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An inverse problem for diffusive logistic equation with free boundary

Jiapeng Pan[a] , Bin Wu[a]*,

[a]*School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China*

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Abstract: This paper considers an inverse problem for a logistic model with free boundary. This inverse problem aims to identify the growth coefficient only depending on time from a fixed point measurement data. Based on a fixed point argument, we prove the local in time existence and uniqueness of our inverse problem.

Key words: diffusive logistic model; inverse problem; free boundary

1 Introduction

Free boundary problems are a kind of mathematical physics models with one unknown function that defines this boundary. It has been found in a broad variety of physical applications, such as the one-phase Stefan problem [18,15,3,14,1,6], the free boundary problems for predator-prey model [13], the information diffusion in online social networks with time varying distance [20], ductal carcinoma in situ mathematical model [21] and so on. In the last twenty years, there has been a few of work related to the related inverse problems, see [7,9,10,13] and the references therein for more details. Such inverse problems are more complicated than the traditional ones because of the unknown free boundary.

In this paper, we consider the following diffusive logistic model with free boundary [11]:

$$\begin{cases} u_t - du_{xx} = r(t)u\left(1-\frac{u}{K}\right), & (x,t)\in Q_{s,T}, \\ u_x(0,t) = u(s(t),t) = 0, & t\in(0,T), \\ s(0) = s_0 > 0, & \\ u(x,0) = u_0(x), & x\in[0,s_0], \\ s'(t) = -\mu u_x(s(t),t), & t\in(0,T), \end{cases} \tag{1.1}$$

where $Q_{s,T} = \{(x,t)|0 < x < s(t), 0 < t < T\}$, $s(t)$ represents the free boundary is unknown function. System (1.1) can be used to depict information diffusion in online social networks, in which $u(x,t)$ denotes the density of influenced users at time t and distance x, K and d indicate the carrying capacity and diffusion rate, respectively.

In this paper we couple to the equations the following additional the boundary observation on u:

$$u(0,t) = f(t) \quad t\in[0,T], \tag{1.2}$$

as our inversion input data to determine the unknown function $r(t)$, which represents the intrinsic growth rate in this model.

In [5], the authors proved showed the local in time existence and uniqueness of logistic model and further obtained blow-up property about a free boundary model. The authors [11] proved a global existence for a logistic equation with free boundary.

The last boundary condition $s'(t) = -\mu u_x(s(t),t)$ on boundary $s(t)$ is called Stefan condition, which is widely used to describe phase transitions between solid and fluid states [2].

Recently, inverse source problems with a free boundary have received much attention. For example, Snitko [11] proved the local in time existence and uniqueness for an inverse problem of determining an unknown time-dependent leading coefficient in a parabolic equation with free boundary. Hussein, Lesnic, Ivanchov and Snitko [8] investigated a multiple time-dependent coefficient identification thermal problem with unknown free boundary under two additional integral conditions.

In this paper, we consider a coefficient inverse problem for system (1.1), which is a semi-linear model with free boundary. On the other hand, we use the measurement at boundary point $x = 0$. In practical applications our measurement data are less than the global measurement data. In this paper we will prove the local existence and uniqueness for our coefficient inverse problem of determining $r(t)$ in (1.1) by the measurement data (1.2).

The rest of our paper is organized as follows. In Section 2, we prove a local in time existence and uniqueness result for the direct free boundary problem and show the solution in suitable Sobolev space continuous dependence on T and r. In Section 3, we first transfer our inverse problem to an equivalent problem. Then a local existence and uniqueness of the equivalent problem is obtained by the contraction mapping.

2 Direct free boundary problem

In this section, we prove the existence local in time of the direct problem (1.1) in a suitable Banach space. Meanwhile, we show a continuous property of the solution with respect to r and T, which is important to consider the inverse problem of determining the unknown r.

Firstly, we make a change of variable to straighten the free boundary. Let $\Omega=(0,1)$, $Q_T=\Omega\times(0,T)$, and

$$\xi=\frac{x}{s(t)},\qquad u(x,t)=v(\xi,t), \tag{2.1}$$

system (1.1) can be rewritten as

$$\begin{cases} v_t-d\frac{1}{s^2(t)}v_{\xi\xi}-\frac{s'(t)}{s(t)}\xi v_\xi=r(t)v\left(1-\frac{v}{K}\right), & (\xi,t)\in Q_T,\\ v_\xi(0,t)=v(1,t)=0, & t\in(0,T),\\ v(\xi,0)=v_0(\xi), & \xi\in\overline{\Omega},\\ s(t)s'(t)=-\mu v_\xi(1,t), & t\in(0,T), \end{cases} \tag{2.2}$$

where $v_0(\xi)=u_0(x)$, and (1.2) is rewritten as

$$v(0,t)=f(t),\qquad t\in[0,T] \tag{2.3}$$

Let

$$h(t)=s(t)s'(t). \tag{2.4}$$

Then, (v,h) further satisfies the following problem:

