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STUDY OF CONVERGENCE OF LAGUERRE WAVELET BASED NUMERICAL METHOD FOR INITIAL AND BOUNDARY VALUE BRATU-TYPE PROBLEMS

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Abstract. Based on Laguerre wavelets, an efficient numerical method is proposed for the numerical solution of initial and boundary value Bratu-type problems arising in fuel ignition of the combustion theory and heat transfer. Convergence of the method for these kinds of problems is addressed in the form of theorems with proof. To illustrate the ability of the method is validated on test problems, numerical results are compared with exact and those from existing methods in the literature. The results demonstrate the accuracy and the efficiency of the Laguerre wavelet based numerical method.

Keywords: Laguerre wavelet method, Bratu's problem, Convergence method, Limit point.

1. Introduction

Wavelet theory is a relatively new and an emerging area in mathematical research. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms. The main advantage of using orthogonal basis is that it reduces the problem into solving a system of algebraic equations. In recent years, the wavelets are dealing with dynamic system problems, especially in solving differential equations with two-point boundary value constraints have been discussed in many papers [13, 19]. By transforming differential equations into algebraic equations, the solution may be found by determining the corresponding coefficients that satisfy the algebraic equations. Some efforts have been made to solve Bratu's problem by using the wavelet collocation method [19].

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either analytically or numerically. Now a day's our attention devoted to the search for better and more efficient solution methods for determining a solution. We mention that the spectral collocation method is very useful in providing highly accurate solutions to nonlinear differential equations [11]. Here, we intend to extend the application of Laguerre wavelet method to solve nonlinear initial value problems and boundary value problems of Bratu type. To the best of our knowledge, there are no results on Laguerre wavelet approximation for Bratu type equations arising in mathematical physics. This partially motivated our interest in such method. The aim of this paper is to study convergence of Laguerre wavelet method for boundary and initial value Bratu-type problems [1, 18, 19].

It is well known that Bratu's boundary value problem in one-dimensional planar coordinates is of the form

$$y'' + \beta e^{y} = 0, \;\; 0 < x < 1 \tag{1.1}$$

with boundary conditions $y(0) = y(1) = 0$. for $\beta > 0$ is constant, the exact solution of (1.1) is given by [5],

$$y(x) = -2ln\left[\frac{cosh\left(\frac{\theta(x-\frac{1}{2})}{2}\right)}{cosh\left(\frac{\theta}{2}\right)}\right] \tag{1.2}$$

where θ satisfies,

$$\theta = \sqrt{2\beta} sinh\left(\frac{\theta}{2}\right) \tag{1.3}$$

The problem has zero, one and two solutions when $\beta > \beta_c, \beta = \beta_c, and\ \beta < \beta_c$, respectively, where the critical value β_c satisfies the equation,

$$1 = \frac{1}{4}\sqrt{2\beta_c}\cosh\left(\frac{\theta}{4}\right).$$

The critical value β_c is given by $\beta_c = 3.513830719$ [3, 7, 8].

In addition, an initial value problem of Bratu's type,

$$y'' + \beta e^y = 0, \quad 0 < x < 1 \tag{1.4}$$

with initial conditions $y(0) = y'(0) = 0$ will be investigated.

Applications of the Bratu type equations are employed in the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the universe, chemical reaction theory, radioactive heat transfer and nanotechnology [6]. A substantial amount of research work has been directed for the study of the Bratu problem [3, 6, 10, 17]. Several numerical techniques, such as the finite difference method, finite element approximation, weighted residual method, and the shooting method, have been implemented independently to handle the Bratu model. In addition, Boyd [6] employed Chebyshev polynomial expansions and the Gegenbauer as base functions. Syam and Hamdan [15] presented the Laplace Adomain decomposition method (LADM) for solving Bratu's problem.

In this paper, Laguerre wavelet based numerical method is presented for the approximate solution of Bratu's problem. The method is based on expanding the solution by Laguerre wavelets with unknown coefficients. The properties of Laguerre wavelets together with the collocation method are utilized to evaluate the unknown coefficients and then an approximate solution to eq. (1.1) is identified.

The organization of the rest of the paper is as follows. In section 2, properties of Laguerre wavelet are described. In section 3, formulation of the method based on Laguerre wavelet is defined for initial and boundary value Bratu-type problems. Analysis of the Laguerre wavelet method for Bratu-type problems is presented in section 4 in the form of theorems with proof. Numerical results are reported in section 5, and finally conclusions are drawn in section 6.

