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A New Coding Theory on Fibonacci n-Step Polynomials

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Abstract. In this paper, we develop a new series of Fibonacci n-step polynomials. Based on these series of polynomials, we introduce a new class of square matrix of order n. Thereby, we define a new coding theory called Fibonacci $\boldsymbol{n}$-step polynomials coding theory. Then we calculate the generalized relations among the code elements for all values of n. It is shown that, for $n = 2$, the correct ability of this method is 93.33% whereas for $n = 3$, the correct ability of this method is 99.80%. The interesting part of this coding/decoding method is that the correct ability does not depend on x and increases as n increases.

Keywords: Fibonacci numbers, Fibonacci n-step numbers, Fibonacci polynomials, Fibonacci n-step polynomials, Fibonacci n-step polynomials coding, Error correction.

1. Introduction

The Fibonacci numbers F_k $(k = 0, \pm1, \pm2, \pm3, \ldots)$ is defined by the second-order linear recurrence relation:

$$F_{k+1} = F_k + F_{k-1} \tag{1}$$

with the initial terms $F_0 = 0, F_1 = 1$.

The Fibonacci polynomials are the extension of the Fibonacci numbers, defined by the recurrence relation

$$F_{k+1}(x) = xF_k(x) + F_{k-1}(x) \tag{2}$$

with the initial terms $F_0(x) = 0, F_1(x) = 1$.

The Fibonacci n-step numbers [15] $F_k^{(n)}$ are the generalizations of the Fibonacci numbers, defined by the recurrence relation

$$F_k^{(n)} = F_{k-1}^{(n)} + F_{k-2}^{(n)} + \ldots + F_{k-n}^{(n)},$$

$$F_0^{(n)} = F_1^{(n)} = \cdots = F_{n-2}^{(n)} = 0, F_{n-1}^{(n)} = 1 \tag{3}$$

for $k = 0, \pm1, \pm2, \pm3, \ldots$ and $n = 1, 2, 3, \ldots$.

The first few sequence of Fibonacci n-step numbers are summarized in the Table 1.

Table 1. Fibonacci n-step Numbers

k	-5	-4	-3	-2	-1	0	1	2	3	4	5	Name
$F_k^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	Degenerate
$F_k^{(2)}$	5	-3	2	-1	1	0	1	1	2	3	5	Fibonacci Numbers
$F_k^{(3)}$	-3	2	0	-1	1	0	0	1	1	2	4	Tribonacci Numbers
$F_k^{(4)}$	2	0	0	-1	1	0	0	0	1	1	2	Tetranacci Numbers
$F_k^{(5)}$	0	0	0	-1	1	0	0	0	0	1	1	Pentanacci Numbers

For $k \geq n-1, r_n = \lim_{k\to\infty} \frac{F_k^{(n)}}{F_{k-1}^{(n)}}$ exists, called n- annaci constant and is the real root ≥ 1 of the equation

$$x^n - x^{n-1} - x^{n-2} - \cdots - x - 1 = 0$$

For even n, there are exactly two real roots, one is > 1 and one is < 1 and for odd n, there is exactly one real root, which is always ≥ 1, the equality sign hold if and only if $n = 1$.

Actually, $r_1 = 1, r_2 = 1.61803$ called Golden mean, $r_3 = 1.83929$ called Tribonacci constant, $r_4 = 1.92756$ called Tetranacci constant, $r_5 = 1.96595$ called Pentanacci constant etc. and $\lim_{n\to\infty} r_n = 2$ [6]. In 2006, A. P. Stakhov introduces a new coding theory on Fibonacci matrices [14]. In 2009, M. Basu and B. Prasad describe the generalized relations among the code elements for Fibonacci coding theory [1] and Coding theory on the m-extension of the Fibonacci p-numbers [2]. In 2010, M. Esmaeili and M. Esmaeili introduce a Fibonacci-polynomial based coding method with error detection and correction [12]. After that, M. Basu and M. Das present a new coding theory on Tribonacci matrices [3], coding theory on Fibonacci n-step numbers [4] and coding theory on constant coefficient Fibonacci n-step numbers [5].

In this paper we define Fibonacci n-step polynomials and Fibonacci n-step polynomial matrix of order n. Thereby, we illustrate a new coding theory called Fibonacci n-step polynomials coding theory along with its properties.

Definition 1.1. The Fibonacci n-step polynomials $F_k^{(n)}(x)$ are defined by the recurrence relation

$$F_k^{(n)}(x) = x^{n-1}F_{k-1}^{(n)}(x) + x^{n-2}F_{k-2}^{(n)}(x) + \dots + F_{k-n}^{(n)}(x), \qquad (4)$$

with the initial terms $F_0^{(n)}(x) = F_1^{(n)}(x) = \cdots = F_{n-2}^{(n)}(x) = 0, F_{n-1}^{(n)}(x) = 1$, for $k = 0, \pm1, \pm2, \pm3, \dots$ and $n = 1, 2, 3, \dots$.

The first few sequence of Fibonacci n-step polynomials are summarized in the Table 2.

Table 1. Fibonacci n-step polynomials

k	-5	-4	-3	-2	-1	0	1	2	3	4	5
$F_k^{(1)}(x)$	1	1	1	1	1	1	1	1	1	1	1
$F_k^{(2)}(x)$	x^4+3x^2+1	$-(x^3+2x)$	x^2+1	$-x$	1	0	1	x	x^2+1	(x^3+2x)	x^4+3x^2+1
$F_k^{(3)}(x)$	$-(x^4+2x)$	x^3+1	0	$-x$	1	0	0	1	x^2	(x^4+x)	x^6+2x^3+1
$F_k^{(4)}(x)$	(x^4+1)	0	0	$-x$	1	0	0	0	1	x^3	x^6+x^2
$F_k^{(5)}(x)$	0	0	0	$-x$	1	0	0	0	0	1	x^4

Definition 1.2. n- annaci polynomial ratio is defined by $\frac{F_k^{(n)}(x)}{F_{k-1}^{(n)}(x)}, k \geq n-1$. Taking $k \to \infty$, we have $\lim_{k\to\infty} \frac{F_k^{(n)}(x)}{F_{k-1}^{(n)}(x)}$ exists [13] and $\lim_{k\to\infty} \frac{F_k^{(n)}(x)}{F_{k-1}^{(n)}(x)} = r_n(x)$, say which depends on x.

For example, $r_2(x) = \frac{x+\sqrt{x^2+4}}{2}$,

$$r_3(x) = \frac{2^{\frac{1}{3}}x^2 + (2x^6+9x^3+27+\sqrt{-19x^6+378x^3+729})^{\frac{1}{3}} + (2x^6+9x^3+27-\sqrt{-19x^6+378x^3+729})^{\frac{1}{3}}}{2^{\frac{1}{3}} \times 3}.$$

Definition 1.3. Fibonacci n-step polynomial matrix $M_n(x)$ is a square matrix of order n and is given by

$$M_n(x) = \begin{pmatrix} \boldsymbol{x} & 1 \\ I_{n-1} & \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} x^{n-1} & x^{n-2} & x^{n-3} & \cdots & x & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} F_n^{(n)}(x) & x^{n-2}F_{n-1}^{(n)}(x)+x^{n-3}F_{n-2}^{(n)}(x)+\cdots+F_1^{(n)}(x) & \cdots & F_{n-1}^{(n)}(x) \\ F_{n-1}^{(n)}(x) & x^{n-2}F_{n-2}^{(n)}(x)+x^{n-3}F_{n-3}^{(n)}(x)+\cdots+F_0^{(n)}(x) & \cdots & F_{n-2}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_2^{(n)}(x) & x^{n-2}F_1^{(n)}(x)+x^{n-3}F_0^{(n)}(x)+\cdots+F_{-n+3}^{(n)}(x) & \cdots & F_1^{(n)}(x) \\ F_1^{(n)}(x) & x^{n-2}F_0^{(n)}(x)+x^{n-3}F_{-1}^{(n)}(x)+\cdots+F_{-n+2}^{(n)}(x) & \cdots & F_0^{(n)}(x) \end{pmatrix} \quad (5)$$

Where x is the row matrix $(x^{n-1} \quad x^{n-2} \quad \ldots \quad x)$, I_{n-1} is the identity matrix of order $n-1$ and 0 is the column matrix of order $n-1$ with each element 0 such that

$$\det M_n(x) = (-1)^{n+1} \quad (6)$$

and

$$M_n^0(x) = I_n \quad (7)$$

The inverse of $M_n(x)$ is

$$M_n^{-1}(x) = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ 1 & -\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ . & . & . & \cdots & \cdots & . \\ . & . & . & \cdots & \cdots & . \\ 0 & 0 & 0 & \cdots & \cdots & 1 \\ 1 & -x^{n-1} & -x^{n-2} & \cdots & \cdots & -x \end{pmatrix}$$

$$= \begin{pmatrix} F_{n-2}^{(n)}(x) & x^{n-2}F_{n-3}^{(n)}(x)+x^{n-3}F_{n-4}^{(n)}(x)+\cdots+F_{-1}^{(n)}(x) & \cdots & F_{n-3}^{(n)}(x) \\ F_{n-3}^{(n)}(x) & x^{n-2}F_{n-4}^{(n)}(x)+x^{n-3}F_{n-5}^{(n)}(x)+\cdots+F_{-2}^{(n)}(x) & \cdots & F_{n-4}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_0^{(n)}(x) & x^{n-2}F_{-1}^{(n)}(x)+x^{n-3}F_{-2}^{(n)}(x)+\cdots+F_{-n+1}^{(n)}(x) & \cdots & F_{-1}^{(n)}(x) \\ F_{-1}^{(n)}(x) & x^{n-2}F_{-2}^{(n)}(x)+x^{n-3}F_{-3}^{(n)}(x)+\cdots+F_{-n}^{(n)}(x) & \cdots & F_{-2}^{(n)}(x) \end{pmatrix} \quad (8)$$

