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Notes on Fuzzy Linear Systems

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Abstract. The paper discusses the detail solution procedure of fuzzy linear systems and fully fuzzy linear system. Each case has been solved by using the concept of Strong and Weak solution with some numerical examples. We have also developed some theorems regarding the existence of the solution.

Keywords: Fuzzy Linear System (FLS), Fully Fuzzy Linear System(FFLS), Generalized Trapezoidal Fuzzy Number (GTrFN), Strong and Weak solutions.

1. Introduction

System of linear equations has so many applications in various areas of mathematical, physical and engineering sciences. It is used to solve several problems in fluid flow, circuit analysis, heat transport, structural mechanics etc. In most of the problems, the measurements and system's parameters are vague or imprecise. We can handle the situation by representing the given data as the fuzzy numbers rather than crisp numbers.

In literature, the concept of fuzzy numbers and arithmetic operations on it introduced by Zadeh [10,11]. Further standard analytical techniques to solve fuzzy linear equation and linear system were proposed by Buckley and Qu [5,6,8,9]. Buckley [6,7] considered the solution of linear fuzzy equations using Classical methods and Zadeh's extension principle. Fuzzy linear systems are the linear systems whose parameters are all or partially represented by fuzzy numbers. A general model for solving a Fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number was first proposed by Friedman et al. [12]. They have used the parametric form of fuzzy numbers and replace the original n×n fuzzy system by a 2n×2n crisp system. Fuzzy linear system has been studied by several authors [1,2,4,13,14,15, 16, 17,18].

In this paper we have used another approach to solve Fuzzy Linear System and Fully Fuzzy Linear System. Here we have solved both systems respectively by using the concept of Strong and Weak solution .We have also developed some conditions following[12] for existence of the strong solution. Each case has been illustrated by numerical examples along the methods.

2. Basic Concept

Definition 2.1: Fuzzy Set: Let X be a universal set. The fuzzy set $\tilde{A} \subseteq X$ is defined by the set of tuples as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \colon \mu_{\tilde{A}} \colon X \to [0,1]\}$.

Definition 2.2: ∝-cut of a fuzzy set: Let X be an universal set. Let $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} (\subseteq X)$ be a fuzzy set. ∝ -cut of the fuzzy set $\tilde{A}$ is a crisp set. It is denoted by $A_\propto$. It is defined as

$$A_\propto = \{x : \mu_{\tilde{A}}(x) \geq \propto \ \forall x \in X\}.$$

Definition 2.3: Convex fuzzy set: A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} \subseteq X$ is called convex fuzzy set if all $A_\propto$ are convex sets i.e. for every element $x_1 \in A_\propto$ and $x_2 \in A_\propto$ and for every $\propto \in [0,1]$

$\lambda x_1 + (1-\lambda)x_2 \in A_\propto \quad \forall \lambda \in [0,1]$. Otherwise the fuzzy set is called non convex fuzzy set.

Definition 2.4: Fuzzy Number: $\tilde{A} \in \mathcal{F}(R)$ is called a fuzzy number where R denotes the set of whole real numbers if

i. $\tilde{A}$ is normal i.e. $x_0 \in R$ exists such that $\mu_{\tilde{A}}(x_0) = 1$.

ii. $\forall \propto \in (0,1]$ $A_\propto$ is a closed interval.

If $\tilde{A}$ is a fuzzy number then $\tilde{A}$ is a convex fuzzy set and if $\mu_{\tilde{A}}(x_0) = 1$ then $\mu_{\tilde{A}}(x)$ is non decreasing for $x \leq x_0$ and non increasing for $x \geq x_0$.

The membership function of a fuzzy number $\tilde{A}\ (a_1, a_2, a_3, a_4)$ is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & x \in [a_2, a_3] \neq \phi \\ L(x), & a_1 \leq x \leq a_2 \\ R(x), & a_3 \leq x \leq a_4 \end{cases}$$

Where L(x) denotes an increasing function and $0 < L(x) \leq 1$ and R(x) denotes a decreasing function and $0 \leq R(x) < 1$.

