}\right)} \leq \frac{\log^{[3]} M\left(r, g\right)}{\log T\left(r, g^{(k)}\right)} + 0(1). \qquad \text{…. } (3.4)$$

Since,

$$\liminf_{r\to\infty} \frac{\log T\left(r, g^{(k)}\right)}{\log r} = \lambda_g,$$

so for all large values of *r* and arbitrary $\varepsilon (0 < \varepsilon < \lambda_g)$ we have

$$\log T\left(r, g^{(k)}\right) > (\lambda_g - \varepsilon)\log r. \qquad \text{…. } (3.5)$$

Since $\varepsilon > 0$ is arbitrary, so from (3.4) and (3.5) we have

$$\lim_{r\to\infty}\sup \frac{\log^{[n+1]} T\left(r, f_{n,g}\right)}{\log T\left(r, g^{(k)}\right)} \le \frac{\bar{\rho}_g}{\lambda_g}. \qquad \text{….} \ (3.6)$$

Combining (3.3) and (3.6) we obtain the first part of the theorem.
Similarly when *n* is odd then we having the second part of the theorem.
This proves the theorem.

Theorem 3.2. Let *f* and *g* be two non-constant entire functions such that $0 < \lambda_f \le \rho_f < \infty$ and $0 < \lambda_g \le \rho_g < \infty$ respectively. Then

$$(i) \quad \frac{\lambda_g}{\rho_g} \le \lim_{r\to\infty}\inf \frac{\log^{[n]} T\left(r, f_{n,g}\right)}{\log T(r,g)} \le 1 \le \lim_{r\to\infty}\sup \frac{\log^{[n]} T\left(r, f_{n,g}\right)}{\log T(r,g)} \le \frac{\rho_g}{\lambda_g}$$

when *n* is even and

$$(ii) \quad \frac{\lambda_f}{\rho_f} \le \lim_{r\to\infty}\inf \frac{\log^{[n]} T\left(r, f_{n,g}\right)}{\log T(r,f)} \le 1 \le \lim_{r\to\infty}\sup \frac{\log^{[n]} T\left(r, f_{n,g}\right)}{\log T(r,f)} \le \frac{\rho_f}{\lambda_f}$$

when *n* is odd.

Proof. First suppose that *n* is even. Then for given $\varepsilon(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$ we get from Lemma 2.7, for all large values of *r*,

$$\log^{[n-1]} T(r, f_{n,g}) \le (\rho_f + \varepsilon)(1 + 0(1)) \log M(r,g) + O(1)$$

$$i.e., \ \frac{\log^{[n]} T\left(r, f_{n,g}\right)}{\log T(r,g)} \le \frac{\log^{[2]} M(r,g)}{\log T(r,g)} + 0(1) \qquad \text{......} \quad (3.7)$$

$$i.e., \ \lim_{r\to\infty}\inf \frac{\log^{[n]} T\left(r, f_{n,g}\right)}{\log T(r,g)} \le 1 \qquad [by\ Lemma\ 2.5]. \quad \text{….} \ (3.8)$$

Also,

$$\log^{[n-1]} T\left(r, f_{n,g}\right) \ge (\lambda_f - \varepsilon)(1 + 0(1)) \log M\left(\frac{r}{4^{n-1}}, g\right) + O(1)$$

$$i.e. \ \log^{[n]} T\left(r, f_{n,g}\right) \ge \log T\left(\frac{r}{4^{n-1}}, g\right) + O(1).$$

So,

$$\frac{\log^{[n]} T\left(r, f_{n,g}\right)}{\log T(r,g)} \ge \frac{\log T\left(\frac{r}{4^{n-1}}, g\right)}{\log \frac{r}{4^{n-1}}} \left(\frac{\log r - \log 4^{n-1}}{\rho_g \log r}\right) + 0(1)$$

$$i.e. \ \lim_{r\to\infty}\inf \frac{\log^{[n]} T\left(r, f_{n,g}\right)}{\log T(r,g)} \ge \frac{\lambda_g}{\rho_g}. \qquad \text{….} \ (3.9)$$