Theorem 1.1. $M_n^k(x) = \begin{pmatrix} F_{k+n-1}^{(n)}(x) & x^{n-2}F_{k+n-2}^{(n)}(x)+x^{n-3}F_{k+n-3}^{(n)}(x)+\cdots+F_k^{(n)}(x) & \cdots & F_{k+n-2}^{(n)}(x) \\ F_{k+n-2}^{(n)}(x) & x^{n-2}F_{k+n-3}^{(n)}(x)+x^{n-3}F_{k+n-4}^{(n)}(x)+\cdots+F_{k-1}^{(n)}(x) & \cdots & F_{k+n-3}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{k+1}^{(n)}(x) & x^{n-2}F_k^{(n)}(x)+x^{n-3}F_{k-1}^{(n)}(x)+\cdots+F_{k-n+2}^{(n)}(x) & \cdots & F_k^{(n)}(x) \\ F_k^{(n)}(x) & x^{n-2}F_{k-1}^{(n)}(x)+x^{n-3}F_{k-2}^{(n)}(x)+\cdots+F_{k-n+1}^{(n)}(x) & \cdots & F_{k-1}^{(n)}(x) \end{pmatrix}$

for $k = 0, \pm 1, \pm 2, \ldots$

Proof: Case 1: $k > 1$.

$$M_n^1 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} F_n^{(n)}(x) & x^{n-2}F_{n-1}^{(n)}(x)+x^{n-3}F_{n-2}^{(n)}(x)+\cdots+F_1^{(n)}(x) & \cdots & F_{n-1}^{(n)}(x) \\ F_{n-1}^{(n)}(x) & x^{n-2}F_{n-2}^{(n)}(x)+x^{n-3}F_{n-3}^{(n)}(x)+\cdots+F_0^{(n)}(x) & \cdots & F_{n-2}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_2^{(n)}(x) & x^{n-2}F_1^{(n)}(x)+x^{n-3}F_0^{(n)}(x)+\cdots+F_{-n+3}^{(n)}(x) & \cdots & F_1^{(n)}(x) \\ F_1^{(n)}(x) & x^{n-2}F_0^{(n)}(x)+x^{n-3}F_{-1}^{(n)}(x)+\cdots+F_{-n+2}^{(n)}(x) & \cdots & F_0^{(n)}(x) \end{pmatrix}$$

$$= \begin{pmatrix} F_{1+n-1}^{(n)}(x) & x^{n-2}F_{1+n-2}^{(n)}(x)+x^{n-3}F_{1+n-3}^{(n)}(x)+\cdots+F_{1}^{(n)}(x) & \cdots & F_{1+n-2}^{(n)}(x) \\ F_{1+n-2}^{(n)}(x) & x^{n-2}F_{1+n-3}^{(n)}(x)+x^{n-3}F_{1+n-4}^{(n)}(x)+\cdots+F_{1-1}^{(n)}(x) & \cdots & F_{1+n-3}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{1+1}^{(n)}(x) & x^{n-2}F_{1}^{(n)}(x)+x^{n-3}F_{1-1}^{(n)}(x)+\cdots+F_{1-n+2}^{(n)}(x) & \cdots & F_{1}^{(n)}(x) \\ F_{1}^{(n)}(x) & x^{n-2}F_{1-1}^{(n)}(x)+x^{n-3}F_{1-2}^{(n)}(x)+\cdots+F_{1-n+1}^{(n)}(x) & \cdots & F_{1-1}^{(n)}(x) \end{pmatrix}$$

The theorem is true for $k = 1$.
Let the theorem is true for $k = m$ then

$$M_n^m(x) = \begin{pmatrix} F_{m+n-1}^{(n)}(x) & x^{n-2}F_{m+n-2}^{(n)}(x)+x^{n-3}F_{m+n-3}^{(n)}(x)+\cdots+F_{m}^{(n)}(x) & \cdots & F_{m+n-2}^{(n)}(x) \\ F_{m+n-2}^{(n)}(x) & x^{n-2}F_{m+n-3}^{(n)}(x)+x^{n-3}F_{m+n-4}^{(n)}(x)+\cdots+F_{m-1}^{(n)}(x) & \cdots & F_{m+n-3}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{m+1}^{(n)}(x) & x^{n-2}F_{m}^{(n)}(x)+x^{n-3}F_{m-1}^{(n)}(x)+\cdots+F_{m-n+2}^{(n)}(x) & \cdots & F_{m}^{(n)}(x) \\ F_{m}^{(n)}(x) & x^{n-2}F_{m-1}^{(n)}(x)+x^{n-3}F_{m-2}^{(n)}(x)+\cdots+F_{m-n+1}^{(n)}(x) & \cdots & F_{m-1}^{(n)}(x) \end{pmatrix}$$

Now,
$M_n^{m+1}(x) = M_n^m(x)M_n^1(x) =$

$$\begin{pmatrix} F_{m+n-1}^{(n)}(x) & x^{n-2}F_{m+n-2}^{(n)}(x)+x^{n-3}F_{m+n-3}^{(n)}(x)+\cdots+F_{m}^{(n)}(x) & \cdots & F_{m+n-2}^{(n)}(x) \\ F_{m+n-2}^{(n)}(x) & x^{n-2}F_{m+n-3}^{(n)}(x)+x^{n-3}F_{m+n-4}^{(n)}(x)+\cdots+F_{m-1}^{(n)}(x) & \cdots & F_{m+n-3}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{m+1}^{(n)}(x) & x^{n-2}F_{m}^{(n)}(x)+x^{n-3}F_{m-1}^{(n)}(x)+\cdots+F_{m-n+2}^{(n)}(x) & \cdots & F_{m}^{(n)}(x) \\ F_{m}^{(n)}(x) & x^{n-2}F_{m-1}^{(n)}(x)+x^{n-3}F_{m-2}^{(n)}(x)+\cdots+F_{m-n+1}^{(n)}(x) & \cdots & F_{m-1}^{(n)}(x) \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Using the recurrence relation (4), we have

$$M_n^{m+1} = \begin{pmatrix} F_{m+n}^{(n)}(x) & x^{n-2}F_{m+n-1}^{(n)}(x)+x^{n-3}F_{m+n-2}^{(n)}(x)+\cdots+F_{m+1}^{(n)}(x) & \cdots & F_{m+n-1}^{(n)}(x) \\ F_{m1+n-1}^{(n)}(x) & x^{n-2}F_{m+n-2}^{(n)}(x)+x^{n-3}F_{m+n-3}^{(n)}(x)+\cdots+F_{m}^{(n)}(x) & \cdots & F_{m+n-2}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{m+2}^{(n)}(x) & x^{n-2}F_{m+1}^{(n)}(x)+x^{n-3}F_{m}^{(n)}(x)+\cdots+F_{m-n+3}^{(n)}(x) & \cdots & F_{m+1}^{(n)}(x) \\ F_{m+1}^{(n)}(x) & x^{n-2}F_{m}^{(n)}(x)+x^{n-3}F_{m-1}^{(n)}(x)+\cdots+F_{m-n+2}^{(n)}(x) & \cdots & F_{m}^{(n)}(x) \end{pmatrix}$$

Hence by induction, for all $k > 0$ we can write

$$M_n^k(x) = \begin{pmatrix} F_{k+n-1}^{(n)}(x) & x^{n-2}F_{k+n-2}^{(n)}(x)+x^{n-3}F_{k+n-3}^{(n)}(x)+\cdots+F_{k}^{(n)}(x) & \cdots & F_{k+n-2}^{(n)}(x) \\ F_{k+n-2}^{(n)}(x) & x^{n-2}F_{k+n-3}^{(n)}(x)+x^{n-3}F_{k+n-4}^{(n)}(x)+\cdots+F_{k-1}^{(n)}(x) & \cdots & F_{k+n-3}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{k+1}^{(n)}(x) & x^{n-2}F_{k}^{(n)}(x)+x^{n-3}F_{k-1}^{(n)}(x)+\cdots+F_{k-n+2}^{(n)}(x) & \cdots & F_{k}^{(n)}(x) \\ F_{k}^{(n)}(x) & x^{n-2}F_{k-1}^{(n)}(x)+x^{n-3}F_{k-2}^{(n)}(x)+\cdots+F_{k-n+1}^{(n)}(x) & \cdots & F_{k-1}^{(n)}(x) \end{pmatrix}$$

Case 2: $k < 0$.
Similarly, we can prove the theorem for all $k < 0$.
Hence the theorem.