$$\begin{cases} v_t-dA(h)v_{\xi\xi}-B(h)\xi v_\xi=r(t)v\left(1-\frac{v}{K}\right), & (\xi,t)\in Q_T,\\ v_\xi(0,t)=v(1,t)=0, & t\in(0,T),\\ v(\xi,0)=v_0(\xi), & \xi\in\overline{\Omega},\\ h(t)=-\mu v_\xi(1,t), & t\in(0,T), \end{cases} \tag{2.5}$$

with

$$\begin{cases} A(h)=\frac{1}{2\int_0^t h(\tau)d\tau+s_0^2},\\ B(h)=\frac{h}{2\int_0^t h(\tau)d\tau+s_0^2}. \end{cases} \tag{2.6}$$

We define

$$X_T=C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)\times C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T], \tag{2.7}$$

and

$$||(v,h)||_{X_T}=||v||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)}+||h||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}. \tag{2.8}$$

Theorem 2.1. Let $v_0\in C^{2+\alpha}(\overline{\Omega})$, $v_0'(1)<0$, $r\in C^{\frac{\alpha}{2}}[0,T]$. Then there exists a sufficient small $T_0>0$ such that the direct problem (2.5) has a unique solution $(v,h)\in X_T$ for any $0<T<T_0$. Furthermore, we have the following estimate

$$||(v,h)||_{X_T}\le C\left[(T+T^2)\|r\|_{C^{\frac{\alpha}{2}}[0,T]}+||v_0||_{C^{2+\alpha}(\overline{\Omega})}\right], \tag{2.9}$$

Where C is a constant depending on Ω, T, μ and s_0.

Proof. Define $D_{M,T}=V_{M,T}\times H_{M,T}$, where

$$V_{M,T}=\left\{\hat{v}\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)\mid \hat{v}(\xi,0)=v_0(\xi),||\hat{v}||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)}\le M\right\},$$

$$H_{M,T}=\left\{\hat{h}\in C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]\mid \hat{h}(t)=h^*,||\hat{h}||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}\le M\right\}, \tag{2.10}$$

where $h^* = -\mu v_0'(1) > 0$. For given $(\hat{v}, \hat{h}) \in D_{M,T}$, we consider

$$\begin{cases} v_t - dA(\hat{h})v_{\xi\xi} - B(\hat{h})\xi v_\xi = r(t)\hat{v}\left(1 - \frac{\hat{v}}{K}\right), & (\xi, t) \in Q_T, \\ v_\xi(0,t) = v(1,t) = 0, & t \in (0,T), \\ v(\xi, 0) = v_0(\xi), & \xi \in \overline{\Omega}, \end{cases} \tag{2.11}$$

and

$$h(t) = -\mu v_\xi(1,t), \quad t \in [0,T]. \tag{2.12}$$

By the theory of linear parabolic equation, there is a unique solution $v \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ of system (2.11). Then $h \in C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]$ by (2.12).

Thus, the mapping

$$F: D_{M,T} \subset X_T \to X_T,$$
$$(\widehat{w}(\xi,t), \hat{h}(t)) \mapsto (w(\xi,t), h(t)) \tag{2.13}$$

is well defined.

We split the following proof into two steps.

Step 1. We prove $F(D_{M,T}) \subset D_{M,T}$.

We use the standard Schauder theory for parabolic differential equation with Neumann boundary, e.g. Theorem 5.3 in [12] to obtain

$$\begin{aligned} ||v||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)} &\le C\left[||r||_{C^{\frac{\alpha}{2}}[0,T]} ||\hat{v}\left(1 - \frac{\hat{v}}{K}\right)||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} + ||v_0||_{C^{2+\alpha}(\overline{\Omega})}\right] \\ &\le C\left[||r||_{C^{\frac{\alpha}{2}}[0,T]} \left(||\hat{v}||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} + \|\hat{v}\|^2_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)}\right) + ||v_0||_{C^{2+\alpha}(\overline{\Omega})}\right] \\ &\le C\left[(TM + T^2M^2)||r||_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\overline{\Omega})}\right]. \end{aligned} \tag{2.14}$$

Also, from (2.12), we have

$$||h||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]} \le C||v_\xi||_{C^{1+\alpha,\frac{1}{2}+\frac{\alpha}{2}}(\overline{Q}_T)} \le C||v||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)}, \tag{2.15}$$

Letting $M = 4C(||v_0||_{C^{2+\alpha}(\overline{\Omega})})$ and choosing $T_0 > 0$ sufficient small such that

$$C(TM + T^2M^2)||r||_{C^{\frac{\alpha}{2}}[0,T]} \le \frac{1}{4}M, \tag{2.16}$$

together with (2.16), we have

$$||(v,h)||_{X_T} \le M. \tag{2.17}$$

So, we have $F(D_{M,T}) \subset D_{M,T}$ for all $0 < T < T_0$.