2. Properties of Laguerre wavelet

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter ***a*** and the translation parameter ***b*** varies continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}}\,\psi\left(\frac{x-b}{a}\right), a, b \in R, a \neq 0.$$

If we restrict the parameters ***a*** and ***b*** to discrete values as

$$a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0,$$

we have the following family of discrete wavelets :

$$\psi_{k,n}(x) = |a_0|^{\frac{1}{2}}\,\psi\left(a_0^k x - nb_0\right).$$

Where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2 \text{ and } b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis.

The Laguerre wavelets $\psi_{k,n}(x) = \psi(k, n, m, x)$ involve four arguments *n =1, 2, 3,...,* 2^{k-1}, *k* is assumed any positive integer, *m* is the degree of the Laguerre polynomials and it is the Normalized time. They are defined on the interval [0, 1) as

$$\psi_{k,n}(x) = \begin{cases} 2^{\frac{k}{2}}\bar{L}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0 & otherwise \end{cases} \tag{2.1}$$

$$\text{where } \bar{L}_m(x) = \frac{L_m}{m!} \tag{2.2}$$

m = 0, 1, 2... M-1. In eq. (1.2) the coefficients are used for orthonormality. Here $L_m(x)$ are the Laguerre polynomials of degree m with respect to the weight function W(x) =1 on the interval [0,∞) and satisfy the following recursive formula $L_0(x) = 1$, $L_1(x) = 1 - x$,

$$L_{m+2}(x) = \frac{(2m + 3 - x)L_{m+1}(x) - (m + 1)L_m(x)}{m + 2}, \qquad m = 0,1,2 \ldots$$

A function $y(x)$ defined over [0, 1) can be expanded as a laguerre wavelet series as follows:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x)$$

where $\psi_{n,m}(x)$ is given by the eq. (2.1). We approximate y(x) by truncated series

$$y(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\psi(x) \quad (2.3)$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C^T = [C_{1,0},\dots,C_{1,M-1,}C_{2,0},\dots,C_{2,M-1},\dots,C_{2^{k-1},0},\dots,C_{2^{k-1},M-1}].$$
$$\psi(x) = [\psi_{1,0},\dots,\psi_{1,M-1,}\psi_{2,0},\dots,\psi_{2,M-1},\dots,\psi_{2^{k-1},0},\dots,\psi_{2^{k-1},M-1}].$$

3. Laguerre wavelet numerical method of solution

Consider Bratu's problem given in eq. (1.1), in order to use Laguerre wavelet, we first approximate $y(x)$ as

$$y(x) = C^T\psi(x) \quad (3.1)$$

Here,

$$C^T = [C_{1,0},\dots,C_{1,M-1,}C_{2,0},\dots,C_{2,M-1},\dots,C_{2^{k-1},0},\dots,C_{2^{k-1},M-1}].$$
$$\psi(x) = [\psi_{1,0},\dots,\psi_{1,M-1,}\psi_{2,0},\dots,\psi_{2,M-1},\dots,\psi_{2^{k-1},0},\dots,\psi_{2^{k-1},M-1}].$$

Applying eq. (3.1) in eq. (1.1) we get,

$$\frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) + \beta\exp\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x)\right) = 0. \quad (3.2)$$

Then a total number of $2^{k-1}M$ equations should exist to determine the $2^{k-1}M$ coefficients such as,

$$C_{10}, C_{11},,\dots,C_{1M-1,}C_{20}, C_{21},\dots,C_{2M-1},\dots,C_{2^{k-1}0}, C_{2^{k-1}1},\dots,C_{2^{k-1}M-1}.$$

Since two conditions are furnished by the boundary conditions, namely

$$\begin{cases} y_{k,M}(0) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(0) = 0 \\ y_{k,M}(1) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(1) = 0 \end{cases} \quad (3.3)$$

or initial conditions, namely

$$\begin{cases} y_{k,M}(0) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(0) = 0 \\ \frac{d}{dx}y_{k,M}(0) = \frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(0) = 0 \end{cases} \quad (3.4)$$

Further we need $2^{k-1}M - 2$ equations to recover the unknown coefficients $C_{n,m}$. These equations can be obtained by collocating eq. (3.2):

We, now assume eq. (3.2) is exact at $2^{k-1}M - 2$ points x_i as follows:

$$\frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_i) + \beta\exp\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_i)\right) = 0. \quad (3.5)$$

Suitable collocation points are limit points of following sequence:

$$\{x_i\} = \{\frac{1}{2}\left(1 + \cos\frac{(i-1)\pi}{2^{k-1}M-1}\right)\} \quad i = 2.3,\dots. \quad (3.6)$$

Combine eqs. (3.5) and (3.3) or (3.5) and (3.4) to obtain $2^{k-1}M$ equations from which we can compute values for the unknown coefficients $C_{n,m}$. Same procedure is repeated for differential equations of higher order also.