2. Some Properties of Fibonacci n-step Polynomial Matrix

$M_n^k(x) = x^{n-2}M_n^{k-1}(x) + x^{n-2}M_n^{k-2}(x) + \ldots + M_n^{k-n}$

Proof: $M_n^k(x) = \begin{pmatrix} F_{k+n-1}^{(n)}(x) & x^{n-2}F_{k+n-2}^{(n)}(x)+x^{n-3}F_{k+n-3}^{(n)}(x)+\cdots+F_k^{(n)}(x) & \cdots & F_{k+n-2}^{(n)}(x) \\ F_{k+n-2}^{(n)}(x) & x^{n-2}F_{k+n-3}^{(n)}(x)+x^{n-3}F_{k+n-4}^{(n)}(x)+\cdots+F_{k-1}^{(n)}(x) & \cdots & F_{k+n-3}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{k+1}^{(n)}(x) & x^{n-2}F_k^{(n)}(x)+x^{n-3}F_{k-1}^{(n)}(x)+\cdots+F_{k-n+2}^{(n)}(x) & \cdots & F_k^{(n)}(x) \\ F_k^{(n)}(x) & x^{n-2}F_{k-1}^{(n)}(x)+x^{n-3}F_{k-2}^{(n)}(x)+\cdots+F_{k-n+1}^{(n)}(x) & \cdots & F_{k-1}^{(n)}(x) \end{pmatrix}$

Now using the recurrence relation (4), we can write

$$F_{k+n-1}^{(n)}(x) = x^{n-1}F_{k+n-2}^{(n)}(x) + x^{n-2}F_{k+n-3}^{(n)}(x) + \ldots + F_{k-1}^{(n)}(x)$$
$$F_{k+n-2}^{(n)}(x) = x^{n-1}F_{k+n-3}^{(n)}(x) + x^{n-2}F_{k+n-4}^{(n)}(x) + \ldots + F_{k-2}^{(n)}(x)$$
$$\ldots$$
$$F_{k-1}^{(n)}(x) = x^{n-1}F_{k-2}^{(n)}(x) + x^{n-2}F_{k-3}^{(n)}(x) + \ldots + F_{k-n}^{(n)}(x)$$

Hence by using the property of matrix addition, we can write

$$M_n^k(x) = x^{n-1}\begin{pmatrix} F_{k+n-2}^{(n)}(x) & x^{n-2}F_{k+n-3}^{(n)}(x)+x^{n-3}F_{k+n-4}^{(n)}(x)+\cdots+F_{k-1}^{(n)}(x) & \cdots & F_{k+n-2}^{(n)}(x) \\ F_{k+n-3}^{(n)}(x) & x^{n-2}F_{k+n-4}^{(n)}(x)+x^{n-3}F_{k+n-5}^{(n)}(x)+\cdots+F_{k-2}^{(n)}(x) & \cdots & F_{k+n-3}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_k^{(n)}(x) & x^{n-2}F_{k-1}^{(n)}(x)+x^{n-3}F_{k-2}^{(n)}(x)+\cdots+F_{k-n+1}^{(n)}(x) & \cdots & F_{k-1}^{(n)}(x) \\ F_{k-1}^{(n)}(x) & x^{n-2}F_{k-2}^{(n)}(x)+x^{n-3}F_{k-3}^{(n)}(x)+\cdots+F_{k-n}^{(n)}(x) & \cdots & F_{k-2}^{(n)}(x) \end{pmatrix}$$

$$+x^{n-2}\begin{pmatrix} F_{k+n-3}^{(n)}(x) & x^{n-2}F_{k+n-4}^{(n)}(x)+x^{n-3}F_{k+n-5}^{(n)}(x)+\cdots+F_{k-2}^{(n)}(x) & \cdots & F_{k+n-4}^{(n)}(x) \\ F_{k+n-4}^{(n)}(x) & x^{n-2}F_{k+n-5}^{(n)}(x)+x^{n-3}F_{k+n-6}^{(n)}(x)+\cdots+F_{k-3}^{(n)}(x) & \cdots & F_{k+n-5}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{k-1}^{(n)}(x) & x^{n-2}F_{k-2}^{(n)}(x)+x^{n-3}F_{k-3}^{(n)}(x)+\cdots+F_{k-n}^{(n)}(x) & \cdots & F_{k-2}^{(n)}(x) \\ F_{k-2}^{(n)}(x) & x^{n-2}F_{k-3}^{(n)}(x)+x^{n-3}F_{k-4}^{(n)}(x)+\cdots+F_{k-n-1}^{(n)}(x) & \cdots & F_{k-3}^{(n)}(x) \end{pmatrix}$$

$$\ldots\ldots$$
$$\ldots\ldots$$

$$+\begin{pmatrix} F_{k-1}^{(n)} & x^{n-2}F_{k-2}^{(n)}+x^{n-3}F_{k-3}^{(n)}+\cdots+F_{k-n}^{(n)} & \cdots & F_{k-2}^{(n)} \\ F_{k-1}^{(n)} & x^{n-2}F_{k-3}^{(n)}+x^{n-3}F_{k-4}^{(n)}+\cdots+F_{k-n-1}^{(n)} & \cdots & F_{k-3}^{(n)} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{k-1}^{(n)} & x^{n-2}F_{k-n}^{(n)}+x^{n-3}F_{k-n-1}^{(n)}+\cdots+F_{k-2n+2}^{(n)} & \cdots & F_{k-n}^{(n)} \\ F_{k-1}^{(n)} & x^{n-2}F_{k-n-1}^{(n)}+x^{n-3}F_{k-n-2}^{(n)}+\cdots+F_{k-2n+1}^{(n)} & \cdots & F_{k-n-1}^{(n)} \end{pmatrix}$$

$$= x^{n-2}M_n^{k-1}(x) + x^{n-2}M_n^{k-2}(x) + \ldots + M_n^{k-n} \quad (9)$$

1. Using the basic property of matrix, we have

   $(10) M_n^k(x)M_n^l(x) = M_n^l(x)M_n^k(x) = M_n^{k+l}(x),\ k = 0, \pm1, \pm2, \pm3, \ldots$

2. $\det M_n^k(x) = (\det M_n^k(x))^k = ((-1)^{n+1})^k = (-1)^{(n+1)k}$ (11)

3. Fibonacci n-step Polynomial Coding/Decoding Method

We take the initial message **P** having at least $n(n-1)+1$ characters. Now we represent **P** in the form of square matrix $P=(p_{ij})_{n\times n}$, where n is any positive integer. p_{ij} $(\geq 0) i,j=1,2,\dots,n$ depends on the decision makers choice considering the fact that for each i at least one $p_{ij} \neq 0$. Otherwise this method is defunct one. We take the matrix $M_n^k(x)$ as a coding matrix and its inverse matrix $M_n^{-k}(x)$ as a decoding matrix for an arbitrary positive integer k. We name the transformation $P \times M_n^k(x) = E$ as coding, the transformation $E \times M_n^{-k}(x) = P$ as decoding and define E as code matrix.

3.1 Determinant of The Code Matrix *E*

We define the code matrix E by the following formula

$$E = P \times M_n^k(x)$$

Using the basic property of determinants, we have

$$\det E = \det\left(P \times M_n^k(x)\right) = \det P \times \det M_n^k(x) = \det P \times (-1)^{(n+1)k} \tag{12}$$

Example:

We consider $n=3$. We represent the initial message $p_{11}p_{12}p_{13}p_{21}p_{22}p_{23}p_{31}p_{32}p_{33}$ in the form of square matrix P of order 3 as

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \tag{13}$$

p_{ij} $(\geq 0) i,j=1,2,3$ depend on the decision makers choice considering the fact that for each i at least one $p_{ij} \neq 0$. We select for any value of k, the matrix $M_3^k(x)$ as the coding matrix.
Without any loss of generality, we assume that $k=3$. Then,

$$M_3^3(x) = \begin{pmatrix} F_5^{(3)}(x) & xF_4^{(3)}(x) + F_3^{(3)}(x) & F_4^{(3)}(x) \\ F_4^{(3)}(x) & xF_3^{(3)}(x) + F_2^{(3)}(x) & F_3^{(3)}(x) \\ F_3^{(3)}(x) & xF_2^{(3)}(x) + F_1^{(3)}(x) & F_2^{(3)}(x) \end{pmatrix}$$
$$= \begin{pmatrix} x^6+2x^3+1 & x^5+2x^2 & x^4+x \\ x^4+x & x^3+1 & x^2 \\ x^2 & x & 1 \end{pmatrix} \tag{14}$$

The inverse of $M_3^k(x)$ is given by

$$M_3^{-3}(x) = \begin{pmatrix} F_{-1}^{(3)}(x) & xF_{-2}^{(3)}(x) + F_{-3}^{(3)}(x) & F_{-2}^{(3)}(x) \\ F_{-2}^{(3)}(x) & xF_{-3}^{(3)}(x) + F_{-4}^{(3)}(x) & F_{-3}^{(3)}(x) \\ F_{-3}^{(3)}(x) & xF_{-4}^{(3)}(x) + F_{-5}^{(3)}(x) & F_{-4}^{(3)}(x) \end{pmatrix} = \begin{pmatrix} 1 & -3x^2 & -x \\ -x & x^3+1 & -2x^2 \\ -2x^2 & 3x^4-x & 1+3x^3 \end{pmatrix}$$