Step 2. We prove $\left\|F(\hat{v}_1, \hat{h}_1) - F(\hat{v}_2, \hat{h}_2)\right\|_{X_T} \le \frac{1}{2}\left\|(\hat{v}_1, \hat{h}_1) - (\hat{v}_2, \hat{h}_2)\right\|_{X_T}$.

Let $(\hat{v}_i, \hat{h}_i) \in D_{M,T}$, $(v_i, h_i) = F(\hat{v}_i, \hat{h}_i)$ $(i = 1,2)$ and $(V, H) := (v_1 - v_2, h_1 - h_2)$. Then, we have

$$\begin{cases} V_t - dA(\hat{h}_1)V_{\xi\xi} - B(\hat{h}_1)\xi V_\xi = dv_{2,\xi\xi}\left(A(\hat{h}_1) - A(\hat{h}_2)\right) \\ \quad +\xi v_{2,\xi}\left(B(\hat{h}_1) - B(\hat{h}_2)\right) + r[(\hat{v}_1 - \hat{v}_2)(1 - \frac{1}{K}(\hat{v}_1 + \hat{v}_2))], & (\xi,t) \in Q_T, \\ V_\xi(0,t) = V(1,t) = 0, & t \in (0,T), \\ V(\xi,0) = 0, & \xi \in \overline{\Omega}, \\ H(t) = -\mu V_\xi(1,t), & t \in [0,T]. \end{cases} \tag{2.18}$$

Then, applying Theorem 5.3 in [12] again, we obtain

$$\begin{aligned} &||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \\ &\le C\left[||v_{2,\xi\xi}||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)} ||A(\hat{h}_1) - A(\hat{h}_2)||_{C^{\frac{\alpha}{2}}[0,T]}\right] \\ &\quad +C\left[||v_{2,\xi}||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)} ||B(\hat{h}_1) - B(\hat{h}_2)||_{C^{\frac{\alpha}{2}}[0,T]}\right] \\ &\quad +C\left[||r||_{C^{\frac{\alpha}{2}}[0,T]} ||(\hat{v}_1 - \hat{v}_2)(1 - \frac{1}{K}(\hat{v}_1 + \hat{v}_2))||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}\right] \\ &\le C\left[||v_2||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} ||A(\hat{h}_1) - A(\hat{h}_2)||_{C^{\frac{\alpha}{2}}[0,T]}\right] \\ &\quad +C\left[T^{\frac{1}{2}} \ ||B(\hat{h}_1) - B(\hat{h}_2)||_{C^{\frac{\alpha}{2}}[0,T]}\right] \end{aligned}$$

$$+C\left[\|r\|_{C^{\frac{\alpha}{2}}[0,T]}\left(||\hat{v}_1-\hat{v}_2||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}+||\hat{v}_1+\hat{v}_2||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}||\hat{v}_1-\hat{v}_2||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}\right)\right]. \quad (2.19)$$

Obviously,

$$\left|A(\hat{h}_1)-A(\hat{h}_2)\right|=\left|\frac{1}{2\int_0^t\hat{h}_1d\tau+s_0^2}-\frac{1}{2\int_0^t\hat{h}_2d\tau+s_0^2}\right|$$
$$=\frac{2\left|\int_0^t(\hat{h}_2-\hat{h}_1)d\tau\right|}{(2\int_0^t\hat{h}_1d\tau+s_0^2)(2\int_0^t\hat{h}_2d\tau+s_0^2)}, \quad (2.20)$$

which leads to

$$||A(\hat{h}_1)-A(\hat{h}_2)||_{C^{\frac{\alpha}{2}}[0,T]}\le C\left\|\int_0^t(\hat{h}_2-\hat{h}_1)d\tau\right\|_{C^{\frac{\alpha}{2}}[0,T]}\left\|\frac{1}{(2\int_0^t\hat{h}_1d\tau+s_0^2)(2\int_0^t\hat{h}_2d\tau+s_0^2)}\right\|_{C^{\frac{\alpha}{2}}[0,T]}$$
$$\le CT||\hat{h}_1-\hat{h}_2||_{C^{\frac{\alpha}{2}}[0,T]}\le CT^{\frac{3}{2}}||\hat{h}_1-\hat{h}_2||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}. \quad (2.21)$$