Definition 2.5: Generalized Fuzzy Number (GFN): Generalized Fuzzy number $\tilde{A}$ as

$$\tilde{A} = (a_1, a_2, a_3, a_4; w),$$

where

$$0 < w \leq 1,$$

and $a_1, a_2, a_3, a_4\ (a_1 < a_2 < a_3 < a_4)$ are real numbers. The Generalized Fuzzy Number $\tilde{A}$ is a fuzzy subset of real line R, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

1) $\mu_{\tilde{A}}(x)$: R → [0, 1]
2) $\mu_{\tilde{A}}(x) = 0$ for $x \leq a_1$
3) $\mu_{\tilde{A}}(x)$is strictly increasing function for $a_1 \leq x \leq a_2$
4) $\mu_{\tilde{A}}(x) = w$ for $a_2 \leq x \leq a_3$
5) $\mu_{\tilde{A}}(x)$is strictly decreasing function for $a_3 \leq x \leq a_4$
6) $\mu_{\tilde{A}}(x) = 0$ for $a_4 \leq x$

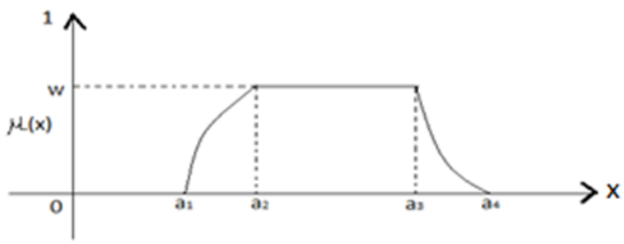

Fig-2.1:-Membership function of a GFN

Definition2.6: Generalized Trapezoidal Fuzzy number (GTrFN):

A Generalized Fuzzy Number $\tilde{A} = (a_1, a_2, a_3, a_4; w)$, is called a Generalized Trapezoidal Fuzzy Number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1 \\ w\dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ w, & a_2 \leq x \leq a_3 \\ w\dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4 \\ 0, & x \geq a_4 \end{cases}$$

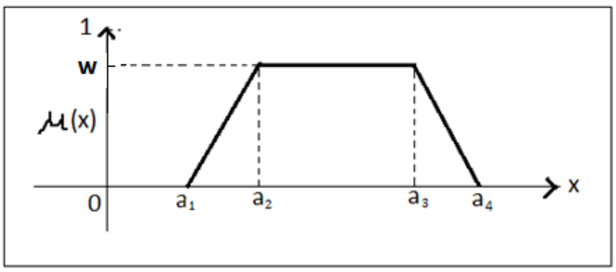

Fig-2.2:-Generalized Trapezoidal Fuzzy Number (GTrFN)

Here $l_s = a_2 - a_1$ is called left spread and $r_s = a_4 - a_3$ is called right spread.

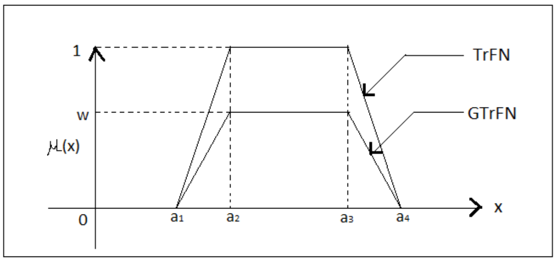

Fig-2.3:Comparison between membership function of TrFN and GTrFN

Note:

1. If $a_2 = a_3$then GTrFN is transformed into a Generalized Triangular Fuzzy Number(GTFN)
2. If $w = 1$ then GTrFN is transformed into a Trapezoidal Fuzzy Number(TrFN)(a_1, a_2, a_3, a_4) and if again here $a_2 = a_3$ then TrFN is transformed into a Triangular Fuzzy Number(TFN)

Definition 2.7: Equality of two GTrFNs: Two GTrFNs $\tilde{A} = (a_1, a_2, a_3, a_4; w_1)$ and $\tilde{B} = (b_1, b_2, b_3, b_4; w_2)$ are equal when $a_1 = b_1, a_2 = b_2, , a_3 = b_3, a_4 = b_4$ and $w_1 = w_2$.

Definition 2.8: Let $\tilde{A} = (a_1, a_2, a_3, a_4; w_1)$ and $\tilde{B} = (b_1, b_2, b_3, b_4; w_2)$ be two positive GTrFNs.

Let $w = min(w_1, w_2)$ where $0 < w \leq 1$

(i) The addition of two GTrFNs $\tilde{A}, \tilde{B}$ is another GTrFN

$$\tilde{C} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; w)$$

(ii) The subtraction of two GTrFNs $\tilde{A}, \tilde{B}$ is another GTrFN

$$\tilde{C} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1; w)$$

(iii) The multiplication of two GTrFNs is a Generalized Tripezoidal Shaped Fuzzy Number

$$\tilde{C} \approx (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4; w)$$

(iv) The division of two GTrFNs is a Generalized Trapezoidal Shaped Fuzzy Number

$$\tilde{C} \approx \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}; w\right)$$

Definition 2.9: A fuzzy number is completely determined by a pair $[X_1(\alpha), X_2(\alpha)]$ of functions $X_1(\alpha), X_2(\alpha)$,$0 \leq \alpha \leq 1$, which satisfy the following requirements:

1. $X_1(\alpha)$ is a bounded monotonic increasing left continuous function over [0,1]. i.e. $\frac{d}{d\alpha}[X_1(\alpha)] > 0$
2. $X_2(\alpha)$ is a bounded monotonic decreasing left continuous function over [0,1]. i.e. $\frac{d}{d\alpha}[X_2(\alpha)] < 0$
3. $X_1(\alpha) \leq X_2(\alpha)\ \forall \alpha \in (0, 1]$

Definition 2.10: Consider the $m \times n$ linear system

$$\begin{aligned} a_{11}\widetilde{x_1} + a_{12}\widetilde{x_2} + \cdots + a_{1n}\widetilde{x_n} &= \widetilde{b_1} \\ a_{21}\widetilde{x_1} + a_{22}\widetilde{x_2} + \cdots + a_{2n}\widetilde{x_n} &= \widetilde{b_2} \\ &\vdots \\ a_{m1}\widetilde{x_1} + a_{m2}\widetilde{x_2} + \cdots + a_{mn}\widetilde{x_n} &= \widetilde{b_n} \end{aligned} \tag{2.1}$$

where the co-efficient matrix $A = (a_{ij})$, $1 \leq i \leq$ m, $1 \leq$ j $\leq$ n is a crisp $m \times n$ matrix and $\widetilde{b_i}, 1 \leq i \leq m$ and $m \leq n$ are known fuzzy numbers and $\widetilde{x_j}$,$1 \leq j \leq n$ are unknown fuzzy numbers.

Then the above system is called a Fuzzy Linear System (FLS).

Theorem-2.1: A fuzzy number vector $(\widetilde{x_1}, \widetilde{x_2}, \dots, \widetilde{x_n})^T$ given by

$x_j = \left(\underline{x_j}(\alpha), \overline{x_j}(\alpha)\right), 1 \leq j \leq n, 0 \leq \alpha \leq 1$ is called a solution of the FLS (2.1) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{b_i} \\ \overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{b_i} \end{cases} \tag{2.2}$$

If for a particular i, $a_{ij} > 0, 1 \leq j \leq n$, we simply get

$$\sum_{j=1}^{n} a_{ij}\underline{x_j} = \underline{b_i}\,, \sum_{j=1}^{n} a_{ij}\,\overline{x_j} = \overline{b_i} \tag{2.3}$$

Definition 2.11: Consider the $m \times n$ linear system

$$\begin{aligned} \widetilde{a_{11}}\widetilde{x_1} + \widetilde{a_{12}}\widetilde{x_2} + \cdots + \widetilde{a_{1n}}\widetilde{x_n} &= \widetilde{b_1} \\ \widetilde{a_{21}}\widetilde{x_1} + \widetilde{a_{22}}\widetilde{x_2} + \cdots + \widetilde{a_{2n}}\widetilde{x_n} &= \widetilde{b_2} \\ &\vdots \\ \widetilde{a_{m1}}\widetilde{x_1} + \widetilde{a_{m2}}\widetilde{x_2} + \cdots + \widetilde{a_{mn}}\widetilde{x_n} &= \widetilde{b_m} \end{aligned} \tag{2.4}$$

where the co-efficient matrix $\tilde{A} = \left(\widetilde{a_{ij}}\right)$, $1 \le i \le$ m, $1 \le$ j $\le$ n is a fuzzy $m \times n$ matrix , $\widetilde{b_i}$,$1 \le i \le m$ are known fuzzy numbers and $\widetilde{x_j}$,$1 \le j \le n$ are unknown fuzzy numbers.

Then the above system is called a Fully Fuzzy Linear System (FFLS).

Theorem-2.2: A fuzzy number vector $(\widetilde{x_1}, \widetilde{x_2}, \ldots, \widetilde{x_n})^T$ given by

$\left(x_j\right)_\alpha = x_j = \left[\underline{x_j}(\alpha), \overline{x_j}(\alpha)\right], 1 \le s \le n, 0 \le \alpha \le 1$ is called a solution of the fuzzy linear system (2.4) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{b_i} \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{b_i} \end{cases} \tag{2.5}$$

If for a particular i, $\widetilde{a_{ij}} > 0, 1 \le j \le n$, we simply get

$$\sum_{j=1}^{n} \underline{a_{ij}}\ \underline{x_j} = \underline{b_i}\,, \sum_{j=1}^{n} \overline{a_{ij}}\ \overline{x_j} = \overline{b_i} \tag{2.6}$$