Then the coding of the message (13) consists of the multiplication of the initial matrix (14) i.e.

$$P \times M_3^3(x) = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \times \begin{pmatrix} x^6+2x^3+1 & x^5+2x^2 & x^4+x \\ x^4+x & x^3+1 & x^2 \\ x^2 & x & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (x^6+2x^3+1)p_{11} + (x^4+x)p_{12} + x^2p_{13} & (x^5+2x^2)p_{11} + (x^3+1)p_{12} + xp_{13} & (x^4+x)p_{11} + x^2p_{12} + p_{13} \\ (x^6+2x^3+1)p_{21} + (x^4+x)p_{22} + x^2p_{23} & (x^5+2x^2)p_{21} + (x^3+1)p_{22} + xp_{23} & (x^4+x)p_{21} + x^2p_{22} + p_{23} \\ (x^6+2x^3+1)p_{31} + (x^4+x)p_{32} + x^2p_{33} & (x^5+2x^2)p_{31} + (x^3+1)p_{32} + xp_{33} & (x^4+x)p_{31} + x^2p_{32} + p_{33} \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = E \tag{15}$$

where

$e_{11} = (x^6 + 2x^3 + 1)p_{11} + (x^4 + x)p_{12} + x^2 p_{13}$, $e_{12} = (x^5 + 2x^2)p_{11} + (x^3 + 1)p_{12} + xp_{13}$, $e_{13} = (x^4 + x)p_{11} + x^2 p_{12} + p_{13}$

$e_{21} = (x^6 + 2x^3 + 1)p_{21} + (x^4 + x)p_{22} + x^2 p_{23}$, $e_{22} = (x^5 + 2x^2)p_{21} + (x^3 + 1)p_{22} + xp_{23}$, $e_{23} = (x^4 + x)p_{21} + x^2 p_{22} + p_{23}$

$e_{31} = (x^6 + 2x^3 + 1)p_{31} + (x^4 + x)p_{32} + x^2 p_{33}$, $e_{32} = (x^5 + 2x^2)p_{31} + (x^3 + 1)p_{32} + xp_{33}$, $e_{33} = (x^4 + x)p_{31} + x^2 p_{32} + p_{33}$.

Solving these we have,

$p_{11} = e_{11} - xe_{12} - 2x^2 e_{13}$, $p_{12} = -3x^2 e_{11} + (x^3 + 1)e_{12} + (3x^4 - x)e_{13}$, $p_{13} = -xe_{11} - 2x^2 e_{12} + (1 + 3x^3)e_{13}$,

$p_{21} = e_{21} - xe_{22} - 2x^2 e_{23}$, $p_{22} = -3x^2 e_{21} + (x^3 + 1)e_{22} + (3x^4 - x)e_{23}$, $p_{23} = -xe_{21} - 2x^2 e_{22} + (1 + 3x^3)e_{23}$,

$p_{31} = e_{31} - xe_{32} - 2x^2 e_{33}$, $p_{32} = -3x^2 e_{31} + (x^3 + 1)e_{32} + (3x^4 - x)e_{33}$, $p_{33} = -xe_{31} - 2x^2 e_{32} + (1 + 3x^3)e_{33}$.

The code matrix E is sent to a channel and the decoding of the code message E is obtained by the following manner:

$$E \times M_3^{-3}(x) = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \times \begin{pmatrix} 1 & -3x^2 & -x \\ -x & x^3 + 1 & -2x^2 \\ -2x^2 & 3x^4 - x & 1 + 3x^3 \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} - xe_{12} - 2x^2 e_{13} & -3x^2 e_{11} + (x^3 + 1)e_{12} + (3x^4 - x)e_{13} & -xe_{11} - 2x^2 e_{12} + (1 + 3x^3)e_{13} \\ e_{21} - xe_{22} - 2x^2 e_{23} & -3x^2 e_{21} + (x^3 + 1)e_{22} + (3x^4 - x)e_{23} & -xe_{21} - 2x^2 e_{22} + (1 + 3x^3)e_{23} \\ e_{31} - xe_{32} - 2x^2 e_{33} & -3x^2 e_{31} + (x^3 + 1)e_{32} + (3x^4 - x)e_{33} & -xe_{31} - 2x^2 e_{32} + (1 + 3x^3)e_{33} \end{pmatrix}$$
$$= P$$

4. Relations Among The Code Matrix Elements

In this paper, we develop the relations among the code matrix elements. We write the code matrix E with the help of the initial matrix P and the coding matrix $M_n^k(x)$ as

$$E = P \times M_n^k(x) = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \times$$

$$\begin{pmatrix} F_{k+n-1}^{(n)}(x) & x^{n-2}F_{k+n-2}^{(n)}(x) + x^{n-3}F_{k+n-3}^{(n)}(x) + \cdots + F_k^{(n)}(x) & \cdots & F_{k+n-2}^{(n)}(x) \\ F_{k+n-2}^{(n)}(x) & x^{n-2}F_{k+n-3}^{(n)}(x) + x^{n-3}F_{k+n-4}^{(n)}(x) + \cdots + F_{k-1}^{(n)}(x) & \cdots & F_{k+n-3}^{(n)}(x) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ F_{k+1}^{(n)}(x) & x^{n-2}F_k^{(n)}(x) + x^{n-3}F_{k-1}^{(n)}(x) + \cdots + F_{k-n+2}^{(n)}(x) & \cdots & F_k^{(n)}(x) \\ F_k^{(n)}(x) & x^{n-2}F_{k-1}^{(n)}(x) + x^{n-3}F_{k-2}^{(n)}(x) + \cdots + F_{k-n+1}^{(n)}(x) & \cdots & F_{k-1}^{(n)}(x) \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix}$$

We choose k in such a manner that $e_{ij} > 0$, for all i, j.
After decoding, we have

$P = E \times M_n^{-k}(x)$

$$= \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix} \times \begin{pmatrix} F_{k+n-1}^{(n)}(x) & x^{n-2}F_{k+n-2}^{(n)}(x) + x^{n-3}F_{k+n-3}^{(n)}(x) + \cdots + F_k^{(n)}(x) & \cdots & F_{k+n-2}^{(n)}(x) \\ F_{k+n-2}^{(n)}(x) & x^{n-2}F_{k+n-3}^{(n)}(x) + x^{n-3}F_{k+n-4}^{(n)}(x) + \cdots + F_{k-1}^{(n)}(x) & \cdots & F_{k+n-3}^{(n)}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ F_{k+1}^{(n)}(x) & x^{n-2}F_k^{(n)}(x) + x^{n-3}F_{k-1}^{(n)}(x) + \cdots + F_{k-n+2}^{(n)}(x) & \cdots & F_k^{(n)}(x) \\ F_k^{(n)}(x) & x^{n-2}F_{k-1}^{(n)}(x) + x^{n-3}F_{k-2}^{(n)}(x) + \cdots + F_{k-n+1}^{(n)}(x) & \cdots & F_{k-1}^{(n)}(x) \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}.$$

Case 1: $n = 2$.

We write the code matrix E and the initial matrix P as:

$$E = P \times M_2^k(x) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \times \begin{pmatrix} F_{k+1}^{(2)}(x) & F_k^{(2)}(x) \\ F_k^{(2)}(x) & F_{k-1}^{(2)}(x) \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \quad (16)$$

and

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = E \times M_2^{-k}(x) = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \times M_2^{-k}(x) \quad (17)$$

Case 1.1 k is an odd integer. Then, we have

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \times \begin{pmatrix} -F_{k-1}^{(2)}(x) & F_k^{(2)}(x) \\ F_k^{(2)}(x) & -F_{k+1}^{(2)}(x) \end{pmatrix} \quad (18)$$

It follows from (18) that the elements of the matrix P can be calculated according to the following formulas :

$$p_{11} = -F_{k-1}^{(2)}(x)e_{11} + F_k^{(2)}(x)e_{12} \quad (19)$$

$$p_{12} = F_k^{(2)}(x)e_{11} - F_{k+1}^{(2)}(x)e_{12} \quad (20)$$

$$p_{21} = -F_{k-1}^{(2)}(x)e_{21} + F_k^{(2)}(x)e_{22} \quad (21)$$

and

$$p_{22} = F_k^{(2)}(x)e_{21} - F_{k+1}^{(2)}(x)e_{22} \quad (22)$$

Since p_{11}, p_{12}, p_{21} and p_{22} are integers, we have

$$-F_{k-1}^{(2)}(x)e_{11} + F_k^{(2)}(x)e_{12} \geq 0, \quad (23)$$

$$F_k^{(2)}(x)e_{11} - F_{k+1}^{(2)}(x)e_{12} \geq 0, \quad (24)$$

$$-F_{k-1}^{(2)}(x)e_{21} + F_k^{(2)}(x)e_{22} \geq 0 \quad (25)$$

and

$$F_k^{(2)}(x)e_{21} - F_{k+1}^{(2)}(x)e_{22} \geq 0. \quad (26)$$

From the equations (23) and (24), we have

$$\frac{F_{k+1}^{(2)}(x)}{F_k^{(2)}(x)} \leq \frac{e_{11}}{e_{12}} \leq \frac{F_k^{(2)}(x)}{F_{k-1}^{(2)}(x)} \quad (27)$$

From the equations (25) and (26), we have

$$\frac{F_{k+1}^{(2)}(x)}{F_k^{(2)}(x)} \leq \frac{e_{21}}{e_{22}} \leq \frac{F_k^{(2)}(x)}{F_{k-1}^{(2)}(x)} \quad (28)$$

Case 1.2 k is an even integer.