Similarly, we have

$$\left|B(\hat{h}_1)-B(\hat{h}_2)\right|=\left|\frac{\hat{h}_1}{2\int_0^t\hat{h}_1d\tau+s_0^2}-\frac{\hat{h}_2}{2\int_0^t\hat{h}_2d\tau+s_0^2}\right|$$
$$=\frac{\left|2\left((\hat{h}_1\int_0^t(\hat{h}_2-\hat{h}_1)d\tau-(\hat{h}_2-\hat{h}_1)\int_0^t\hat{h}_1d\tau)\right)+s_0^2(\hat{h}_1-\hat{h}_2)\right|}{(2\int_0^t\hat{h}_1d\tau+s_0^2)(2\int_0^t\hat{h}_2d\tau+s_0^2)}, \quad (2.22)$$

and then

$$||B(\hat{h}_1)-B(\hat{h}_2)||_{C^{\frac{\alpha}{2}}[0,T]}$$
$$\le C||\hat{h}_1\int_0^t(\hat{h}_2-\hat{h}_1)d\tau+(\hat{h}_2-\hat{h}_1)\int_0^t\hat{h}_1d\tau+s_0^2(\hat{h}_1-\hat{h}_2)\,||_{C^{\frac{\alpha}{2}}[0,T]}$$
$$\le C\left(T||\hat{h}_1||_{C^{\frac{\alpha}{2}}[0,T]}||\hat{h}_1-\hat{h}_2||_{C^{\frac{\alpha}{2}}[0,T]}+||\hat{h}_1-\hat{h}_2||_{C^{\frac{\alpha}{2}}[0,T]}\right)$$
$$\le C\left(T^2||\hat{h}_1||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}||\hat{h}_1-\hat{h}_2||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}+T^{\frac{1}{2}}||\hat{h}_1-\hat{h}_2||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}\right)$$
$$\le C\left(T^2M+T^{\frac{1}{2}}\right)||\hat{h}_1-\hat{h}_2||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]} \quad (2.23)$$

Therefore, from (2.14), (2.19), (2.21) and (2.23), we have

$$||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}\le C\left[\left((TM+T^2M^2)||r||_{C^{\frac{\alpha}{2}}[0,T]}+||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right)\right]\times$$
$$\left(T^{\frac{3}{2}}+T^{\frac{5}{2}}M+T\right)||\hat{h}_1-\hat{h}_2||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}$$
$$+C(T+T^2M)||r||_{C^{\frac{\alpha}{2}}[0,T]}||\hat{v}_1-\hat{v}_2||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}$$
$$\le C\left\{\left[\left(T^{\frac{3}{2}}+T^{\frac{5}{2}}M+T\right)\left((TM+T^2M^2)||r||_{C^{\frac{\alpha}{2}}[0,T]}+||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right)\right]\right.$$
$$\left.+(T+T^2M)\,||r||_{C^{\frac{\alpha}{2}}[0,T]}\right\}\times\left[||\hat{h}_1-\hat{h}_2||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}\right.$$
$$\left.+||\hat{v}_1-\hat{v}_2||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}\right]$$
$$\le C\beta_M(T)\left(||r||_{C^{\frac{\alpha}{2}}[0,T]}+||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right)\left\|(\hat{v}_1,\hat{h}_1)-(\hat{v}_2,\hat{h}_2)\right\|_{X_T}, \quad (2.24)$$

where

$$\beta_M(T)=\left(T+T^{\frac{3}{2}}+T^{\frac{5}{2}}M+T^2M+T^{\frac{7}{2}}M^2+T^3M^2+T^{\frac{9}{2}}M^3\right), \quad (2.25)$$

satisfies

$$\lim_{T\to0}\beta_M(T)=0. \quad (2.26)$$

Also, we have

$$||H||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}\le C||V_\xi||_{C^{1+\alpha,\frac{1}{2}+\frac{\alpha}{2}}(\bar{Q}_T)}\le C||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}. \quad (2.27)$$

So, from (2.24) and (2.27), we have

$$||(V,H)||_{X_T} \leq C\beta_M(T)\left(||r||_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right)\left\|(\hat{v}_1,\hat{h}_1)-(\hat{v}_2,\hat{h}_2)\right\|_{X_T}. \quad (2.28)$$

We can choose T sufficiently small such that

$$C\beta_M(T)\left(||r||_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right) \leq \frac{1}{2}$$

to obtain $\left\|F(\hat{v}_1,\hat{h}_1)-F(\hat{v}_2,\hat{h}_2)\right\|_{X_T} \leq \frac{1}{2}\left\|(\hat{v}_1,\hat{h}_1)-(\hat{v}_2,\hat{h}_2)\right\|_{X_T}$.

Therefore the Banach fixed point theorem concludes that for a small time T, there exists a unique solution $(v,h) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T) \times C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]$. This completes the proof of Theorem 2.1. ■

In order to study our inverse problem, we also need the following better regularity for v under suitable condition on v_0.