4. Analysis of the Laguerre wavelet method for Bratu-type problems

Theorem 1. The series of Laguerre wavelet $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x)$ is converges to $y(x)$.

Proof: $L^2(R)$ be the infinite dimensional Hilbert space and $\psi_{n,m}(x)$ is defined as

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}}\bar{L}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}} \\ 0 & \text{otherwise} \end{cases}$$

forms orthonormal basis,

Let $y(x) = \sum_{i=0}^{M-1} C_{\eta,i}\psi_{\eta,i}(x)$ where $C_{\eta,i} = < y(x), \psi_{\eta,i}(x) >$ for fixed η.

Let us define the sequence of partial sums S_n of { $C_{\eta,i}\psi_{\eta,i}(x)$ }, let S_n and S_m are the partial sums with n ≥ m. We have to prove S_n is Cauchy sequence in Hilbert space $L^2(R)$.

Choose $S_n = \sum_{i=0}^{n} C_{\eta,i}\psi_{\eta,i}(x)$,

Now $< y(x),\ S_n > = < y(x), \sum_{i=0}^{n} C_{\eta,i}\psi_{\eta,i}(x) > = \sum_{i=m+1}^{n} |C_{\eta,i}|^2$

We claim that $||\, S_n - S_m||^2 = \sum_{i=m+1}^{n} |C_{\eta,i}|^2 \quad \forall\ n > m$

Now $||\sum_{i=m+1}^{n} C_{\eta,i}\psi_{\eta,i}(x)\, ||^2 =< \sum_{i=m+1}^{n} C_{\eta,i}\psi_{\eta,i}(x), \sum_{i=m+1}^{n} C_{\eta,i}\psi_{\eta,i}(x) >$

$$||\sum_{i=m+1}^{n} C_{\eta,i}\psi_{\eta,i}(x)\, ||^2 = \sum_{i=m+1}^{n} |C_{\eta,i}|^2 \qquad \forall\ n > m$$

By Bessel's Inequality

Since $\sum_{i=m+1}^{n} |C_{\eta,i}|^2 \le ||y(x)||^2$,

Therefore $\sum_{i=1}^{n} |C_{\eta,i}|^2$ is bounded and convergent.

Hence $||\sum_{i=m+1}^{n} C_{\eta,i}\psi_{\eta,i}(x)\, ||^2 \to 0\ as\ m, n \to \infty$.

This implies $||\sum_{i=m+1}^{n} C_{\eta,i}\psi_{\eta,i}(x)\, || \to 0$.

Therefore { S_n } is a Cauchy sequence and it converges to *K* (say)

We assert that $y(x) = K$

Now $< K - y(x), \psi_{\eta,i}(x) >=< K, \psi_{\eta,i}(x) > -< y(x), \psi_{\eta,i}(x) >$

$< K - y(x), \psi_{\eta,i}(x) >=< K, \psi_{\eta,i}(x) > -< \lim_{n\to\infty} S_n, \psi_{\eta,i}(x) >= 0$

Therefore $< K - y(x), \psi_{\eta,i}(x) >= 0$

Hence $y(x) = K$ and $\sum_{i=0}^{n} C_{\eta,i}\psi_{\eta,i}(x)$ converges to $y(x)$ as $n \to \infty$ and hence proved.

Theorem 2. If $\Psi_{i,j}$'s are Laguerre wavelets then $\Psi_{i,j}$'s are uniformly continuous on interval I= [0, 1).

Proof: Laguerre Wavelets are lipschitz functions.

Then given $\varepsilon > 0$, we can choose $\delta = \frac{\varepsilon}{k}$.

If $x_1, x_2 \in I$ with $|x_1 - x_2| < \delta$ then $\left|\Psi_{i,j}(x_1) - \Psi_{i,j}(x_2)\right| < k \cdot \frac{\varepsilon}{k} = \varepsilon$.

Therefore $\Psi_{i,j}$ are uniformly continuous on interval I.

Corollary: If Laguerre wavelets $\Psi_{i,j}$ are uniformly continuous on interval I then they are continuous.