Similarly, we have

$$\frac{F_{k+1}^{(2)}(x)}{F_k^{(2)}(x)} \geq \frac{e_{11}}{e_{12}} \geq \frac{F_k^{(2)}(x)}{F_{k-1}^{(2)}(x)} \tag{29}$$

and

$$\frac{F_{k+1}^{(2)}(x)}{F_k^{(2)}(x)} \geq \frac{e_{21}}{e_{22}} \geq \frac{F_k^{(2)}(x)}{F_{k-1}^{(2)}(x)} \tag{30}$$

Hence from (27), (28), (29) and (30), we have

$$\frac{F_{k+1}^{(2)}(x)}{F_k^{(2)}(x)} \leq \frac{e_{11}}{e_{12}} \leq \frac{F_k^{(2)}(x)}{F_{k-1}^{(2)}(x)} \; and \; \frac{F_{k+1}^{(2)}(x)}{F_k^{(2)}(x)} \leq \frac{e_{21}}{e_{22}} \leq \frac{F_k^{(2)}(x)}{F_{k-1}^{(2)}(x)}$$

or

$$\frac{F_{k+1}^{(2)}(x)}{F_k^{(2)}(x)} \geq \frac{e_{11}}{e_{12}} \geq \frac{F_k^{(2)}(x)}{F_{k-1}^{(2)}(x)} \; and \; \frac{F_{k+1}^{(2)}(x)}{F_k^{(2)}(x)} \geq \frac{e_{21}}{e_{22}} \geq \frac{F_k^{(2)}(x)}{F_{k-1}^{(2)}(x)}$$

For large k, we have

$$\frac{e_{11}}{e_{12}} \approx r_2(x) \tag{31}$$

and

$$\frac{e_{21}}{e_{22}} \approx r_2(x) \tag{32}$$

where , $r_2(x) = \frac{x+\sqrt{x^2+4}}{2}$.

Case 2: $n = 3$.

In this case, we can write the code matrix E and the initial matrix P as:

$$E = P \times M_2^k(x) = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \times \begin{pmatrix} F_{k+2}^{(3)}(x) & xF_{k+1}^{(3)}(x) + F_k^{(3)}(x) & F_{k+1}^{(3)}(x) \\ F_{k+1}^{(3)}(x) & xF_k^{(3)}(x) + F_{k-1}^{(3)}(x) & F_k^{(3)}(x) \\ F_k^{(3)}(x) & xF_{k-1}^{(3)}(x) + F_{k-2}^{(3)}(x) & F_{k-1}^{(3)}(x) \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

and

$$P = E \times M_2^{-k}(x) = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \times \begin{pmatrix} F_{k+2}^{(3)}(x) & xF_{k+1}^{(3)}(x) + F_k^{(3)}(x) & F_{k+1}^{(3)}(x) \\ F_{k+1}^{(3)}(x) & xF_k^{(3)}(x) + F_{k-1}^{(3)}(x) & F_k^{(3)}(x) \\ F_k^{(3)}(x) & xF_{k-1}^{(3)}(x) + F_{k-2}^{(3)}(x) & F_{k-1}^{(3)}(x) \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

Now

$$\begin{aligned} \det M_3^k(x) = {} & F_{k+2}^{(3)}(x)\left(xF_k^{(3)}(x)F_{k-1}^{(3)}(x) + \left(F_{k-1}^{(3)}(x)\right)^2 - xF_k^{(3)}(x)F_{k-1}^{(3)}(x) - F_k^{(3)}(x)F_{k-2}^{(3)}(x)\right) \\ & + \left(xF_{k+1}^{(3)}(x) + F_k^{(3)}(x)\right)\left(\left(F_k^{(3)}(x)\right)^2 - F_{k+2}^{(3)}(x)F_{k+2}^{(3)}(x)\right) \\ & + F_{k+1}^{(3)}(x)\left(xF_{k+1}^{(3)}(x)F_{k-1}^{(3)}(x) + F_{k+1}^{(3)}(x)F_{k-2}^{(3)}(x) - x\left(F_k^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k-1}^{(3)}(x)\right) \\ = {} & 1 \end{aligned} \tag{33}$$

and

$$M_3^{-k}(x) = \frac{1}{\det M_3^k(x)} adj\, M_3^k(x)$$
$$= \begin{pmatrix} (F_{k-1}^{(3)}(x))^2 - F_k^{(3)}(x)F_{k-2}^{(3)}(x) & F_{k+1}^{(3)}(x)F_{k-2}^{(3)}(x) - F_k^{(3)}(x)F_{k-1}^{(3)}(x) & (F_k^{(3)}(x))^2 - F_{k-1}^{(3)}(x)F_{k+1}^{(3)}(x) \\ (F_k^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k-1}^{(3)}(x) & F_{k-1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_{k+1}^{(3)}(x)F_k^{(3)}(x) & (F_{k+1}^{(3)}(x))^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x) \\ (F_{k+1}^{(3)}(x))^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x) & F_k^{(3)}(x)F_{k+3}^{(3)}(x) - F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x) & (F_{k+2}^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x) \end{pmatrix}$$

We choose k in such a manner that $e_{ij} > 0$ for $i, j = 1, 2, 3$.
Also, we have $p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23}, p_{31}, p_{32}, p_{33} \geq 0$.
Therefore,

$$p_{11} = e_{11}\left(\left(F_{k-1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k-2}^{(3)}(x)\right) + e_{12}\left(\left(F_k^{(3)}(x)\right)^2 - F_{k+1}^{(3)}(x)F_{k-1}^{(3)}(x)\right) + e_{13}\left(\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)\right) \geq 0 \quad (34)$$

$$p_{12} = e_{11}\left(F_{k+1}^{(3)}(x)F_{k-2}^{(3)}(x) - F_k^{(3)}(x)F_{k-1}^{(3)}(x)\right) + e_{12}\left(F_{k-1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_{k+1}^{(3)}(x)F_k^{(3)}(x)\right) + e_{13}\left(F_k^{(3)}(x)F_{k+3}^{(3)}(x) - F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x)\right) \geq 0 \quad (35)$$

$$p_{13} = e_{11}\left(\left(F_k^{(3)}(x)\right)^2 - F_{k-1}^{(3)}(x)F_{k+1}^{(3)}(x)\right) + e_{12}\left(\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)\right) + e_{13}\left((F_{k+2}^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)\right) \geq 0 \quad (36)$$

$$p_{21} = e_{21}\left(\left(F_{k-1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k-2}^{(3)}(x)\right) + e_{22}\left(\left(F_k^{(3)}(x)\right)^2 - F_{k+1}^{(3)}(x)F_{k-1}^{(3)}(x)\right) + e_{23}\left(\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)\right) \geq 0 \quad (37)$$

$$p_{22} = e_{21}\left(F_{k+1}^{(3)}(x)F_{k-2}^{(3)}(x) - F_k^{(3)}(x)F_{k-1}^{(3)}(x)\right) + e_{22}\left(F_{k-1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_{k+1}^{(3)}(x)F_k^{(3)}(x)\right) + e_{23}\left(F_k^{(3)}(x)F_{k+3}^{(3)}(x) - F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x)\right) \geq 0 \quad (38)$$

$$p_{23} = e_{21}\left(\left(F_k^{(3)}(x)\right)^2 - F_{k-1}^{(3)}(x)F_{k+1}^{(3)}(x)\right) + e_{22}\left(\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)\right) + e_{23}\left((F_{k+2}^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)\right) \geq 0 \quad (39)$$

$$p_{31} = e_{31}\left(\left(F_{k-1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k-2}^{(3)}(x)\right) + e_{32}\left(\left(F_k^{(3)}(x)\right)^2 - F_{k+1}^{(3)}(x)F_{k-1}^{(3)}(x)\right) + e_{33}\left(\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)\right) \geq 0 \quad (40)$$

$$p_{32} = e_{31}\left(F_{k+1}^{(3)}(x)F_{k-2}^{(3)}(x) - F_k^{(3)}(x)F_{k-1}^{(3)}(x)\right) + e_{32}\left(F_{k-1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_{k+1}^{(3)}(x)F_k^{(3)}(x)\right) + e_{33}\left(F_k^{(3)}(x)F_{k+3}^{(3)}(x) - F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x)\right) \geq 0 \quad (41)$$

and

$$p_{33} = e_{31}\left(\left(F_k^{(3)}(x)\right)^2 - F_{k-1}^{(3)}(x)F_{k+1}^{(3)}(x)\right) + e_{32}\left(\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)\right) + e_{33}\left((F_{k+2}^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)\right) \geq 0 \quad (42)$$