Theorem 2.2. Let $v_0 \in C^{3+\alpha}(\bar{\Omega})$, $v_0'(1) < 0$, $r \in C^{\frac{\alpha}{2}}[0,T]$. Then for all $0 < T < T_0$ we have the following estimate for :

$$\begin{aligned}
&||v||_{C^{3+\alpha,\frac{3}{2}+\frac{\alpha}{2}}(\bar{Q}_T)} \\
&\leq C\left[\left(T^{\frac{9}{2}} + T^{\frac{11}{2}}\right)\|r\|^3_{C^{\frac{\alpha}{2}}[0,T]}\right] \\
&+C\left[\left(T^{\frac{3}{2}} + T^2 + T^{\frac{5}{2}} + T^4\right)\|r\|^2_{C^{\frac{\alpha}{2}}[0,T]}\right] \\
&+C\left[T^{\frac{1}{2}}||r||_{C^{\frac{\alpha}{2}}[0,T]}||v_0||_{C^{2+\alpha}(\bar{\Omega})} + T^{\frac{3}{2}}||r||_{C^{\frac{\alpha}{2}}[0,T]}\|v_0\|^2_{C^{2+\alpha}(\bar{\Omega})}\right] \\
&+C\left[\|v_0\|^2_{C^{2+\alpha}(\bar{\Omega})} + ||v_0||_{C^{3+\alpha}(\bar{\Omega})}\right]
\end{aligned} \quad (2.30)$$

where C is a constant depending on Ω, T, μ and s_0.

Proof. Let $v_\xi = g$. Then by (2.5) we obtain

$$\begin{cases}
g_t - dA(h)g_{\xi\xi} - B(h)g - B(h)\xi g_\xi = rv_\xi - \frac{2}{K}rvv_\xi, & (\xi,t) \in Q_T, \\
g(0,t) = 0, g_\xi(1,t) = \frac{h^2}{\mu d}, & t \in (0,T), \\
g(\xi,0) = v_0'(\xi), & \xi \in \bar{\Omega},
\end{cases} \quad (2.31)$$

Then, by Theorem 5.3 in [12], we obtain

$$\begin{aligned}
||g||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} &\leq C\left[||r||_{C^{\frac{\alpha}{2}}[0,T]}||v_\xi||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)} + ||r||_{C^{\frac{\alpha}{2}}[0,T]}\|v\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}||v_\xi||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}\right] \\
&+C\left[\|h\|^2_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]} + ||v_0'||_{C^{2+\alpha}(\bar{\Omega})}\right] \\
&\leq C\left[T^{\frac{1}{2}}||r||_{C^{\frac{\alpha}{2}}[0,T]}||v||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} + T^{\frac{3}{2}}||r||_{C^{\frac{\alpha}{2}}[0,T]}\|v\|^2_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}\right] \\
&+C\left[\|h\|^2_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]} + ||v_0'||_{C^{2+\alpha}(\bar{\Omega})}\right].
\end{aligned} \quad (2.32)$$

Furthermore, substituting (2.9) into (2.32), we have

$$\begin{aligned}
||g||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} &\leq C\left[T^{\frac{1}{2}}||r||_{C^{\frac{\alpha}{2}}[0,T]}\left((T+T^2)\|r\|_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right)\right] \\
&+C\left[T^{\frac{3}{2}}||r||_{C^{\frac{\alpha}{2}}[0,T]}\left((T+T^2)\|r\|_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right)^2\right] \\
&+C\left[\left((T+T^2)\|r\|_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right)^2 ||v_0'||_{C^{2+\alpha}(\bar{\Omega})}\right] \\
&\leq C\left[\left(T^{\frac{9}{2}} + T^{\frac{11}{2}}\right)\|r\|^3_{C^{\frac{\alpha}{2}}[0,T]}\right] + C\left[\left(T^{\frac{3}{2}} + T^2 + T^{\frac{5}{2}} + T^4\right)\|r\|^2_{C^{\frac{\alpha}{2}}[0,T]}\right] \\
&+C\left[T^{\frac{1}{2}}||r||_{C^{\frac{\alpha}{2}}[0,T]}||v_0||_{C^{2+\alpha}(\bar{\Omega})}T^{\frac{3}{2}}||r||_{C^{\frac{\alpha}{2}}[0,T]}\|v_0\|^2_{C^{2+\alpha}(\bar{\Omega})}\right] \\
&+C\left[\|v_0\|^2_{C^{2+\alpha}(\bar{\Omega})} + ||v_0||_{C^{3+\alpha}(\bar{\Omega})}\right].
\end{aligned} \quad (2.33)$$

This completes the proof.

3. Inverse Problem

We are now in a position to prove the local existence and uniqueness of our inverse problem (1.1) and (1.2), or its equivalent form (2.2) and (2.3). We will proceed the proof by the contraction mapping.

From (2.2) and (2.3), we deduce

$$v_t(0,t) - d\frac{1}{s^2(t)}v_{\xi\xi}(0,t) = r(t)f(t)(1-\frac{f(t)}{K}). \tag{3.1}$$

Therefore,

$$r(t) = \frac{1}{p(t)}\left[f'(t) - d\frac{1}{s^2(t)}v_{\xi\xi}(0,t)\right] \tag{3.2}$$

with $p(t) = f(t)(1-\frac{f(t)}{K})$.

Now we show the original inverse problem is equivalent to the inverse problem (2.2) and (3.2), i.e. the following theorem.