Theorem 3. If $\Psi_{i,j}: I \to R$ is uniformly continuous on subset I of R and $\{x_n\}$ is a Cauchy sequence in I then $\{\Psi_{i,j}(x_n)\}$ is Cauchy sequence in R.

Proof. Let $\{x_n\}$ is Cauchy sequence in I. $\varepsilon > 0$ being given.

Choose $\delta > 0$ such that If $x_1, x_2 \epsilon I$ satisfy $|x_1 - x_2| < \delta \quad \forall\, n, m \ge \delta_1$ (by choice of δ). We have $\left|\Psi_{i,j}(x_1) - \Psi_{i,j}(x_2)\right| < \varepsilon$.

Hence $\{\Psi_{i,j}(x_n)\}$ is a Cauchy sequence.

Theorem 4. Suppose that $y(x) \in C^m[0,1]$ and $C^T\psi(x)$ is the approximate solution using Laguerre wavelet. Then the error bound would be given by

$$||E(x)|| \le ||\frac{2}{m!\, 4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\, ||.$$

Proof. Applying the definition of norm in the inner product space, we have,

$$||E(x)||^2 = \int_0^1 [y(x) - C^T\psi(x)]^2 dx.$$

Divide interval [0, 1] into 2^{k-1} subintervals $I_n = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right], n = 1,2,3, \dots, 2^{k-1}$.

$$||E(x)||^2 = \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [y(x) - C^T\psi(x)]^2 dx.$$

$$||E(x)||^2 \le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [y(x) - p_m(x)]^2 dx.$$

Where $p_m(x)$ is the interpolating polynomial of degree m which approximates $y(x)$ on I_n.
By using the maximum error estimate for the polynomial on I_n, then

$$||E(x)||^2 \le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!\,4^m 2^{m(k-1)}} \max_{x\in I_n.} |y^m(x)|\right]^2 dx$$

$$||E(x)||^2 \le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!\,4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\right]^2 dx$$

$$||E(x)||^2 = \int_0^1 \left[\frac{2}{m!\,4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\right]^2 dx$$

$$||E(x)|| \le ||\frac{2}{m!\,4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\,||$$

Where we have used the well-known maximum error bound for the interpolation.

5. Numerical case studies

Example 5.1 First, consider the initial value problem [12],

$$y'' - 2e^y = 0,\ 0 < x < 1, \tag{5.1}$$

$$y(0) = 0,\ y'(0) = 0. \tag{5.2}$$

The exact solution is $y(x) = -2\ln(\cos(x))$.

Here, solving it using Laguerre wavelet, with *k* =1, *M*= 5, 10. First we assume that the unknown function $y(x)$ is given by,

$$y(x) = C^T\psi(x). \tag{5.3}$$

Collocating the above initial value problem with collocation point, we get,

$$\frac{d^2}{dx^2} C^T\psi(x_i) + \beta \exp\left(C^T\psi(x_i)\right) = 0 \tag{5.4}$$

Using the initial conditions, we obtain,

$$C^T\psi(0) = 0, \qquad \frac{d}{dx}[C^T\psi(0)] = 0 \tag{5.5}$$

Equations (5.4) and (5.5) generate a system of nonlinear equations. These equations can be solved for unknown coefficients of the vector *C*. A comparison between the exact and the approximate solutions is demonstrated in Fig. 5.1. From this, it can be found that the obtained approximate solutions are very close to the exact solution. In addition, Tables 5.1, 5.2, and 5.3 shows that how approximate solutions obtained by this method is nearer to exact solution and comparison of present method with other methods. The results of the proposed method are more accurate.

Table 5.1 Numerical results of the example 5.1.

X	Exact solution	Laguerre wavelet solution	
		k=1, M=5	k=1, M=10
0.1	0.010016711246471	0.010945019574018	0.010015723120887
0.2	0.040269546104817	0.042843068920563	0.040268130498589
0.3	0.091383311852116	0.095192376400338	0.091380154233775
0.4	0.164458038150111	0.168696160272670	0.164454336540075
0.5	0.261168480887445	0.265262628695516	0.261164512747200
0.6	0.383930338838875	0.388004979725458	0.383924231282435

0.7	0.536171515135862	0.541241401317702	0.536163668075460
0.8	0.722781493622688	0.730495071326083	0.722773470872198
0.9	0.950884887171629	0.962494157503059	0.950873803594732
1	1.231252940772028	1.245171817499718	1.231239850909333