Dividing both sides of (34), (35) and (36) by $e_{11}(> 0)$, we have

$$\left(\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)\right)\frac{e_{13}}{e_{11}} \geq \left(F_{k+1}^{(3)}(x)F_{k-1}^{(3)}(x) - \left(F_k^{(3)}(x)\right)^2\right)\frac{e_{12}}{e_{11}} + \left(F_k^{(3)}(x)F_{k-2}^{(3)}(x) - \left(F_{k-1}^{(3)}(x)\right)^2\right) \quad (43)$$

$$\left(F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_k^{(3)}(x)F_{k+3}^{(3)}(x)\right)\frac{e_{13}}{e_{11}} \leq \left(F_{k-1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_{k+1}^{(3)}(x)F_k^{(3)}(x)\right)\frac{e_{12}}{e_{11}} + \left(F_{k+1}^{(3)}(x)F_{k-2}^{(3)}(x) - F_k^{(3)}(x)F_{k-1}^{(3)}(x)\right) \quad (44)$$

and

$$\left((F_{k+2}^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)\right)\frac{e_{13}}{e_{11}} \geq \left(F_k^{(3)}(x)F_{k+2}^{(3)}(x) - \left(F_{k+1}^{(3)}(x)\right)^2\right)\frac{e_{12}}{e_{11}} + \left(F_{k-1}^{(3)}(x)F_{k+1}^{(3)}(x) - \left(F_k^{(3)}(x)\right)^2\right) \quad (45)$$

Let $a = \left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x), b = F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_k^{(3)}(x)F_{k+3}^{(3)}(x), c = (F_{k+2}^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)$.

Now $3^3 = 27$ cases arise for $a >=< 0, b >=< 0, c >=< 0$.

Case 2.1: $a > 0,\ b > 0, c > 0$. Then from (43), we have

$$\frac{e_{13}}{e_{11}} \geq u, \text{ where } u = \frac{e_{12}}{e_{11}}\left(\frac{F_{k+1}^{(3)}(x)F_{k-1}^{(3)}(x) - \left(F_k^{(3)}(x)\right)^2}{\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)}\right) + \frac{F_k^{(3)}(x)F_{k-2}^{(3)}(x) - \left(F_{k-1}^{(3)}(x)\right)^2}{\left(F_{k+1}^{(3)}(x)\right)^2 - F_k^{(3)}(x)F_{k+2}^{(3)}(x)} \quad (46)$$

Then from (44), we have

$\frac{e_{13}}{e_{11}} \leq v$, where

$$v = \frac{e_{12}}{e_{11}}\left(\frac{F_{k-1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_{k+1}^{(3)}(x)F_k^{(3)}(x)}{F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_k^{(3)}(x)F_{k+3}^{(3)}(x)}\right) + \frac{F_{k+1}^{(3)}(x)F_{k-2}^{(3)}(x) - F_k^{(3)}(x)F_{k-1}^{(3)}(x)}{F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x) - F_k^{(3)}(x)F_{k+3}^{(3)}(x)} \quad (47)$$

Then from (45), we have

$$\frac{e_{13}}{e_{11}} \geq w, \text{ where } w = \frac{e_{12}}{e_{11}}\left(\frac{F_k^{(3)}(x)F_{k+2}^{(3)}(x) - \left(F_{k+1}^{(3)}(x)\right)^2}{(F_{k+2}^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)}\right) + \frac{F_{k-1}^{(3)}(x)F_{k+1}^{(3)}(x) - \left(F_k^{(3)}(x)\right)^2}{(F_{k+2}^{(3)}(x))^2 - F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)} \quad (48)$$

From (46) and (47), we have

$$\frac{e_{11}}{e_{12}} \geq \min\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_k^{(3)}(x)}, \frac{F_{k+1}^{(3)}(x)}{xF_k^{(3)}(x)+F_{k-1}^{(3)}(x)}, \frac{F_k^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}, \text{ using (33)} \quad (49)$$

From (47) and (48), we have

$$\frac{e_{11}}{e_{12}} \leq \max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_k^{(3)}(x)}, \frac{F_{k+1}^{(3)}(x)}{xF_k^{(3)}(x)+F_{k-1}^{(3)}(x)}, \frac{F_k^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}, \text{ using (33)} \quad (50)$$

From (49) and (50), we have

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x) + F_k^{(3)}(x)}, \frac{F_{k+1}^{(3)}(x)}{xF_k^{(3)}(x) + F_{k-1}^{(3)}(x)}, \frac{F_k^{(3)}(x)}{xF_{k-1}^{(3)}(x) + F_{k-2}^{(3)}(x)}\right\} \leq \frac{e_{11}}{e_{12}}$$

$$\leq \max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_k^{(3)}(x)}, \frac{F_{k+1}^{(3)}(x)}{xF_k^{(3)}(x)+F_{k-1}^{(3)}(x)}, \frac{F_k^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\} \quad (51)$$

Similarly, we have

$$\min\left\{\frac{xF_{k+1}^{(3)}(x) + F_k^{(3)}(x)}{F_{k+1}^{(3)}(x)}, \frac{xF_k^{(3)}(x) + F_{k-1}^{(3)}(x)}{F_k^{(3)}(x)}, \frac{xF_{k-1}^{(3)}(x) + F_{k-2}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\} \leq \frac{e_{12}}{e_{13}}$$

$$\leq \max\left\{\frac{xF_{k+1}^{(3)}(x) + F_k^{(3)}(x)}{F_{k+1}^{(3)}(x)}, \frac{xF_k^{(3)}(x) + F_{k-1}^{(3)}(x)}{F_k^{(3)}(x)}, \frac{xF_{k-1}^{(3)}(x) + F_{k-2}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}$$

and

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}\leq\frac{e_{11}}{e_{13}}\leq\max\left\{\frac{F_{k+2}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}$$

Case 2.2: $a=0,\ b>0, c>0$.
Then from (43), we have

$$\frac{e_{11}}{e_{12}}\geq\min\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}0, since\ a=0 \tag{52}$$

From (44) and (45), we have

$$\frac{e_{11}}{e_{12}}\geq\max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}, \text{using (33) and } a=0 \tag{53}$$

From (49) and (50), we have

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}\leq\frac{e_{11}}{e_{12}}$$

$$\leq\max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\} \tag{54}$$

Case 2.3: $a<0,\ b<0, c<0$. Then from (43), we have

$$\frac{e_{13}}{e_{11}}\leq u, \text{ where } u=\frac{e_{12}}{e_{11}}\left(\frac{F_{k+1}^{(3)}(x)F_{k-1}^{(3)}(x)-\left(F_{k}^{(3)}(x)\right)^2}{\left(F_{k+1}^{(3)}(x)\right)^2-F_{k}^{(3)}(x)F_{k+2}^{(3)}(x)}\right)+\frac{F_{k}^{(3)}(x)F_{k-2}^{(3)}(x)-\left(F_{k-1}^{(3)}(x)\right)^2}{\left(F_{k+1}^{(3)}(x)\right)^2-F_{k}^{(3)}(x)F_{k+2}^{(3)}(x)} \tag{55}$$

Then from (44), we have

$$\frac{e_{13}}{e_{11}}\geq v, \text{ where } v=\frac{e_{12}}{e_{11}}\left(\frac{F_{k-1}^{(3)}(x)F_{k+2}^{(3)}(x)-F_{k+1}^{(3)}(x)F_{k}^{(3)}(x)}{F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x)-F_{k}^{(3)}(x)F_{k+3}^{(3)}(x)}\right)+\frac{F_{k+1}^{(3)}(x)F_{k-2}^{(3)}(x)-F_{k}^{(3)}(x)F_{k-1}^{(3)}(x)}{F_{k+1}^{(3)}(x)F_{k+2}^{(3)}(x)-F_{k}^{(3)}(x)F_{k+3}^{(3)}(x)} \tag{56}$$

Then from (45), we have

$$\frac{e_{13}}{e_{11}}\leq w, \text{ where } w=\frac{e_{12}}{e_{11}}\left(\frac{F_{k}^{(3)}(x)F_{k+2}^{(3)}(x)-\left(F_{k+1}^{(3)}(x)\right)^2}{(F_{k+2}^{(3)}(x))^2-F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)}\right)+\frac{F_{k-1}^{(3)}(x)F_{k+1}^{(3)}(x)-\left(F_{k}^{(3)}(x)\right)^2}{(F_{k+2}^{(3)}(x))^2-F_{k+1}^{(3)}(x)F_{k+3}^{(3)}(x)} \tag{57}$$

From (55) and (56), we have

$$\frac{e_{11}}{e_{12}}\leq\max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}, \text{using (33)} \tag{58}$$

From (56) and (57), we have

$$\frac{e_{11}}{e_{12}}\geq\max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}, \text{using (33)} \tag{59}$$

From (58) and (59), we have

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}\leq\frac{e_{11}}{e_{12}}$$

$$\leq\ \max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\} \tag{60}$$

Similarly, we have

$$\min\left\{\frac{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}\leq\frac{e_{12}}{e_{13}}$$

$$\leq\max\left\{\frac{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}$$

and

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}\le\frac{e_{11}}{e_{13}}\le\max\left\{\frac{F_{k+2}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}$$

