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Wavelet Preconditioners of Electrohydrodynamic flow problem

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Abstract. In this paper wavelet preconditioned method is used for the solution of Electrohydrodynamic flow problem. A finite difference method is used for the solution of Electrohydrodynamic flow equation. The method comprise the nonlinear Newton iteration on the outer loop and a linear iteration on the inside loop where wavelet based preconditioned GMRES (Generalized Minimum Residual) method is used. In the scheme the Jacobian vector product is approximated accurately with much ease (without forming Jacobian explicitly and requiring no extra storage). It overcomes the limitations of conventional schemes for the numerical solution of Electrohydrodynamic flow problem, for computing fluid velocity, covering wide range of Hartmann electric number (*Ha*) parameter with constant α of practical interest. To confirm and validate the solutions obtained, by the present method, are compared with those obtained by GMRES method.

Keywords: Classical preconditioners, Wavelet preconditioners, Electrohydrodynamic flow, Hartmann electric number.

1. Introduction

Wavelet analysis is a new numerical tool that allows one to represent a function as a linear combination of building blocks (a basis), called wavelets, which are localized in both translation and dilation. Good wavelet localization properties in physical and wave number spaces are to be contrasted with the spectral approach, which employs infinitely differentiable functions but with global support and small discrete changes in the resolution. The various types of wavelets have been used in current research areas in which haar wavelet is the simplest wavelet because of simple applicability, orthogonality and compact support. The haar wavelet based techniques has been successfully used in various applications such as time–frequency analysis, signal de-noising, numerical approximation and solving differential equations (Chen and Hsiao [1], Hsiao and Wang [2], Hsiao [3] and Lepik [4-6]).

Most of the technical problems and engineering phenomena are frequently defined based on the nonlinear differential equations. It is essential to remember that except a limited number of these problems, finding efficient solution is difficult. Therefore, researchers used some classical numerical methods. The classical numerical methods such as the Newtons method (NM), ILU, GMRES are useful numerical techniques ones, which present the numerical solutions for nonlinear differential equations. In this paper, numerical solution of electrohydrodynamic flow problem of a fluid in an ion drag configuration in a circular cylindrical conduit is presented using wavelet preconditioners. The electrohydrodynamic flow of a fluid and its governing equations is considered [7]. The electrohydrodynamic flow is important in analysis of the flow meters, accelerators, pumps and magnetohydrodynamic generators. The differential equation of the problem is the nonlinear singular boundary value problem.

$$y''+\frac{1}{x}y'+Ha\left(1-\frac{y}{1-\alpha y}\right)=0, \quad 0<x<1 \tag{1.1}$$

$$y'(0)=0, \quad y(1)=0 \tag{1.2}$$

where y is the velocity of the fluid, x is the radial distance from the center of cylindrical conduit, Ha is a constant (Hartmann electric number) and α is also a constant that shows the nonlinearity of the problem. Existence and uniqueness of solution of Eqs. (1.1) and (1.2) are discussed by Paullet [8].

As we know, most of the flow and heat transfer equations are nonlinear and usually have not an efficient solution. So, numerical techniques are used many researchers to solve such equations. The most known numerical methods used to solve electrohydrodynamic flow problem is the Newtons, ILU and

GMRES etc. these classical methods gives slow convergence as well as more computational time. To overcome these limitations, a significant challenge is to solve the problem efficiently with faster convergence and less computational time, because Eq. (1.1) is a singular nonlinear BVP, and the type of nonlinearity is in the form of a rational function. Numerical solution of this problem is presented using wavelet based preconditioners.

The present work is organized as follows; Preliminaries are given in section 2. Method of solution with numerical experiment is discussed in section 3. Finally, conclusion of the proposed work is drawn in section 4.

2. Preliminaries

Wavelets are functions generated from one single function called the mother wavelet by the simple operations of dilation and translation. A mother wavelet gives rise to a decomposition of the Hilbert space $L^2(\mathrm{R})$, into a direct sum of closed subspaces $W_j,\ j\in \mathrm{Z}$. [9]

Let

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

and

$$W_j = clos_{L^2(\mathrm{R})}\left[\psi_{j,k} : k\in \mathrm{Z}\right] \tag{2.1}$$