Table 5.2. Comparison of the absolute error of the example 5.1.

x	**PIA(1,3) algorithm [2]**	**Legendre wavelet [18]**	**Present Method (M=10,k=1)**
0.1	6.71×10^{-6}	9.00×10^{-8}	9.88×10^{-7}
0.2	9.55×10^{-6}	1.5×10^{-7}	1.41×10^{-6}
0.3	3.11×10^{-6}	6.14×10^{-7}	3.15×10^{-6}
0.4	8.04×10^{-6}	8.88×10^{-6}	3.70×10^{-6}
0.5	8.48×10^{-6}	5.67×10^{-5}	3.96×10^{-6}
0.6	2.03×10^{-5}	2.55×10^{-4}	6.10×10^{-6}
0.7	7.15×10^{-5}	9.24×10^{-4}	7.84×10^{-6}
0.8	2.91×10^{-4}	2.90×10^{-3}	8.02×10^{-6}
0.9	1.05×10^{-3}	7.90×10^{-3}	1.10×10^{-5}
1	3.53×10^{-3}	1.00×10^{-3}	1.30×10^{-5}

Table 5.3 Maximum absolute error of the example 5.1 for different values of M.

k	***M***	**Maximum absolute error**
1	5	1.39×10^{-2}
	10	1.30×10^{-5}

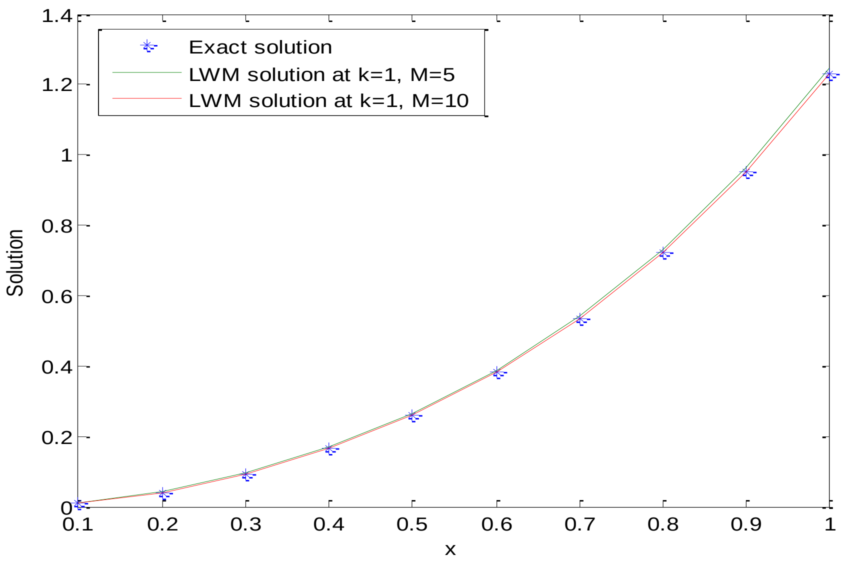

Fig. 5.1 Comparison of exact and approximate solutions for Example 5.1 for different value of M.

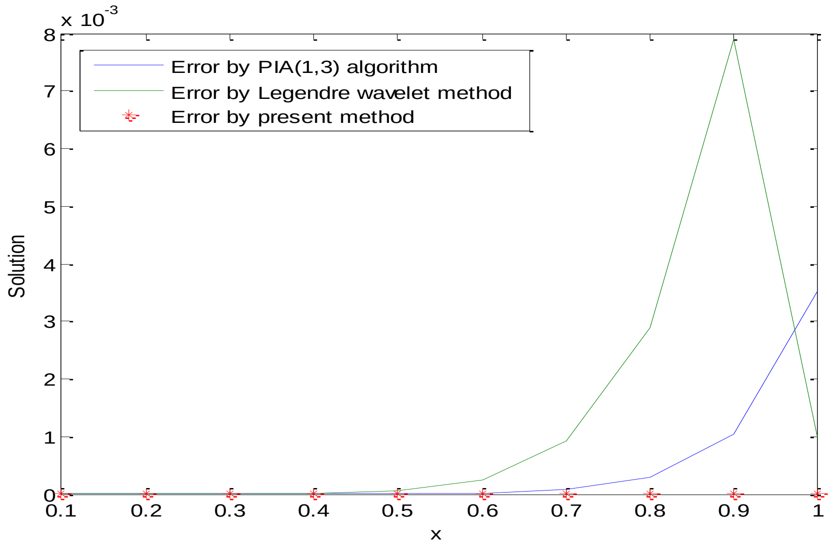

Fig. 5.2 Graphical representation of absolute error of the example 5.1 and other methods with Exact solution.