3. Solution procedures

3.1. Solution method for m×n FLS

Consider the $m \times n$ FLS

$$\begin{aligned} a_{11}\widetilde{x_1} + a_{12}\widetilde{x_2} + \cdots + a_{1n}\widetilde{x_n} &= \widetilde{b_1} \\ a_{21}\widetilde{x_1} + a_{22}\widetilde{x_2} + \cdots + a_{2n}\widetilde{x_n} &= \widetilde{b_2} \\ &\vdots \\ a_{m1}\widetilde{x_1} + a_{m2}\widetilde{x_2} + \cdots + a_{mn}\widetilde{x_n} &= \widetilde{b_n} \end{aligned} \tag{3.1.1}$$

where the co-efficient matrix $A = \left(a_{ij}\right)$, $1 \le i \le$ m, $1 \le$ j $\le$ n is a crisp $m \times n$ matrix and $\widetilde{b_i}, 1 \le i \le m$ and $m \le n$ are known fuzzy numbers and $\widetilde{x_j}$,$1 \le j \le n$ are unknown fuzzy numbers.

In order to solve the above FLS (3.1.1) we must solve a $(2m) \times (2n)$ crisp linear system where the right hand side column is the function vector $\left(\underline{b_1}, \underline{b_2}, \ldots, \underline{b_m}, \overline{b_1}, \overline{b_2}, \ldots, \overline{b_m}\right)^T$.

Let us now rearrange the linear system so that the unknowns are $\underline{x_j}, (-\overline{x_j})$,

$1 \le j \le n$, and the right hand side column is $b = \left(\underline{b_1}, \underline{b_2}, \ldots, \underline{b_m}, -\overline{b_1}, -\overline{b_2}, \ldots, -\overline{b_m}\right)^T$.

We get the $(2m) \times (2n)$ crisp linear system

$$\begin{gathered} S_{11}\underline{x_1} + S_{12}\underline{x_2} + \cdots + S_{1n}\underline{x_n} + S_{1,n+1}(-\overline{x_1}) \\ +S_{1,n+2}(-\overline{x_2}) + \cdots + S_{1,2n}(-\overline{x_n}) = \underline{b_1} \\ \vdots \\ S_{m1}\underline{x_1} + S_{m2}\underline{x_2} + \cdots + S_{mn}\underline{x_n} + S_{m,n+1}(-\overline{x_1}) \\ +S_{m,n+2}(-\overline{x_2}) + \cdots + S_{m,2n}(-\overline{x_n}) = \underline{b_m} \\ S_{m+1,1}\underline{x_1} + S_{m+1,2}\underline{x_2} + \cdots + S_{m+1,n}\underline{x_n} + S_{m+1,n+1}(-\overline{x_1}) \\ +S_{m+1,n+2}(-\overline{x_2}) + \cdots + S_{m+1,2n}(-\overline{x_n}) = -\overline{b_1} \\ \vdots \\ S_{2m,1}\underline{x_1} + S_{2m,2}\underline{x_2} + \cdots + S_{2m,n}\underline{x_n} + S_{2m,n+1}(-\overline{x_1}) \\ +S_{2m,n+2}(-\overline{x_2}) + \cdots + S_{2m,2n}(-\overline{x_n}) = -\overline{b_m} \end{gathered}$$

where $S_{i,j}$ are determined as follows:

$$\begin{gathered} a_{i,j} \ge 0 \Longrightarrow S_{i,j} = a_{i,j}\,, S_{i+n,j} = a_{i,j} \\ a_{i,j} < 0 \Longrightarrow S_{i,j+n} = -a_{i,j}\,, S_{i+n,j} = -a_{i,j} \end{gathered}$$

and any $S_{i,j}$ which is not determined by the above equations is zero.

Using matrix notation we get

$$Sx = b \tag{3.1.2}$$

where

$$S = (S_{i,j}),\ \ 1 \le i \le 2m, 1 \le j \le 2\text{n}\ ,\ x = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}, -\overline{x_1}, -\overline{x_2}, \dots, -\overline{x_n}\right)^T$$

and

$$b = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_m}, -\overline{b_1}, -\overline{b_2}, \dots, -\overline{b_m}\right)^T.$$

Case3.1.1: when m=n then we see that S is a square matrix
i.e.

$$S = (S_{i,j}),\ \ 1 \le i, j \le 2\text{n}$$

and

$$x = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}, -\overline{x_1}, -\overline{x_2}, \dots, -\overline{x_n}\right)^T$$

and

$$b = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_n}, -\overline{b_1}, -\overline{b_2}, \dots, -\overline{b_n}\right)^T$$

So we can write

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}\begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} \underline{b} \\ \overline{b} \end{pmatrix} \tag{3.1.3}$$

Theorem3.1.1: (Friedman et al 1998): The matrix S is non singular if and only if the matrices $S_1 - S_2$ and $S_1 + S_2$ both are nonsingular.
Proof: By adding $(n+i)$th row of S to its ith row for $1 \le i \le n$ we obtain