Also from (3.7), we get for all large values of *r*,

$$\frac{\log^{[n]} T\left(r, fn,g\right)}{\log T(r,g)} \leq \frac{\log^{[2]} M(r,g)}{\log r}\frac{\log r}{\log T(r,g)} + 0(1)$$

$$i.e., \ \lim_{r\to\infty} \sup \frac{\log^{[n]} T\left(r, fn,g\right)}{\log T(r,g)} \leq \frac{\rho_g}{\lambda_g}. \qquad \text{…. } (3.10)$$

Again from Lemma 2.7,

$$\log^{[n-1]} T\left(r, fn,g\right) \geq (\lambda_f - \varepsilon)(1+0(1))\log M\left(\frac{r}{4^{n-1}}, g\right) + O(1)$$

$$i.e. \ \log^{[n]} T\left(r, fn,g\right) \geq \log^{[2]} M\left(\frac{r}{4^{n-1}}, g\right) + O(1). \qquad \text{…. } (3.11)$$

Let $0 < \varepsilon < \min\{1, \lambda_f, \lambda_g\}.$

Since

$$\lim_{r\to\infty}\inf \frac{\log T(r,g)}{r^{\lambda_g(r)}} = 1,$$

there is a sequence of values of *r* tending to infinity for which

$$T(r,g) < (1+\varepsilon) r^{\lambda_g(r)} \qquad \text{…. } (3.12)$$

and for all large values of r,

$$\log^{+} M(r,g) > T(r,g) > (1-\varepsilon) r^{\lambda_g(r)}. \qquad \text{.....}(3.13)$$

From (3.13) we obtain for all large values of *r* and for $\delta > 0$ and $\varepsilon(0 < \varepsilon < 1)$

$$\log M\left(\frac{r}{4^{n-1}}, g\right) > (1-\varepsilon)\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4^{n-1}}\right)}}$$

$$\geq \frac{1-\varepsilon}{\left(4^{n-1}\right)^{\lambda_g+\delta}} r^{\lambda_g(r)}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of *r*.

So by (3.12) we get for a sequence of value of *r* tending to infinity,

$$\log M\left(\frac{r}{4^{n-1}}, g\right) \geq \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{\left(4^{n-1}\right)^{\lambda_g+\delta}} T(r,g)$$

$$i.e., \ \log^{[2]} M\left(\frac{r}{4^{n-1}}, g\right) \geq \log T(r,g) + O(1). \qquad \text{.....}(3.14)$$

Now from (3.11) and (3.14)

$$\lim_{r\to\infty}\sup \frac{\log^{[n]} T\left(r, fn,g\right)}{\log T(r,g)} \geq 1. \qquad \text{....}(3.15)$$

So the theorem follows from (3.8), (3.9), (3.10) and (3.15), when n is even.

Similarly, when n is odd we get (ii).

This proves the theorem.

Corollary 3.3. Using the hypothesis of Theorem 3.2 if *f* and *g* are of regular growth then

$$\lim_{r\to\infty}\frac{\log^{[n]}T\left(r,fn,g\right)}{\log T\left(r,g\right)}=\lim_{r\to\infty}\frac{\log^{[n]}T\left(r,fn,g\right)}{\log T\left(r,f\right)}=1.$$

Remark 3.4. The conditions $\lambda_f, \lambda_g > 0$ and $\rho_f, \rho_g < \infty$ are necessary for Theorem 3.2 and Corrollary 3.3, which are shown by the following examples.

Example 3.5. Let *f =z, g = exp z* and $\alpha = 1.$ Then $\lambda_f = \rho_f = 0 \ and \ 0 < \lambda_g = \rho_g < \infty.$

Now when n is even then $f_n = \exp^{\left[\frac{n}{2}\right]} z.$
Therefore

$$T(r, f_n) \le \log M(r, f_n) = \exp^{\left[\frac{n}{2}-1\right]} r.$$

So,

$$\log^{[n]} T(r, f_n) \le \log^{[n]}\left(\exp^{\left[\frac{n}{2}-1\right]} r\right)$$

$$= \log^{\left[\frac{n}{2}+1\right]} r.$$

Also when n is odd, $f_n = \exp^{\left[\frac{n-1}{2}\right]} z.$
Therefore

$$T(r, f_n) \le \log M(r, f_n) = \exp^{\left[\frac{n-1}{2}-1\right]} r.$$

So,

$$\log^{[n]} T(r, f_n) \le \log^{[n]}\left(\exp^{\left[\frac{n-1}{2}-1\right]} r\right)$$