Theorem 3.1. Let $u(x,t) = v(\xi,t)$, $s(t) = \left(2\int_0^t h(\tau)d\tau + s_0^2\right)^{\frac{1}{2}}$, $\xi = \frac{x}{s(t)}$. Then for sufficient small $T > 0$, the inverse problem (1.1) and (1.2) has a solution $(u,s,r) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T) \times C^{\frac{3}{2}+\frac{\alpha}{2}}[0,T] \times C^{\frac{\alpha}{2}}[0,T]$ if and only if the inverse problem (2.2) and (3.2) has a solution $(v,h,r) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T) \times C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T] \times C^{\frac{\alpha}{2}}[0,T]$.

Proof. This theorem can be proved by using the same method in [19]. For brevity, the detailed proof is omitted.

Theorem 3.2. Let $v_0 \in C^{4+\alpha}(\bar{\Omega})$, $f(t) \in C^{1+\frac{\alpha}{2}}[0,T]$ such that $0 < f(t) < K$. Then there exists sufficient small $T_1 > 0$ such that the problem (2.2) and (3.2) has a solution $(v,h,r) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T) \times C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T] \times C^{\frac{\alpha}{2}}[0,T]$.

Remark 3.1. K indicates the carrying capacity. By the maximum principle, we can obtain $0 \le u \le K$. So our assumption $0 < f(t) < K$ is acceptable in pratice.

Proof. Step 1. We define the mapping.

Let

$$R_{N,T} = \left\{\hat{r} \in C^{\frac{\alpha}{2}}[0,T] \,|\, \|\hat{r}\|_{C^{\frac{\alpha}{2}}[0,T]} \le N\right\}. \tag{3.3}$$

For given $\hat{r} \in R_{N,T}$, we consider

$$\begin{cases} v_t - dA(h)v_{\xi\xi} - B(h)\xi v_\xi = \hat{r}(t)v\left(1-\frac{v}{K}\right), & (\xi,t) \in Q_T, \\ v_\xi(0,t) = v(1,t) = 0, & t \in (0,T), \\ v(\xi,0) = v_0(\xi), & \xi \in \bar{\Omega}, \end{cases} \tag{3.4}$$

and

$$h(t) = -\mu v_\xi(1,t), \quad t \in [0,t]. \tag{3.5}$$

By Theorem 2.1, we know that there exists a sufficient small $T_0 > 0$ such that the problem (3.4) and (3.5) has a unique solution$(v,h) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T) \times C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]$. Then we obtain $r \in C^{\frac{\alpha}{2}}[0,T]$ by (3.2). So, the mapping

$$\begin{aligned} G: R_{N,T} \subset C^{\frac{\alpha}{2}}[0,T] &\to C^{\frac{\alpha}{2}}[0,T], \\ \hat{r}(t) &\mapsto r(t) \end{aligned} \tag{3.6}$$

is well defined.

Step 2. We prove

$$||w||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le C[(T^2+T^4)N^3 + N + 1] \tag{3.7}$$

for sufficiently small T, where $w = v_{\xi\xi}$.

Obviously, by (3.4) we know that w satisfies

$$\begin{cases} w_t - dA(h)w_{\xi\xi} - 2B(h)w - B(h)\xi w_\xi + \frac{2}{K}\hat{r}vw - \hat{r}w = -\frac{2}{K}\hat{r}v_\xi^2, & (\xi,t) \in Q_T, \\ w_\xi(0,t) = 0, w(1,t) = \frac{h^2}{\mu d}, & t \in (0,T), \\ w(\xi,0) = v_0''(\xi), & \xi \in \bar{\Omega}, \end{cases} \tag{3.8}$$

Then, by Theorem 5.3 in [12] again, we obtain

$$||w||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le C\left[||\hat{r}||_{C^{\frac{\alpha}{2}}[0,T]}||v_\xi||^2_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)} + \|h\|^2_{C^{1+\frac{\alpha}{2}}[0,T]} + ||v_0''||_{C^{2+\alpha}(\bar{\Omega})}\right]. \quad (3.9)$$

Applying Theorem 2.1 to (3.4), we have

$$||v||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le C\left[(T+T^2)\|\hat{r}\|_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right]. \quad (3.10)$$

By (3.5), we obtain the following estimate for h:

$$||h||_{C^{1+\frac{\alpha}{2}}[0,T]} \le C||v_\xi||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le CT^{\frac{1}{2}}||w||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}. \quad (3.11)$$

Substituting (3.10) and (3.11) into (3.9) yields

$$\begin{aligned} ||w||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} &\le C\left[||\hat{r}||_{C^{\frac{\alpha}{2}}[0,T]}\left((T+T^2)\|r\|_{C^{\frac{\alpha}{2}}[0,T]} + ||v_0||_{C^{2+\alpha}(\bar{\Omega})}\right)^2\right] \\ &\quad +C\left[T^{\frac{1}{2}}||h||_{C^{1+\frac{\alpha}{2}}[0,T]}||w||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} + ||v_0||_{C^{4+\alpha}(\bar{\Omega})}\right] \\ &\le C\left[(T^2+T^4)N^3 + N + T^{\frac{1}{2}}||h||_{C^{1+\frac{\alpha}{2}}[0,T]} + 1\right] \end{aligned} \quad (3.12)$$