Then every $f\in L^2(\mathrm{R})$ has a unique decomposition

$$f(x) = \ldots + s_{-1} + s_0 + s_1 + \ldots \tag{2.2}$$

Where $s_j \in W_j$ for all $j\in \mathrm{Z}$, it is

$$L^2(\mathrm{R}) = \sum_{j\in \mathrm{Z}} W_j = \ldots \oplus W_{-1}\oplus W_0 \oplus W_1 \oplus \ldots \tag{2.3}$$

Using this decomposition of $L^2(\mathrm{R})$, a nested sequence of closed subspaces $V_j,\ j\in \mathrm{Z}$ of $L^2(\mathrm{R})$ can be obtained, defined by

$$V_j = \ldots \oplus W_{j-2}\oplus W_{j-1} \tag{2.4}$$

These closed subspaces $\{V_j,\ j\in \mathrm{Z}\}$ of $L^2(\mathrm{R})$, form a "multiresolution analysis" with the following properties:

i) $\ldots \subset V_{-1}\subset V_0 \subset V_1 \subset \ldots$

ii) $\mathrm{clos}_{L^2}\left(\bigcup V_j\right) = L^2(\mathrm{R})$

iii) $\bigcap V_j = \{0\}$

iv) $V_{j+1} = V_j \oplus W_j$

v) $f(x)\in V_j \Leftrightarrow f(2x)\in V_{j+1},\ j\in \mathrm{Z}$

Let $\phi\in V_0$ the so-called "scaling function" that generates the multiresolution analysis $\{V_j\}_{j\in \mathrm{Z}}$ of $L^2(\mathrm{R})$. Then

$$\{\phi(-k) : k\in \mathrm{Z}\} \tag{2.5}$$

is a basis of V_0, and by setting

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \tag{2.6}$$

it follows that, for each $j\in \mathrm{Z}$, the family

$$\{\phi_{j,k} : k\in \mathrm{Z}\} \tag{2.7}$$

is also a basis of V_j.

Then, since $\phi\in V_0$ is in V_1 and since $\{\phi_{1,k} : k\in \mathrm{Z}\}$ is a basis of V_1, there exists a unique sequence a_k that describes the following "two-scale relation":

$$\phi(x) = \sum_{k=-\infty}^{\infty} a_k\,\phi(2x-k) \tag{2.8}$$

of the scaling function ϕ.

Different choices for ϕ may yield different multiresolution analyses, and the most useful scaling functions are those that have compact support. As an example of multiresolution analysis, a family of orthogonal Daubechies wavelets with compact support has been constructed by Daubechies [9].

A wavelet basis is orthonormal if any two translated or dilated wavelets satisfy the condition

$$\int_{-\infty}^{\infty} \psi_{n,k}(x)\psi_{m,l}(x)\,dx = \delta_{n,m}\,\delta_{k,l} \tag{2.9}$$

where δ is the Kronecker Delta function.

Each wavelet family is governed by a set of N (an even integer) coefficients $a_k : k = 0, 1, \ldots, N-1$ through the two-scale relation

$$\phi_N(x) = \sum_{k=0}^{N-1} a_k \phi_N(2x-k) \tag{2.10}$$

Based on the scaling function $\phi_N(x)$, the mother wavelet can be written as,

$$\psi_N(x) = \sum_{k=2-N}^{1} b_k \phi_N(2x-k) \tag{2.11}$$

Since the wavelets are orthonormal to the scaling basis the coefficients of the scaling function and the mother wavelet for the two-scale equation are related by:

$$b_k = (-1)^k a_{1-k} \tag{2.12}$$

In her work, Daubechies [10] found and exploited the link between vanishing moments of the wavelet ψ and regularity of wavelet and scaling functions, ψ and ϕ. The wavelet function ψ has K vanishing moments if

$$\int x^k \psi(x)\,dx = 0 \text{ for } 0 \le k \le K \tag{2.13}$$

and a necessary and sufficient condition for this to hold is that integer translates of the scaling function ϕ exactly interpolate polynomials of degree up to *K*. That is, for each *k*, $0 \le k \le K$ there exists constants c_l such that

$$x^k = \sum_l c_l^k \phi_l(x) \tag{2.14}$$

Daubechies introduced scaling functions satisfying this property and distinguished by having the shortest possible support. The scaling function ϕ_N (where *N* is an even integer) has support $[0, N-1]$, while the corresponding wavelet ψ_N has support in the interval $[1-N/2, N/2]$ and has $(N/2-1)$ vanishing wavelet moments [10]. Thus, according to Eqn. (2.14) Daubechies scaling functions of order *N* can exactly represent any polynomial of order up to, but not greater than $N / 2 - 1$.