$$= \log^{\left[\frac{n+1}{2}+1\right]} r.$$

Now $\log T(r, f) = \log\log r \ and \ \log T(r,g) = \log r - \log \pi.$
Therefore when n is even

$$\frac{\log^{[n]}T\left(r,fn\right)}{\log T\left(r,g\right)} \le \frac{\log^{\left[\frac{n}{2}+1\right]} r}{\log r - \log \pi} \to 0 \ \ as \ r \to \infty,$$

and when n is odd

$$\frac{\log^{[n]}T\left(r,fn\right)}{\log T\left(r,f\right)} \le \frac{\log^{\left[\frac{n+1}{2}+1\right]} r}{\log\log r} \to 0 \ \ as \ r \to \infty.$$

Example 3.6. Let $f = \exp^{[2]} z,$ *g= exp z* and $\alpha = 1.$ Then $\lambda_f = \rho_f = \infty \ and \ \lambda_g = \rho_g = 1.$

Now when n is even then $f_n = \exp^{\left[\frac{3n}{2}\right]} z$.

Therefore

$$3T(2r, f_n) \geq \log M(r, f_n) = \exp^{\left[\frac{3n}{2}-1\right]} r$$

$$i.e.,\ T(r, f_n) \geq \frac{1}{3}\exp^{\left[\frac{3n}{2}-1\right]} \frac{r}{2}$$

$$i.e.,\ l\log^{[n]} T(r, f_n) \geq \exp^{\left[\frac{n}{2}-1\right]} \frac{r}{2} + 0(1).$$

Also when n is odd, $f_n = \exp^{\left[\frac{3n-1}{2}\right]} z$.

Therefore

$$3T(2r, f_n) \geq \log M(r, f_n) = \exp^{\left[\frac{3n-1}{2}-1\right]} r$$

$$i.e.,\ T(r, f_n) \geq \frac{1}{3}\exp^{\left[\frac{3n-1}{2}-1\right]} \frac{r}{2}$$

$$i.e.,\ \log^{[n]} T(r, f_n) \geq \exp^{\left[\frac{n-3}{2}\right]} \frac{r}{2} + 0(1).$$

Also, $T(r, f) \leq e^r\ and\ T(r,g) = \dfrac{r}{\pi}$.

Therefore when n is even

$$\frac{\log^{[n]} T(r, fn)}{\log T(r,g)} \geq \frac{\exp^{\left[\frac{n}{2}-1\right]} r + 0(1)}{\log r - \log \pi} \to \infty\ \ as\ r \to \infty,$$

and when n is odd

$$\frac{\log^{[n]} T(r, fn)}{\log T(r, f)} \geq \frac{\exp^{\left[\frac{n-1}{2}\right]} r + 0(1)}{r} \to \infty\ \ as\ r \to \infty.$$

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References

[1] D. Banerjee and N. Mandal, Maximum modulus and maximum term of generalized iterated entire functions, Bulletin of the Allahabad Mathematical Society, 2012, 27(1): 117-131.
[2] D. Banerjee and R. K. Dutta, The growth of iterated entire functions, Bulletin of Mathematical Analysis and Applications, 2011, 3(3): 35-49.
[3] J. Clunie, The composition of entire and meromorphic functions, Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, 1970, 75-92.
[4] W. K. Hayman, Meromorphic Functions, Oxford University Press, 1964.
[5] I. Lahiri, Generalized proximate order of meromorphic functions, Matematnykn Bechnk, 1989, 41: 9-16.

[6] I. Lahiri, Growth of composite integral functions, Indian J. Pure Appl. Math., 1997, 20(9): 899-907.
[7] I. Lahiri and S. K. Datta, On the growth of composite entire and meromorphic functions, Indian J. Pure and Appl. Math., 2004, 35(4): 525-543.
[8] K. Niino and N. Smita, Growth of a composite function of entire functions, Kodai Math. J., 1980, 3: 374-379.
[9] K. Niino and C. C. Yang, Some growth relationships on factors of two composite entire functions, factorization theory of meromorphic functions and related topics, Marcel Dekker Inc. (New York and Basel). 1982, 95-99.
[10] D. Sato, On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., 1963, 69: 411-414.
[11] A. P. Singh, Growth of composite entire functions, Kodai Math. J., 1985, 8: 99-102.