By Theorem 2.2, we have

$$\begin{aligned} T^{\frac{1}{2}}||h||_{C^{1+\frac{\alpha}{2}}[0,T]} &\le CT^{\frac{1}{2}}||v_\xi||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le CT^{\frac{1}{2}}||v||_{C^{3+\alpha,\frac{3}{2}+\frac{\alpha}{2}}(\bar{Q}_T)} \\ &\le CT^{\frac{1}{2}}\left[\left(T^{\frac{9}{2}}+T^{\frac{11}{2}}\right)\|\hat{r}\|^3_{C^{\frac{\alpha}{2}}[0,T]}\right] + CT^{\frac{1}{2}}\left[\left(T^{\frac{3}{2}}+T^2+T^{\frac{5}{2}}+T^4\right)\|\hat{r}\|^2_{C^{\frac{\alpha}{2}}[0,T]}\right] \\ &\quad +CT^{\frac{1}{2}}\left[T^{\frac{1}{2}}||\hat{r}||_{C^{\frac{\alpha}{2}}[0,T]}||v_0||_{C^{2+\alpha}(\bar{\Omega})} + T^{\frac{3}{2}}||\hat{r}||_{C^{\frac{\alpha}{2}}[0,T]}\|v_0\|^2_{C^{2+\alpha}(\bar{\Omega})}\right] \\ &\quad +CT^{\frac{1}{2}}\left[\|v_0\|^2_{C^{2+\alpha}(\bar{\Omega})} + ||v_0||_{C^{3+\alpha}(\bar{\Omega})}\right] \\ &\le CT^{\frac{1}{2}}(N^3+N^2+N+1) \end{aligned} \quad (3.13)$$

So, for given N we can choose sufficient small T such that $CT^{\frac{1}{2}}(N^3+N^2+N+1) < \frac{1}{2}$ and then obtain

$$||w||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le C[(T^2+T^4)N^3+N+1] \quad (3.14)$$

Step 3. We prove the existence of the fixed point of mapping G.

First, we prove $G(R_{N,T}) \subset R_{N,T}$.

By (3.2) and noticing that $0 < f(t) < K$, we obtain

$$\begin{aligned} ||r||_{C^{\frac{\alpha}{2}}[0,T]} &\le C||f'(t) - d\frac{1}{s^2(t)}v_{\xi\xi}(0,t)||_{C^{\frac{\alpha}{2}}[0,T]} \\ &\le C\left(||f||_{C^{1+\frac{\alpha}{2}}[0,T]} + ||v_{\xi\xi}||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}\right) \\ &\le C\left(1+||w||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}\right) \\ &\le C\left(1+T||w||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}\right) \end{aligned} \quad (3.15)$$

By substituting (3.14) into (3.15), we have

$$||r||_{C^{\frac{\alpha}{2}}[0,T]} \le C + CT[(T^2+T^4)N^3+N+1]. \quad (3.16)$$

Then we can choose $N > 2C$ and sufficient small $T > 0$ to obtain

$$||r||_{C^{\frac{\alpha}{2}}[0,T]} \le N, \quad (3.17)$$

i.e. $G(R_{N,T}) \subset R_{N,T}$.

Next, we prove

$$\|G(\hat{r}_1) - G(\hat{r}_2)\|_{C^{\frac{\alpha}{2}}[0,T]} \le \frac{1}{2}\|\hat{r}_1 - \hat{r}_2\|_{C^{\frac{\alpha}{2}}[0,T]} \quad (3.18)$$

for given $\hat{r}_1$ and $\hat{r}_2 \in R_{N,T}$.

Let $(v_i, h_i)(i = 1,2)$ be two solutions of the system (2.5) corresponding to $\hat{r}_1$ and $\hat{r}_2$ respectively, and let $r_1 = G(\hat{r}_1), r_2 = G(\hat{r}_2)$. Then by (3.2) we know that $r_1 - r_2$ satisfies

$$r_1 - r_2 = -\frac{d}{p(t)}\left[A(h_1)v_{1,\xi\xi}(0,t) - A(h_2)v_{2,\xi\xi}(0,t)\right]$$

$$= -\frac{d}{p(t)}\left[A(h_1)v_{1,\xi\xi}(0,t) - v_{2,\xi\xi}(0,t) - \big(A(h_1) - A(h_2)\big)v_{2,\xi\xi}(0,t)\right] \tag{3.19}$$