Example 5.2. Next, consider the another initial value problem,

$$y'' - e^{y} = 0,\ 0 < x < 1 \tag{5.6}$$

$$y(0) = 0,\ y'(0) = 1. \tag{5.7}$$

The exact solution is $y(x) = -\ln(1 - \sin(x))$.

Solving the above equation by using the Laguerre wavelet method with k = 1, M =5, 10. The numerical results obtained are presented in Table 5.4 & Table 5.5 and also in Figure 5.2, which shows the comparison between the exact and approximate solutions for various values of *M* (with *k* = 1). Moreover, higher accuracy can be achieved by taking higher order approximations.

Table 5.4 Numerical results of the example 5.2.

X	**Exact solution**	**Laguerre wavelet solution**	
		k=1, M=5	***k=1, M=10***
0.1	0.105175440170716	0.106162141310651	0.105174456039094
0.2	0.221481596620136	0.224204367051520	0.221480195643380
0.3	0.350295630327762	0.354307936760044	0.350292515140609
0.4	0.493343238943652	0.497784185739087	0.493339627370005
0.5	0.652822343722163	0.657074525056935	0.652818539922009
0.6	0.831587685227202	0.835750441547304	0.831581852777163
0.7	1.033436216165614	1.038513497809331	1.033428798997193
0.8	1.263567445357214	1.271195332207581	1.263560062376200
0.9	1.529365557326900	1.540757658872044	1.529355398932239
1	1.841817641269531	1.855292267698133	1.841805866322829

Table 5.5 Maximum absolute error of the example 5.2 for different values of *M*.

k	**M**	**Maximum absolute error**
1	5	1.34×10^{-2}
	10	1.17×10^{-5}

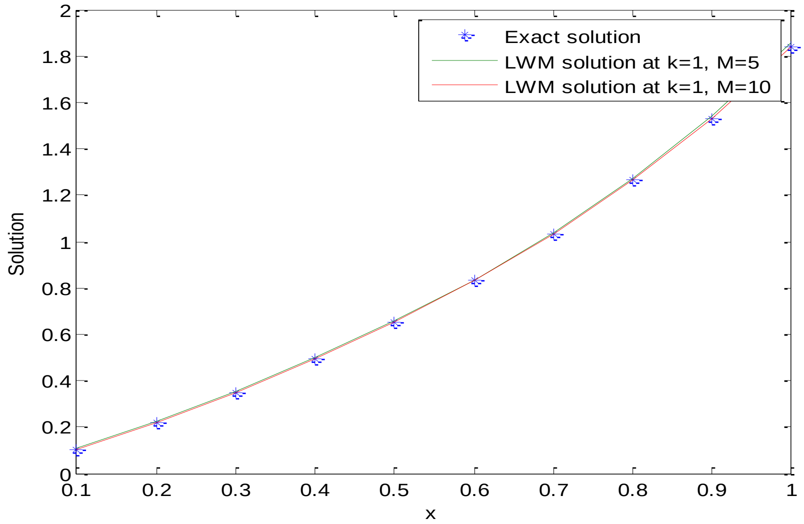

Fig. 5.3 Comparison of exact and approximate solutions for Example 5.2

Example 5.3. Now, consider the boundary value problem as first case for Bratu's equation is as follows, when $\beta = 2$ [9, 18],

$$y'' + 2e^{y} = 0, \; 0 < x < 1 \tag{5.8}$$

$$y(0) = 0, \; y(1) = 0. \tag{5.9}$$

We solve the equation by using the Laguerre wavelets method with $k = 1$, $M = 5$, 10. The numerical results obtained are presented in Table 5.6. It shows the comparison between the exact and approximate solutions for various values of M (with $k = 1$). Table 5.7 & 5.8 shows Laguerre wavelet method is better than Laplace, Decomposition, and B-spline methods.