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \to \begin{pmatrix} S_1 + S_2 & S_1 + S_2 \\ S_2 & S_1 \end{pmatrix} = M_1$$

Now we subtract the j th column of S, from its $(n+j)$th column for $1 \le i \le n$ we obtain

$$M_1 = \begin{pmatrix} S_1 + S_2 & S_1 + S_2 \\ S_2 & S_1 \end{pmatrix} \to \begin{pmatrix} S_1 + S_2 & 0 \\ S_2 & S_1 - S_2 \end{pmatrix} = M_2$$

Clearly $|S| = |M_1| = |M_2| = |S_1 + S_2||S_1 - S_2|$
Therefore $|S| \ne 0$ if and only if $|S_1 - S_2| \ne 0$ and $|S_1 + S_2| \ne 0$
which concludes the proof.

Definition3.1.1: If $x = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}, -\overline{x_1}, -\overline{x_2}, \dots, -\overline{x_n}\right)^T$ is a solution set of (3.1.2) and for any i , $1 \le i \le n$, the inequality $\underline{x_i} \le \overline{x_i}$ holds, then that solution $\widetilde{x_i}$ is called a strong solution of the system (3.1.2).

Definition3.1.2: If $x = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}, -\overline{x_1}, -\overline{x_2}, \dots, -\overline{x_n}\right)^T$, is a solution of (3.1.2) and for some $i \in [1, n]$ the inequality $\underline{x_i} > \overline{x_i}$ holds, then that solution $\widetilde{x_i}$ is called a weak solution of the system (3.1.2).

Theorem3.1.2: The necessary and sufficient conditions for the existence of a strong solution:
Let $S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$ be a nonsingular matrix. The system (3.1.3) has a strong solution if and only if

$$(S_1 + S_2)^{-1}\left(\underline{b} - \overline{b}\right) \le 0$$

Proof: let us define $\underline{x} = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}\right)^T$, $\overline{x} = (\overline{x_1}, \overline{x_2}, \dots, \overline{x_n})^T$ and

$$\underline{b} = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_n}\right)^T, \overline{b} = \left(\overline{b_1}, \overline{b_2}, \dots, \overline{b_n}\right)^T$$

from the system (3.1.3) we obtain

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}\begin{pmatrix} \underline{x} \\ -\overline{x} \end{pmatrix} = \begin{pmatrix} \underline{b} \\ -\overline{b} \end{pmatrix}$$

Hence

$$S_1\underline{x} - S_2\overline{x} = \underline{b} \text{ and } -S_2\underline{x} + S_1\overline{x} = \overline{b}$$

Adding the above two equations we get

$$(S_1 + S_2)\underline{x} - (S_1 + S_2)\overline{x} = \underline{b} - \overline{b}$$

$$(S_1 + S_2)\left(\underline{x} - \overline{x}\right) = \underline{b} - \overline{b}$$

By the **Theorem3.1.1** the matrix $S_1 + S_2$ is non singular.

Therefore

$$\underline{x} - \overline{x} = (S_1 + S_2)^{-1}\left(\underline{b} - \overline{b}\right)$$

If the system (3.1.3) has a strong solution then by the **Definition3.1.1**, we have $\underline{x} - \overline{x} \leq 0$

Hence $(S_1 + S_2)^{-1}\left(\underline{b} - \overline{b}\right) \leq 0$ holds.

Conversely if the above inequality holds then from the relation

$$(S_1 + S_2)\left(\underline{x} - \overline{x}\right) = \underline{b} - \overline{b} \text{ we get } \underline{x} - \overline{x} \leq 0.$$

In order to solve the above linear system (3.1.2) we must now calculate S^{-1}(if exists).

Assuming that S is nonsingular we get

$$x = S^{-1}b$$

Now

$$S^{-1} = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}^{-1}.$$

Let

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}^{-1} = \begin{pmatrix} B & C \\ C & B \end{pmatrix}.$$

So

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}\begin{pmatrix} B & C \\ C & B \end{pmatrix} = \begin{pmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0} & \mathrm{I} \end{pmatrix}.$$

where **0** is the null matrix and I is the unit matrix.

or,

$$S_1 B + S_2 C = \mathrm{I} \qquad \text{(a)}$$

$$S_2 B + S_1 C = \mathbf{0} \qquad \text{(b)}$$

Now if $(S_1 + S_2)^{-1}$and $(S_1 - S_2)^{-1}$exists then adding (a) and (b) we get

$$(S_1 + S_2)(B + C) = I \Longrightarrow B + C = (S_1 + S_2)^{-1}$$

and subtracting (b) from (a) we get

$$(S_1 - S_2)(B - C) = I \Longrightarrow B - C = (S_1 - S_2)^{-1}$$

Now if only ${S_1}^{-1}$ exists,

from (b) we get, $C = -{S_1}^{-1}S_2B$

Again from (1.1) we get, $S_1B + S_2\left(-{S_1}^{-1}S_2B\right) = \mathrm{I} \Longrightarrow B = \left(S_1 - S_2{S_1}^{-1}S_2\right)^{-1}$