Let $(V,H) \coloneqq (v_1 - v_2, h_1 - h_2)$. By (3.4) and (3.5), we obtain

$$\begin{cases} V_t - dA(h_1)V_{\xi\xi} - B(h_1)\xi V_\xi = d\left(A(h_1) - A(h_2)\right)v_{2,\xi\xi} \\ +\big(B(h_1) - B(h_2)\big)\xi v_{2,\xi} + \hat{r}_1 V + (\hat{r}_1 - \hat{r}_2)v_2 - \frac{1}{K}(\hat{r}_1 v_1^2 - \hat{r}_2 v_2^2), & (\xi,t) \in Q_T, \\ V_\xi(0,t) = 0, V(1,t) = 0, & t \in (0,T), \\ V(\xi,0) = 0, & \xi \in \bar{\Omega}, \\ H(t) = -\mu V_\xi(1,t), & t \in [0,T]. \end{cases} \tag{3.20}$$

Then, by the standard theory for the linear parabolic equation we obtain

$$\begin{aligned} ||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le{} & C\left[||v_{2,\xi\xi}||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}||A(h_1) - A(h_2)||_{C^{\frac{\alpha}{2}}[0,T]}\right] \\ & +C\left[||v_{2,\xi}||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}||B(h_1) - B(h_2)||_{C^{\frac{\alpha}{2}}[0,T]}\right] \\ & +C\left[||\hat{r}_1 V + (\hat{r}_1 - \hat{r}_2)v_2 - \frac{1}{K}\left((\hat{r}_1 - \hat{r}_2)v_1^2 + \hat{r}_1(v_1^2 - v_2^2)\right)||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)}\right] \end{aligned} \tag{3.21}$$

Similar (2.21) and (2.23), we have

$$||A(h_1) - A(h_2)||_{C^{\frac{\alpha}{2}}[0,T]} \le CT^{\frac{3}{2}}||h_1 - h_2||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}, \tag{3.22}$$

$$||B(h_1) - B(h_2)||_{C^{\frac{\alpha}{2}}[0,T]} \le C\left(T^2 + T^{\frac{1}{2}}\right)||h_1 - h_2||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]} \tag{3.33}$$

Additionally,

$$||H||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]} \le C||V_\xi||_{C^{1+\alpha,\frac{1}{2}+\frac{\alpha}{2}}(\bar{Q}_T)} \le C||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \tag{3.24}$$

Then, by (3.21)-(3.24) and $||(v_i,h_i)|| \le C(i = 1,2)$ due to Theorem 2.1, where C is depending on Ω, T, N, μ and v_0, we further have

$$\begin{aligned} ||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le{} & C\left[(T^{\frac{3}{2}} + T^2 + T^{\frac{1}{2}})||H||_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}\right] \\ & +C\left[T||\hat{r}_1||_{C^{\frac{\alpha}{2}}[0,T]}||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} + T||\hat{r}_1 - \hat{r}_2||_{C^{\frac{\alpha}{2}}[0,T]}||v_1||^2_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}\right] \\ & +C\left[T||\hat{r}_1||_{C^{\frac{\alpha}{2}}[0,T]}||v_1 - v_2||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}||v_1 + v_2||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)}\right]] \\ \le{} & C\left[\left(T^{\frac{3}{2}} + T^2 + T^{\frac{1}{2}}\right) + TN||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} + T||\hat{r}_1 - \hat{r}_2||_{C^{\frac{\alpha}{2}}[0,T]}\right]. \end{aligned} \tag{3.25}$$

Therefore, for sufficient small T, we have

$$||V||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} \le CT||\hat{r}_1 - \hat{r}_2||_{C^{\frac{\alpha}{2}}[0,T]} \tag{3.26}$$

On the other hand, by (3.19) we have

$$\begin{aligned} ||r_1 - r_2||_{C^{\frac{\alpha}{2}}[0,T]} & \le C\left(||v_{1,\xi\xi} - v_{2,\xi\xi}||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)} + ||A(h_1) - A(h_2)||_{C^{\frac{\alpha}{2}}[0,T]}\right) \\ & \le C\left(\|V\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)} + T^{\frac{3}{2}}\|H\|_{C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]}\right) \end{aligned} \tag{3.27}$$

Thus, by (3.24), (3.26) and (3.27) we find that

$$||r_1 - r_2||_{C^{\frac{\alpha}{2}}[0,T]} \le C\left(T + T^{\frac{5}{2}}\right)||\hat{r}_1 - \hat{r}_2||_{C^{\frac{\alpha}{2}}[0,T]}, \tag{3.28}$$

which implies (3.18), if we choose T sufficient small such that

$$C\left(T + T^{\frac{5}{2}}\right) \le \frac{1}{2}.$$

Therefore the Banach fixed point theorem concludes that for a small time T, there exists a unique solution $r \in C^{\frac{\alpha}{2}}[0,T]$. For given $r \in C^{\frac{\alpha}{2}}[0,T]$, we have a unique solution $(v,h) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T) \times C^{\frac{1}{2}+\frac{\alpha}{2}}[0,T]$ by Theorem 2.1, which completes the proof of Theorem 3.2. ■

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