The coefficients a_k in Eqn. (2.10) are called scaling function filter coefficients and satisfy the following conditions based on the orthonormality and moment conditions.

$$\begin{aligned} &\sum_{k=0}^{N-1} a_k = 2 \\ &\sum_{k=0}^{N-1} a_k\, a_{k+2m} = 2\delta_{0,m} \\ &\sum_{k=0}^{N-1} (-1)^k k^m a_k = 0 \quad for\ m = 0,1,\ldots, N/2-1 \end{aligned} \tag{2.15}$$

Solving Eqn. (2.15) we get the scaling function coefficients a_k. If *N* = *4*, we get the scaling coefficients: $a_0 = 0.68301270,\ a_1 = 1.18301270,\ a_2 = 0.31698729,\ a_3 = -0.18301270$. Using $h(k) = a_k / \sqrt{2}$, we get, $h(0) = 0.482962913,\ h(1) = 0.836516303,\ h(2) = -0.129409522,\ h(3) = 0.224143868$.

Once we obtain coefficients *{ h (k)}* , we can find *{ g (k) }* . Just reverse the coefficients and change the sign at the alternate positions. Therefore $g(0)=h(3), g(1)=-h(2), g(2)=h(1), g(3)=-h(0)$ using these coefficients in wavelet preconditioner matrix as given in Chen [11].

3. Method of solution and Numerical experiment

The finite difference discretization of the equations (1),

$$\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}+\frac{1}{x_i}\left(\frac{y_{i-1}-y_{i+1}}{h}\right)+Ha\left(1-\frac{y_i}{1-\alpha y_i}\right)=0, \quad 0<x<1, \ i=1,2,3,...,n, \ h=\frac{1-0}{n+1}$$

with boundary conditions, $y'(0)=0, \ y(1)=0$. After getting system of nonlinear equations are solved using Newton-GMRES method. For each Newton iteration (outer loop) there is a linear system to be solved using restarted GMRES(*m*) (inner loop). For implementing GMRES(*m*) restart GMRES every *m* steps using latest iterate as the initial guess for the next GMRES cycle. For the convergence of GMRES (25) use Daubechies wavelet based preconditioners (details given in Bujurke et al. [12]). The solution scheme consists of the following steps.

Step-1: take initial values of y_i and constant values $Ha, \ \alpha$

Step-2: using Newton's scheme $y(x_{i+1})=y(x_i)-\dfrac{f(x_i)}{f'(x_i)}$

Step-3: we assume $c=\dfrac{f(x_i)}{f'(x_i)} \Rightarrow f'(x_i)c=f(x_i)$

Step-4: solve Step-3 we get c

Step-5: substitute c value in Step-3 then go to Step-2, we get $y(x_{i+1})$

Step-6: numerical underrelaxation for stability and convergence of iteration

$$y_{i+1}^{new}=y_i^{old}+C(y_i^{new\ predictor}-y_i^{old}), \quad 0.01<C<0.5$$

Step-7: while not converged go to Step-2

The choice of the stopping criteria for linear system is guided by the behavior of the solution of the nonlinear system at the previous Newton iteration. To optimize the cost of implementation and ensure accurate solution a constant tolerance of $1E-06$ is taken in our computations. Once accurate solution of linear system is obtained Newton-inexact scheme (requiring just 4-6 iterations) enables in finding solution of fluid velocity $y(x)$ accurately with error norm less than $1E-08$. This computation is repeated for different set of Hartmann electric number (*Ha*) and fixed constant α . All the computations are done in MATLAB in double precision, as shown in table 1 and figures 1, 2, 3 and 4.

Table 1. The sensitivity of convergence of four schemes with Hartmann electric number ($Ha=1$) and constant $\alpha=0.5$.

n	Newton – GMRES with ILU Preconditioner [43] (Iterations)	Newton – GMRES with Wavelet Preconditioner		
		DWT (Iterations)	DWTPer (Iterations)	DWTPerMod (Iterations)
64	Yes (78)	Yes (43)	Yes (18)	Yes (11)
128	Yes (165)	Yes (75)	Yes (20)	Yes (24)
256	Yes (237)	Yes (189)	Yes (201)	Yes (55)
512	Yes (312)	Yes (201)	Yes (201)	Yes (76)
1024	Yes (350)	Yes (245)	Yes (201)	Yes (81)