Similarly it can be proved for the rest cases.
Hence, we have

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}\le\frac{e_{11}}{e_{12}}$$

$$\le\max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\} \tag{61}$$

$$\min\left\{\frac{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}\le\frac{e_{12}}{e_{13}}$$

$$\le\max\left\{\frac{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\} \tag{62}$$

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}\le\frac{e_{11}}{e_{13}}\le\max\left\{\frac{F_{k+2}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\} \tag{63}$$

Similarly, we have

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\}\le\frac{e_{i1}}{e_{i2}}$$

$$\le\max\left\{\frac{F_{k+2}^{(3)}(x)}{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}\right\},$$

$$\min\left\{\frac{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}\le\frac{e_{i2}}{e_{i3}}$$

$$\le\max\left\{\frac{xF_{k+1}^{(3)}(x)+F_{k}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{xF_{k}^{(3)}(x)+F_{k-1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{xF_{k-1}^{(3)}(x)+F_{k-2}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}$$

and

$$\min\left\{\frac{F_{k+2}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\}\le\frac{e_{i1}}{e_{i3}}\le\max\left\{\frac{F_{k+2}^{(3)}(x)}{F_{k+1}^{(3)}(x)},\frac{F_{k+1}^{(3)}(x)}{F_{k}^{(3)}(x)},\frac{F_{k}^{(3)}(x)}{F_{k-1}^{(3)}(x)}\right\},\text{for } i=2,3$$

For large k, we have

$$\frac{e_{i1}}{e_{i2}}\approx\frac{r_3^2(x)}{1+xr_3(x)},\frac{e_{i2}}{e_{i3}}\approx\frac{1+xr_3(x)}{r_3(x)},\frac{e_{i2}}{e_{i3}}\approx r_3(x)\quad\text{for } i=1,2,3 \tag{64}$$

Case 3: Generalized relations among the code matrix elements
In general, we can establish the relations among the code matrix elements as:

$$\min\left\{\frac{F_{k+r}^{(n)}(x)}{x^{n-2}F_{k+r-1}^{(n)}(x)+x^{n-3}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+1}^{(n)}(x)}:r=0,1,\dots n-1\right\}\le\frac{e_{i1}}{e_{i2}}$$

$$\le\max\left\{\frac{F_{k+r}^{(n)}(x)}{x^{n-2}F_{k+r-1}^{(n)}(x)+x^{n-3}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+1}^{(n)}(x)}:r=0,1,\dots n-1\right\},$$

$$\min\left\{\frac{F_{k+r}^{(n)}(x)}{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}:r=0,1,\dots n-1\right\}\le\frac{e_{i1}}{e_{i3}}$$

$$\le\max\left\{\frac{F_{k+r}^{(n)}(x)}{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}:r=0,1,\dots n-1\right\},$$

$$\cdots$$

$$\min\left\{\frac{F_{k+r}^{(n)}(x)}{F_{k+r-1}^{(n)}(x)}:r=0,1,\dots n-1\right\}\le\frac{e_{i1}}{e_{in}}\le\max\left\{\frac{F_{k+r}^{(n)}(x)}{F_{k+r-1}^{(n)}(x)}:r=0,1,\dots n-1\right\}$$

$$\min\left\{\frac{x^{n-2}F_{k+r-1}^{(n)}(x)+x^{n-3}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+1}^{(n)}(x)}{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}: r=0,1,\ldots n-1\right\}\le\frac{e_{i2}}{e_{i3}}$$
$$\le\max\left\{\frac{x^{n-2}F_{k+r-1}^{(n)}(x)+x^{n-3}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+1}^{(n)}(x)}{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}: r=0,1,\ldots n-1\right\},$$

$$\min\left\{\frac{x^{n-2}F_{k+r-1}^{(n)}(x)+x^{n-3}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+1}^{(n)}(x)}{x^{n-4}F_{k+r-1}^{(n)}(x)+x^{n-5}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+3}^{(n)}(x)}: r=0,1,\ldots n-1\right\}\le\frac{e_{i2}}{e_{i4}}$$
$$\le\max\left\{\frac{x^{n-2}F_{k+r-1}^{(n)}(x)+x^{n-3}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+1}^{(n)}(x)}{x^{n-4}F_{k+r-1}^{(n)}(x)+x^{n-5}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+3}^{(n)}(x)}: r=0,1,\ldots n-1\right\},$$

$$\cdots$$

$$\min\left\{\frac{x^{n-2}F_{k+r-1}^{(n)}(x)+x^{n-3}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+1}^{(n)}(x)}{F_{k+r-1}^{(n)}(x)}: r=0,1,\ldots n-1\right\}\le\frac{e_{i2}}{e_{in}}$$
$$\le\max\left\{\frac{x^{n-2}F_{k+r-1}^{(n)}(x)+x^{n-3}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+1}^{(n)}(x)}{F_{k+r-1}^{(n)}(x)}: r=0,1,\ldots n-1\right\},$$

$$\min\left\{\frac{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}{x^{n-4}F_{k+r-1}^{(n)}(x)+x^{n-5}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+3}^{(n)}(x)}: r=0,1,\ldots n-1\right\}\le\frac{e_{i3}}{e_{i4}}$$
$$\le\max\left\{\frac{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}{x^{n-4}F_{k+r-1}^{(n)}(x)+x^{n-5}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+3}^{(n)}(x)}: r=0,1,\ldots n-1\right\},$$

$$\min\left\{\frac{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}{x^{n-5}F_{k+r-1}^{(n)}(x)+x^{n-6}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+4}^{(n)}(x)}: r=0,1,\ldots n-1\right\}\le\frac{e_{i3}}{e_{i5}}$$
$$\le\max\left\{\frac{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}{x^{n-4}F_{k+r-1}^{(n)}(x)+x^{n-5}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+4}^{(n)}(x)}: r=0,1,\ldots n-1\right\},$$

$$\cdots$$

$$\min\left\{\frac{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}{F_{k+r-1}^{(n)}(x)}: r=0,1,\ldots n-1\right\}\le\frac{e_{i3}}{e_{i5}}$$
$$\le\max\left\{\frac{x^{n-3}F_{k+r-1}^{(n)}(x)+x^{n-4}F_{k+r-2}^{(n)}(x)+\cdots+F_{k+r-n+2}^{(n)}(x)}{F_{k+r-1}^{(n)}(x)}: r=0,1,\ldots n-1\right\},$$

$$\cdots$$

$$\min\left\{\frac{xF_{k+r-1}^{(n)}(x)+F_{k+r-2}^{(n)}(x)}{F_{k+r-1}^{(n)}(x)}: r=0,1,\ldots n-1\right\}\le\frac{e_{in-1}}{e_{in}}$$
$$\le\max\left\{\frac{xF_{k+r-1}^{(n)}(x)+F_{k+r-2}^{(n)}(x)}{F_{k+r-1}^{(n)}(x)}: r=0,1,\ldots n-1\right\},$$

for $i=1,2,\ldots,n$.

Hence the large value of k, we have

$$\frac{e_{i1}}{e_{i2}}\approx\frac{(r_n(x))^{n-1}}{1+xr_n(x)+x^2(r_n(x))^2+\cdots+x^{n-2}(r_n(x))^{n-2}},\frac{e_{i1}}{e_{i2}}$$
$$\approx\frac{(r_n(x))^{n-2}}{1+xr_n(x)+x^2(r_n(x))^2+\cdots+x^{n-3}(r_n(x))^{n-3}},\ldots,\qquad \frac{e_{i1}}{e_{in}}\approx r_n(x)$$

$$\frac{e_{i2}}{e_{i3}} \approx \frac{1 + xr_n(x) + x^2(r_n(x))^2 + \cdots + x^{n-2}(r_n(x))^{n-2}}{r_n(x) + x^2(r_n(x))^2 + \cdots + x^{n-3}(r_n(x))^{n-2}}, \frac{e_{i2}}{e_{i4}} \approx \frac{1 + xr_n(x) + x^2(r_n(x))^2 + \cdots + x^{n-2}(r_n(x))^{n-2}}{(r_n(x))^2 + x(r_n(x))^3 + \cdots + x^{n-4}(r_n(x))^{n-2}}, \cdots, \frac{e_{i2}}{e_{in}} \approx \frac{1 + xr_n(x) + x^2(r_n(x))^2 + \cdots + x^{n-2}(r_n(x))^{n-2}}{(r_n(x))^{n-2}}$$

$$\frac{e_{i3}}{e_{i4}} \approx \frac{1 + xr_n(x) + x^2(r_n(x))^2 + \cdots + x^{n-3}(r_n(x))^{n-3}}{r_n(x) + x(r_n(x))^2 + \cdots + x^{n-4}(r_n(x))^{n-3}}, \frac{e_{i3}}{e_{i5}} \approx \frac{1 + xr_n(x) + x^2(r_n(x))^2 + \cdots + x^{n-3}(r_n(x))^{n-3}}{(r_n(x))^2 + x(r_n(x))^3 + \cdots + x^{n-5}(r_n(x))^{n-3}}, \cdots, \frac{e_{i3}}{e_{in}} \approx \frac{1 + xr_n(x) + x^2(r_n(x))^2 + \cdots + x^{n-3}(r_n(x))^{n-3}}{(r_n(x))^{n-2}}$$

$$\cdots$$

$$\frac{e_{in-1}}{e_{in}} \approx \frac{1 + xr_n(x)}{r_n(x)}, \text{for } i = 1, 2, \ldots n \tag{65}$$

5. Error detection and correction

To develop a new coding theory it is mandatory to show the corrected ability of this method. The most important feature of this method is the error detection and correction ability. This coding/decoding method gives property to detect and correct errors in the code message E. The error detection and correction is based on the property of the determinant of matrix given by (12). At first we calculate the determinant of the initial matrix P, then we determine the code matrix elements of the code matrix E and send it to a communication channel. We treat Det P as the checking elements of the code matrix E received from the communication channel. We calculate the determinant of the matrix E received and compare it with the given $\det P$ by the relation (12). If they satisfy the relation (12), then we conclude that the elements of the code matrix E received are without errors otherwise there are errors. If there are errors, then we try to correct these errors using the relations (12) and (65).