So

$$C = -{S_1}^{-1}S_2\left(S_1 - S_2{S_1}^{-1}S_2\right)^{-1}.$$

Hence

$$S^{-1} = \begin{pmatrix} B & C \\ C & B \end{pmatrix}.$$

where $B = \left(S_1 - S_2{S_1}^{-1}S_2\right)^{-1}$ and $C = -{S_1}^{-1}S_2\left(S_1 - S_2{S_1}^{-1}S_2\right)^{-1}$.

Again if only ${S_2}^{-1}$ exists then

from (b) we get, $B = -{S_2}^{-1}S_1C$

Again from (1.1) we get, $S_2C + S_1\left(-{S_2}^{-1}S_1C\right) = \mathrm{I} \Longrightarrow C = \left(S_2 - S_1{S_2}^{-1}S_1\right)^{-1}$

So

$$B = -{S_2}^{-1}S_1\left(S_2 - S_1{S_2}^{-1}S_1\right)^{-1}.$$

Hence

$$S^{-1} = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$$

Where $C = \left(S_2 - S_1{S_2}^{-1}S_1\right)^{-1}$ and $B = -{S_2}^{-1}S_1\left(S_2 - S_1{S_2}^{-1}S_1\right)^{-1}$.

Note: further if all $a_{ij}, 1 \leq i, j \leq n$ are positive then we get

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}$$

$S^{-1} = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}^{-1} = \begin{pmatrix} {S_1}^{-1} & 0 \\ 0 & {S_1}^{-1} \end{pmatrix}$ when $S_2 = 0$.

Therefore the solution is

$$x = S^{-1}b$$

i.e.

$$\begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} {S_1}^{-1} & O \\ O & {S_1}^{-1} \end{pmatrix} \begin{pmatrix} \underline{b} \\ \overline{b} \end{pmatrix}$$

where

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}, x = \begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix}, b = \begin{pmatrix} \underline{b} \\ \overline{b} \end{pmatrix}$$

and

$$\underline{x} = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}\right)^T, \overline{x} = (-\overline{x_1}, -\overline{x_2}, \dots, -\overline{x_n})^T$$

and

$$\underline{b} = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_n}\right)^T, \overline{b} = \left(-\overline{b_1}, -\overline{b_2}, \dots, -\overline{b_n}\right)^T$$

or, $\underline{x} = {S_1}^{-1}\underline{b}$ and $\overline{x} = {S_1}^{-1}\overline{b}$

Case3.1.2: When $m \neq n$ then S is not square matrix. It is a rectangular matrix so we calculate inverse of S in terms of generalized inverse.
Let the generalized inverse of S be S^+
Where

$$S^+ = \begin{cases} (S^TS)^{-1}S^T & \text{if S has full column rank i.e. rank}(S) = n < m \\ S^T(SS^T)^{-1} & \text{if S has full row rank i.e. rank}(S) = m < n \end{cases}$$

Therefore the solution is

$$x = S^+b.$$

3.2 Solution method for 2×2 FLS (Friedman et al 1998):
Consider the linear fuzzy system

$$A\tilde{X} = \tilde{B} \qquad (3.2.1)$$

Where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tilde{X} = \begin{pmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{pmatrix}, \tilde{B} = \begin{pmatrix} \widetilde{b_1} \\ \widetilde{b_2} \end{pmatrix},$$

$a_{ij}, i, j = 1{,}2$ are real numbers, $\widetilde{b_1}, \widetilde{b_2}$ are GTrFNs and $\widetilde{x_1}, \widetilde{x_2}$ are unknown fuzzy numbers.

$$\widetilde{b_1} = (b_{11}, b_{12}, b_{13}, b_{14}, w), \widetilde{b_2} = (b_{21}, b_{22}, b_{23}, b_{24}, w)$$

So

$$\left(\widetilde{b_1}\right)_\propto = \left[\underline{b_1}(\propto), \overline{b_1}(\propto)\right] = \left[\underline{b_1}, \overline{b_1}\right] = \left[b_{11} + \frac{\propto}{w}(b_{12} - b_{11}), b_{14} - \frac{\propto}{w}(b_{14} - b_{13})\right],$$

$$\left(\widetilde{b_2}\right)_\propto = \left[\underline{b_2}(\propto), \overline{b_2}(\propto)\right] = \left[\underline{b_2}, \overline{b_2}\right] = \left[b_{21} + \frac{\propto}{w}(b_{22} - b_{21}), b_{24} - \frac{\propto}{w}(b_{24} - b_{23})\right]$$