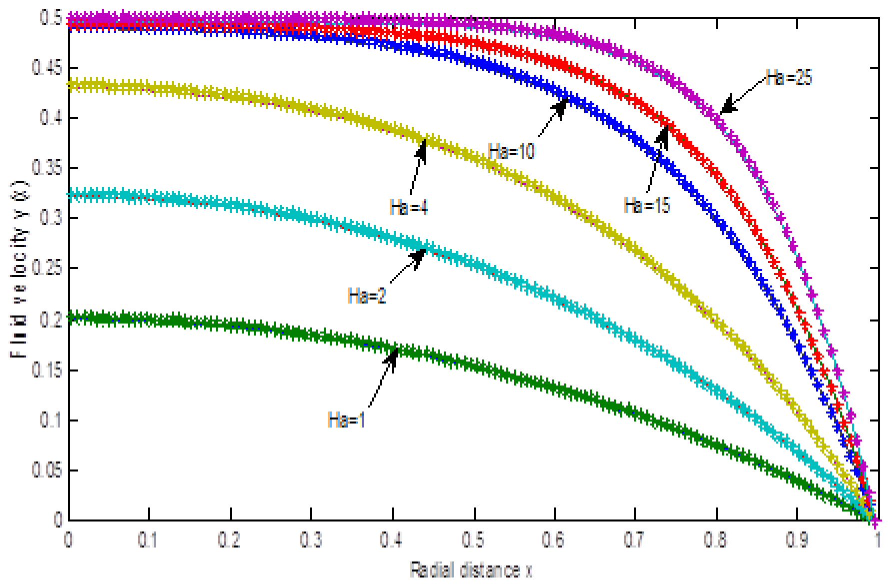

Fig. 1. Numerical solution of fluid velocity $y(x)$ for $n=128$, $\alpha=0.5$ of Eq. (1.1).

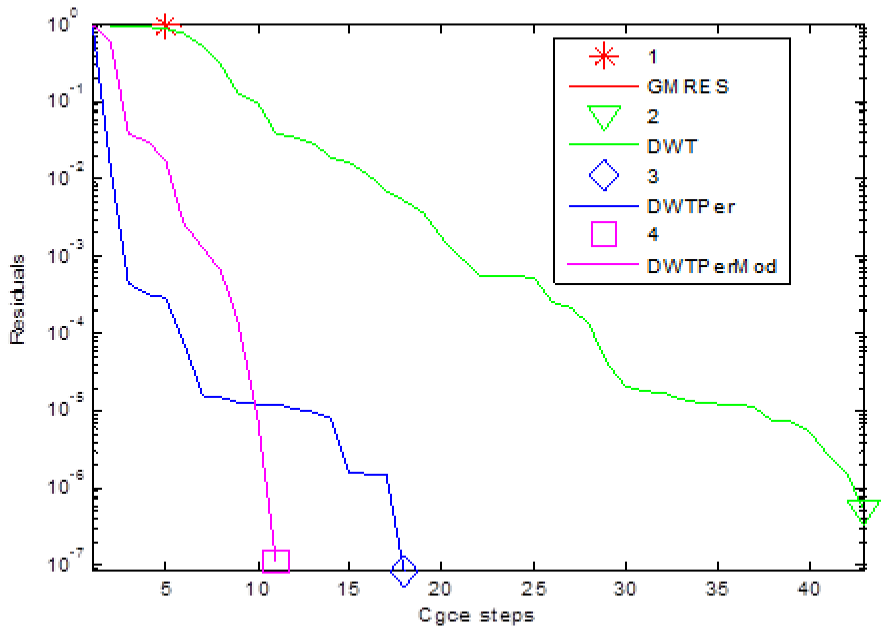

Fig. 2. Convergence of the fluid velocity $y(x)$ using different schemes with Hartmann electric number ($Ha=1$) and constant $\alpha=0.5$ & $n=64$.