Next we consider the error detection and correction.

Case 1: $n = 2$.

Our first hypothesis is that we have the case of "single error" in the code matrix E received from the communication channel. It is clear that there are four variants of the "single error" in the code matrix E:

(a) $\begin{pmatrix} x_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$ (b) $\begin{pmatrix} e_{11} & x_{12} \\ e_{21} & e_{22} \end{pmatrix}$ (c) $\begin{pmatrix} e_{11} & e_{12} \\ x_{21} & e_{22} \end{pmatrix}$ (d) $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & x_{22} \end{pmatrix}$

where x_{11}, x_{12}, x_{21} and x_{22} are possible "destroyed" elements.

For checking the hypothesis (a), (b), (c) and (d) we can write the following algebraic equations based on the "checking relations" (12):

$$x_{11}e_{22} - e_{12}e_{21} = \det P \times (-1)^{3k} \text{ (a possible single error in the (1,1)cell)} \tag{66}$$

$$e_{11}e_{22} - x_{12}e_{21} = \det P \times (-1)^{3k} \text{ (a possible single error in the (1,2)cell)} \tag{67}$$

$$e_{11}e_{22} - e_{12}x_{21} = \det P \times (-1)^{3k} \text{ (a possible single error in the (2,1)cell)} \tag{68}$$

and

$$e_{11}x_{22} - e_{12}e_{21} = \det P \times (-1)^{3k} \text{ (a possible single error in the (2,2)cell)} \tag{69}$$

It follows from $(66) - (69)$ that four variants for calculation of the possible "single error" are

$$x_{11} = \frac{(-1)^{3k} + e_{12}e_{21}}{e_{22}} \tag{70}$$

$$x_{12} = \frac{(-1)^{3k} + e_{11}e_{22}}{e_{21}} \tag{71}$$

$$x_{21} = \frac{(-1)^{3k} + e_{11}e_{22}}{e_{12}} \tag{72}$$

and

$$x_{22} = \frac{(-1)^{3k} + e_{12}e_{21}}{e_{22}} \tag{73}$$

To obtain the correct variant, we have to choose the integer solutions of x_{11}, x_{12}, x_{21} and x_{22} satisfying the additional "checking relations" (31) and (32). If this fails then we have to conclude that our hypothesis about "single error" is incorrect or in other words there is more than one error in the code matrix E received.

Now, we consider one of the "double errors" in the code matrix E received as

$$\begin{pmatrix} x_{11} & x_{12} \\ e_{21} & e_{22} \end{pmatrix} \tag{74}$$

Using the "checking relation" (12) we have the following algebraic equation for the matrix (74):

$$x_{11}e_{22} - x_{12}e_{21} = \det P \times (-1)^{3k} \tag{75}$$

It is important to emphasize that the equation (75) is "Diophantine" equation in two variables. As the "Diophantine" equation has many solutions, we must select such solutions of x_{11} and x_{12} which satisfy the additional "checking relations" (31) and (32). In this way we can correct all possible "double errors" in the code matrix E received which satisfy the "checking relations" $(12), (31)$ and (32). Otherwise, there may be "triple errors" in the code matrix E received and we try to correct all possible "triple errors" by the similar approach.

Let one form of the incorrect code matrices E received having "triple errors" be $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & e_{22} \end{pmatrix}$.

Thus, we try to correct the errors in the code matrix E received based on different hypotheses "single error", "double errors" and "triple errors" using the "checking relations" $(12), (31)$ and (32) together with considering the integral elements of the code matrix E received.

Code matrix E received is "erroneous" or the case of "fourfold errors" if at least one of the previous solutions is not integer. In this case we reject the code matrix E received.

Thus we have

$$\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^{2^2} - 1 = 15$$

cases of errors in the code matrix E received.

Our method shows that it allows correcting all possible "single error", "double errors" and "triple errors" i.e. 14 cases among them. Hence the correct ability of errors of this method is $\frac{14}{15} = 0.9333 = 93.33\%$ which does not depend on the value of x.

Case 2: $n = 3$.

At first, we consider "single error" in the code matrix E received. It is clear that there are nine variants of "single error" in the code matrix E received. For example, one of them is

$$\begin{pmatrix} x_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \tag{76}$$

where x_{11} is the possible destroyed element in the $(1,1)$ cell.

Using relation (12), we have the algebraic equation of the matrix (76):

$$x_{11}(e_{22}e_{33} - e_{23}e_{32}) + e_{12}(e_{23}e_{31} - e_{21}e_{33}) + e_{13}(e_{21}e_{32} - e_{22}e_{31}) = \det P \tag{77}$$

There are nine equations similar to (77) for nine possible variants of "single error" $x_{ij}, i, j = 1, 2, 3$. But we have to select the correct variant only among these cases of the integer solutions of $x_{ij}, i, j = 1, 2, 3$ satisfying the relations (64). If there is no integer solution, we conclude that our hypothesis about "single error" is incorrect or we have more than one error in the code matrix E received.

Now we check all hypotheses of "double errors" in the code matrix E received. We consider one of the "double errors" cases in the code matrix E received as:

$$\begin{pmatrix} x_{11} & x_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \tag{78}$$

Using relation (12), we have the algebraic equation of the matrix (78):

$$x_{11}(e_{22}e_{33} - e_{23}e_{32}) + x_{12}(e_{23}e_{31} - e_{21}e_{33}) + e_{13}(e_{21}e_{32} - e_{22}e_{31}) = \det P \tag{79}$$

According to the relation (64) there is the following relation between x_{11} and x_{12}:

$$x_{11} \approx \frac{r_3^2(x)}{r_3(x)+1} x_{12} \tag{80}$$

Equation (79) is "Diophantine". As the "Diophantine" equation (79) has many solutions, we have to choose integer solutions x_{11} and x_{12}, which satisfy the relation (80).

It is clear that there are $\binom{9}{2}$= 36 variants of "double errors" in the code matrix E received and by using similar approach, we try to correct all "double errors" in the code matrix E received.

Now there are

$$\binom{9}{1} + \binom{9}{2} + \cdots + \binom{9}{9} = 2^{3^2} - 1 = 511$$

possible cases of errors in the code matrix E received. We try to correct all possible triple, fourfold ..., eightfold errors in the code matrix E received using this approach. But we know that it is not possible to correct the "nine fold errors" in the code matrix E received.

Hence the correct ability of errors in this method is $\frac{510}{511} = 0.9980 = 99.80\%$ which is independent on the value of x.

Case 2: $n = m$, where m is large.

In this case, the correct ability of the method is $\frac{2^{m^2}-2}{2^{m^2}-1}$ which depends on the value of n but not on the value of

x. The interesting feature of this method is for large value of n the possibility to correct errors is

$$\frac{2^{n^2}-2}{2^{n^2}-1} \approx 1 = 100\%$$

which does not depend on the value of x.

6. Conclusion

The Fibonacci n-step polynomials coding/decoding method is the main application of the $M_n(x)$ matrices. This coding/decoding method differs from the classical algebraic codes by the following peculiarities:

1. Fibonacci n-step polynomials coding/decoding method converts to matrix multiplication which is very well known method and not time consuming via modern computers.
2. The correct ability of errors of this method for the case $n = 2$ is 93.33% which corrects up to triple errors among four fold errors, for the case $n = 3$ is 98.80% which corrects up to eight fold errors among nine fold errors.
3. The correct ability of errors of this method increases as n increases and for large value of n the correct ability is approximately equal to 1 which does not depend on the value of x.
4. M.S. EL Naschie [7] shows that the Fibonacci series and the golden mean plays a very important role in the construction of a relatively new space-time theory, which is referred to as $\in^{(\infty)}$ Cantorian-fractal space time or E-infinity theory [8, 9]. He [10] also shows that there are relations among E-infinity

theory, string theory, exceptional lie symmetry group and various physical quantities. He [11] explains the existence of these relations using the geometric and topology of space-time as claimed by E-infinity theory. It is certain that based on these theories, Fibonacci n-step polynomials will give a better result in E-infinity theory, string theory etc

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