Let

$$(\widetilde{x_1})_\propto = \left[\underline{x_1}(\propto), \overline{x_1}(\propto)\right] = \left[\underline{x_1}, \overline{x_1}\right], (\widetilde{x_2})_\propto = \left[\underline{x_2}(\propto), \overline{x_2}(\propto)\right] = \left[\underline{x_2}, \overline{x_2}\right]$$

So system (3.2.1) becomes

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \left[\underline{x_1}, \overline{x_1}\right] \\ \left[\underline{x_2}, \overline{x_2}\right] \end{pmatrix} = \begin{pmatrix} \left[b_{11} + \frac{\propto}{w}(b_{12} - b_{11}), b_{14} - \frac{\propto}{w}(b_{14} - b_{13})\right] \\ \left[b_{21} + \frac{\propto}{w}(b_{22} - b_{21}), b_{24} - \frac{\propto}{w}(b_{24} - b_{23})\right] \end{pmatrix}$$

or,

$$a_{11}\left[\underline{x_1}, \overline{x_1}\right] + a_{12}\left[\underline{x_2}, \overline{x_2}\right] = \left[b_{11} + \frac{\propto}{w}(b_{12} - b_{11}), b_{14} - \frac{\propto}{w}(b_{14} - b_{13})\right]$$

$$a_{21}\left[\underline{x_1}, \overline{x_1}\right] + a_{22}\left[\underline{x_2}, \overline{x_2}\right] = \left[b_{21} + \frac{\propto}{w}(b_{22} - b_{21}), b_{24} - \frac{\propto}{w}(b_{24} - b_{23})\right]$$

Assuming all a_{ij} are positive real numbers we get,
or,

$$a_{11}\underline{x_1} + a_{12}\underline{x_2} = b_{11} + \frac{\propto}{w}(b_{12} - b_{11})$$

$$a_{21}\underline{x_1} + a_{22}\underline{x_2} = b_{21} + \frac{\propto}{w}(b_{22} - b_{21})$$
$$a_{11}\overline{x_1} + a_{12}\overline{x_2} = b_{14} - \frac{\propto}{w}(b_{14} - b_{13})$$
$$a_{21}\overline{x_1} + a_{22}\overline{x_2} = b_{24} - \frac{\propto}{w}(b_{24} - b_{23})$$

or,

$$a_{11}\underline{x_1} + a_{12}\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = b_{11} + \frac{\propto}{w}(b_{12} - b_{11})$$
$$a_{21}\underline{x_1} + a_{22}\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = b_{21} + \frac{\propto}{w}(b_{22} - b_{21})$$
$$0.\underline{x_1} + 0.\underline{x_2} + a_{11}.(-\overline{x_1}) + a_{12}.(-\overline{x_2}) = -\left\{b_{14} - \frac{\propto}{w}(b_{14} - b_{13})\right\}$$
$$0.\underline{x_1} + 0.\underline{x_2} + a_{21}.(-\overline{x_1}) + a_{22}.(-\overline{x_2}) = -\left\{b_{24} - \frac{\propto}{w}(b_{24} - b_{23})\right\}$$

or,

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} b_{11} + \frac{\propto}{w}(b_{12} - b_{11}) \\ b_{21} + \frac{\propto}{w}(b_{22} - b_{21}) \\ -b_{14} + \frac{\propto}{w}(b_{14} - b_{13}) \\ -b_{24} + \frac{\propto}{w}(b_{24} - b_{23}) \end{pmatrix}$$

i.e.

$$\begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{pmatrix}\begin{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \end{pmatrix} \\ \begin{pmatrix} -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} b_{11} + \frac{\propto}{w}(b_{12} - b_{11}) \\ b_{21} + \frac{\propto}{w}(b_{22} - b_{21}) \end{pmatrix} \\ \begin{pmatrix} -b_{14} + \frac{\propto}{w}(b_{14} - b_{13}) \\ -b_{24} + \frac{\propto}{w}(b_{24} - b_{23}) \end{pmatrix} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \end{pmatrix} \\ \begin{pmatrix} -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} & \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \end{pmatrix}\begin{pmatrix} \begin{pmatrix} b_{11} + \frac{\propto}{w}(b_{12} - b_{11}) \\ b_{21} + \frac{\propto}{w}(b_{22} - b_{21}) \end{pmatrix} \\ \begin{pmatrix} -b_{14} + \frac{\propto}{w}(b_{14} - b_{13}) \\ -b_{24} + \frac{\propto}{w}(b_{24} - b_{23}) \end{pmatrix} \end{pmatrix}$$

or

$$\begin{pmatrix} \underline{x_1} \\ \underline{x_2} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}\begin{pmatrix} b_{11} + \frac{\propto}{w}(b_{12} - b_{11}) \\ b_{21} + \frac{\propto}{w}(b_{22} - b_{21}) \end{pmatrix}$$
$$= \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} b_{11} + \frac{\propto}{w}(b_{12} - b_{11}) \\ b_{21} + \frac{\propto}{w}(b_{22} - b_{21}) \end{pmatrix}$$