Table 5.6 Numerical results of the example 5.3.

x	Exact solution	Laguerre wavelet solution	
		***k*=1, *M*=5.**	***k*=1, *M*=10**
0.1	0.1144107440	0.114477030814153	0.114410762963416
0.2	0.2064191156	0.206747974250747	0.206419080092473
0.3	0.2738793116	0.274387890707259	0.273879379325990
0.4	0.3150893646	0.315664680467602	0.315089388134766
0.5	0.3289524214	0.329539083702122	0.328952356854643
0.6	0.3150893646	0.315664680467602	0.315089388134766
0.7	0.2738793116	0.274387890707260	0.273879379325535
0.8	0.2064191156	0.206747974250748	0.206419080092473
0.9	0.1144107440	0.114477030814155	0.114410762962962

Table 5.7 Comparison of numerical solutions with exact solution of the Example 5.3.

x	Exact solution	Laplace[13]	Decomposition [12]	B-spline [9]	Present Method[*k*=1,*M*=10]
0.1	0.1144107440	0.1122817141	0.0991935000	0.1143935651	0.1144107629
0.2	0.2064191156	0.2022094162	0.1917440000	0.2063865190	0.2064190800
0.3	0.2738793116	0.2676925058	0.2679915000	0.2738344125	0.2738793793
0.4	0.3150893646	0.3070874506	0.3183360000	0.3150365062	0.3150893881
0.5	0.3289524214	0.3193532294	0.3359375000	0.3288968072	0.3289523568

0.6	0.3150893646	0.3041598403	0.3183360000	0.3150365062	0.3150893881
0.7	0.2738793116	0.2619458909	0.2679915000	0.2738344125	0.2738793793
0.8	0.2064191156	0.1940413072	0.1917440000	0.2063865190	0.2064190800
0.9	0.1144107440	0.1035373785	0.0991935000	0.1143935651	0.1144107629

Table 5.8 Maximum absolute error of the example 5.3 for different values of *M*.

Laplace[13]	**Decomposition [12]**	**B-spline [9]**	**Present Method[k=1,M=5]**	**Present Method[k=1,M=10]**
1.08×10^{-2}	1.52×10^{-1}	5.56×10^{-5}	5.90×10^{-4}	6.80×10^{-8}

Example 5.4. Next consider the another boundary value Bratu-type problem [16],

$$y'' + \pi^2 e^{-y} = 0, 0 < x < 1 \quad (5.10)$$

$$y(0) = 0, \ y(1) = 0. \quad (5.11)$$

Given equation is different from the standard Bratu-type Problem by the term e^{-y} and $\beta = \pi^2 > \beta_c$ the effect of these changes will be examined. Using Laguerre wavelets method, the exact solution can be obtained by considering the boundary condition $y(0) = 0, \ y(1) = 0$, and exact solution is $y(x) = ln\,[1 + sin\,(\pi x)]$. Numerical findings are presented in Table 5.9, 5.10 and in Fig. 5.3.

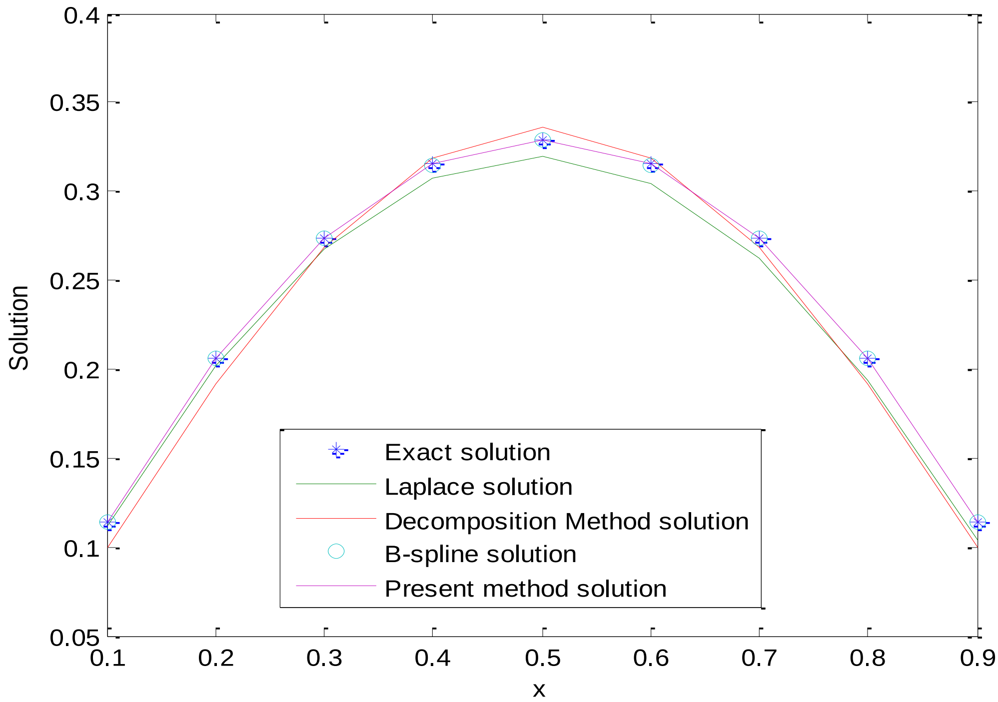

Fig. 5.4 Comparison of exact with different methods approximate solutions for Example 5.3.