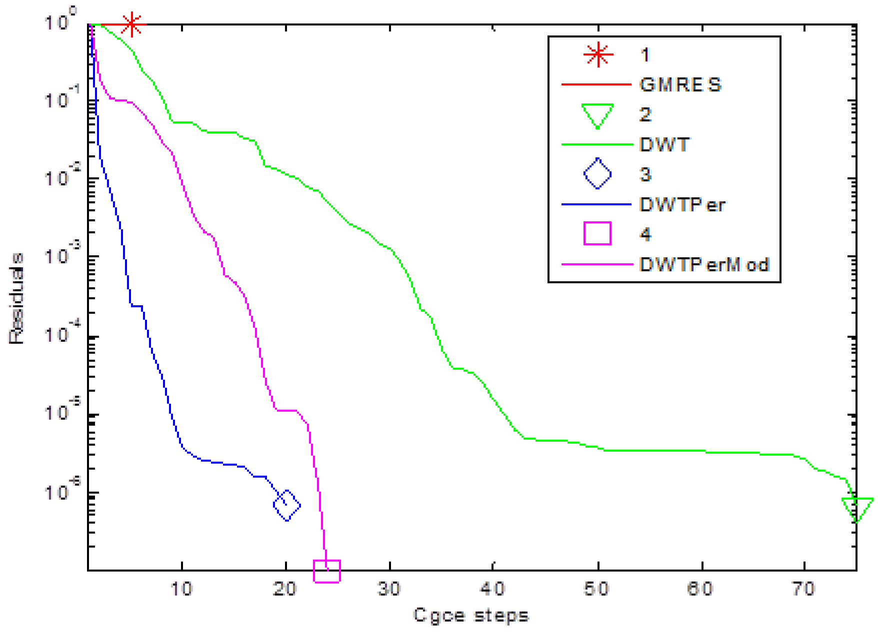

Fig. 3. Convergence of the fluid velocity $y(x)$ using different schemes with Hartmann electric number ($Ha = 1$) and constant $\alpha = 0.5$ & $n = 128$.

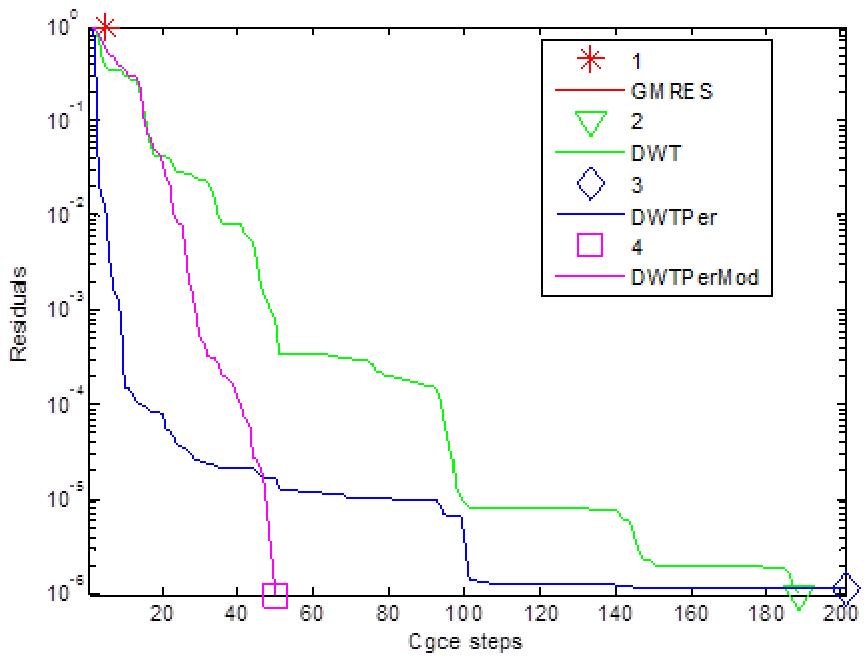

Fig. 4. Convergence of the fluid velocity $y(x)$ using different schemes with Hartmann electric number ($Ha = 1$) and constant $\alpha = 0.5$ & $n = 256$.

4. Conclusions

An important issue of obtaining converging solution of fluid velocity $y(x)$ using Jacobian free Newton-Krylov (GMRES (25)) method, with Daubechies wavelet based preconditioners, is presented here.

Fluid velocity $y(x)$ solution is computed accurately. Later, the fluid velocity $y(x)$ is presented for various parameters of Hartmann electric number. The robustness of these methods overcomes the sensitivity of classical iteration schemes in fluid velocity solutions to slight variations of parameters, Hartmann electric number. Wavelet based preconditioners reduce the number of iterations and accelerate the convergence of GMRES in solving the linear system, almost exactly. Also, the computations confirm these schemes to be equally reliable and attractive alternatives to established GMRES methods used in fluid flow problems.

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