Table 5.9 Numerical results of the example 5.4.

x	**Exact solution**	**LWM solution Error with Exact solution** k=1, M=5.	k=1, M=10
0.1	0.269276469559262	0.268594715178942	0.269277891551610
0.2	0.462340122126475	0.463242870933833	0.462338071054546
0.3	0.592783600716708	0.594638600969557	0.592786404384242
0.4	0.668371029081564	0.670420572218170	0.668372071530030
0.5	0.693147180559945	0.695171984838904	0.693144450611726

0.6	0.668371029081564	0.670420572218170	0.668372071537306
0.7	0.592783600716708	0.594638600969557	0.592786404398794
0.8	0.462340122126475	0.463242870933829	0.462338071069098
0.9	0.269276469559262	0.268594715178935	0.269277891573438

Table 5.9 Maximum absolute error of the example 5.4 for different values of M.

k	M	Absolute error
1	5	2.04×10^{-3}
	10	2.80×10^{-6}

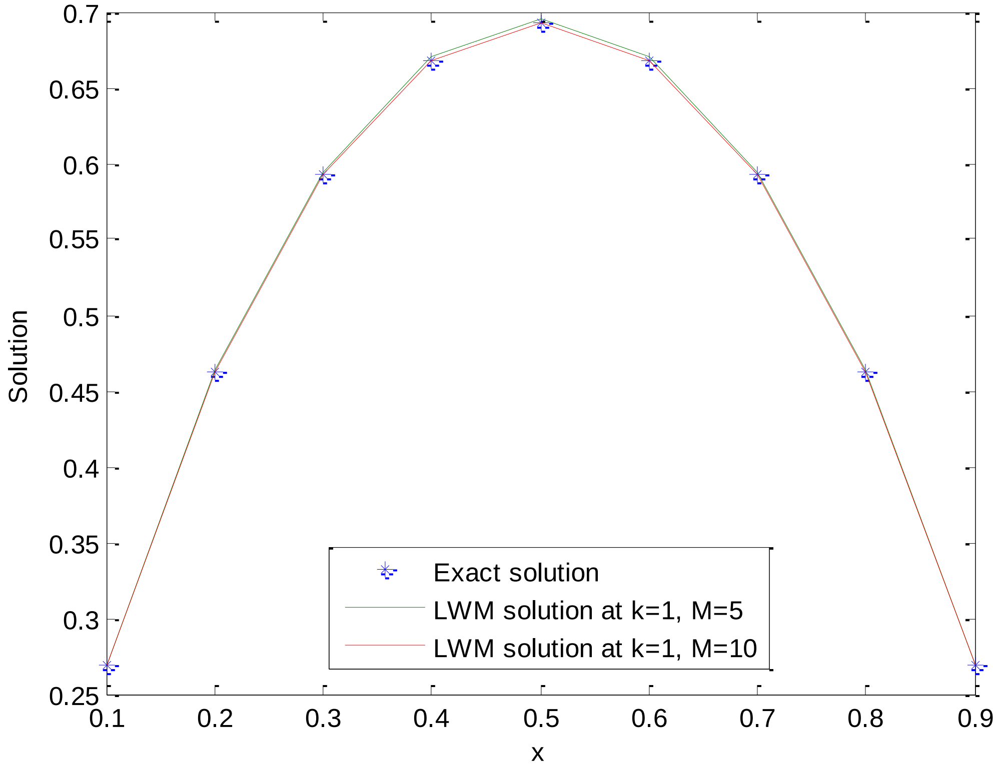

Fig. 5.5 Comparison of exact and approximate solutions for Example 5.4

6. Conclusions

Laguerre wavelet method has been successfully utilized to initial and boundary value Bratu-type problems. Convergence and an error estimate are also studied. Different types of Bratu-type problems can be solved numerically by the present method more accurately than the others. It can be concluded that the Laguerre wavelet method is a powerful and efficient technique in finding very good solutions for these kinds of Bratu's problems.

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