And

$$\begin{pmatrix} -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}\begin{pmatrix} -b_{14} + \frac{\propto}{w}(b_{14} - b_{13}) \\ -b_{24} + \frac{\propto}{w}(b_{24} - b_{23}) \end{pmatrix}$$
$$= \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} -b_{14} + \frac{\propto}{w}(b_{14} - b_{13}) \\ -b_{24} + \frac{\propto}{w}(b_{24} - b_{23}) \end{pmatrix}$$

It implies

$$\underline{x_1} = \frac{a_{22}}{(a_{11}a_{22} - a_{12}a_{21})}\left\{b_{11} + \frac{\propto}{w}(b_{12} - b_{11})\right\} - \frac{a_{12}}{(a_{11}a_{22} - a_{12}a_{21})}\left\{b_{21} + \frac{\propto}{w}(b_{22} - b_{21})\right\}$$

$$\underline{x_2} = \frac{-a_{21}}{(a_{11}a_{22} - a_{12}a_{21})}\left\{b_{11} + \frac{\propto}{w}(b_{12} - b_{11})\right\} + \frac{a_{11}}{(a_{11}a_{22} - a_{12}a_{21})}\left\{b_{21} + \frac{\propto}{w}(b_{22} - b_{21})\right\}$$

$$\overline{x_1} = \frac{a_{22}}{(a_{11}a_{22} - a_{12}a_{21})}\left\{b_{14} + \frac{\propto}{w}(b_{14} - b_{13})\right\} - \frac{a_{12}}{(a_{11}a_{22} - a_{12}a_{21})}\left\{b_{24} + \frac{\propto}{w}(b_{24} - b_{23})\right\}$$

$$\overline{x_2} = \frac{-a_{21}}{(a_{11}a_{22} - a_{12}a_{21})}\left\{b_{14} + \frac{\propto}{w}(b_{14} - b_{13})\right\} + \frac{a_{11}}{(a_{11}a_{22} - a_{12}a_{21})}\left\{b_{24} + \frac{\propto}{w}(b_{24} - b_{23})\right\}$$

Now if

$$\frac{\partial \underline{x_1}}{\partial \propto} = \frac{\{a_{22}(b_{12} - b_{11}) - a_{12}(b_{22} - b_{21})\}}{w(a_{11}a_{22} - a_{12}a_{21})} = \frac{\begin{vmatrix} b_{12} & a_{12} \\ b_{22} & a_{22} \end{vmatrix} - \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}}{w\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \geq 0$$

$$\frac{\partial \underline{x_2}}{\partial \propto} = \frac{\{-a_{21}(b_{12} - b_{11}) + a_{11}(b_{22} - b_{21})\}}{w(a_{11}a_{22} - a_{12}a_{21})} = \frac{\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{vmatrix}}{w\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \geq 0$$

$$\frac{\partial \overline{x_1}}{\partial \propto} = \frac{\{a_{22}(b_{14} - b_{13}) - a_{12}(b_{24} - b_{23})\}}{w(a_{11}a_{22} - a_{12}a_{21})} = \frac{\begin{vmatrix} b_{14} & a_{12} \\ b_{24} & a_{22} \end{vmatrix} - \begin{vmatrix} b_{13} & a_{12} \\ b_{23} & a_{22} \end{vmatrix}}{w\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \leq 0$$

$$\frac{\partial \overline{x_2}}{\partial \propto} = \frac{\{-a_{21}(b_{14} - b_{13}) + a_{11}(b_{24} - b_{23})\}}{w(a_{11}a_{22} - a_{12}a_{21})} = \frac{\begin{vmatrix} a_{11} & b_{14} \\ a_{21} & b_{24} \end{vmatrix} - \begin{vmatrix} a_{11} & b_{13} \\ a_{21} & b_{23} \end{vmatrix}}{w\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \leq 0$$

And

$$\underline{x_1}\Big|_{\propto=w} = \frac{a_{22}b_{12} - a_{12}b_{22}}{(a_{11}a_{22} - a_{12}a_{21})} = \frac{\begin{vmatrix} b_{12} & a_{12} \\ b_{22} & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \leq \frac{a_{22}b_{14} - a_{12}b_{24}}{(a_{11}a_{22} - a_{12}a_{21})} = \frac{\begin{vmatrix} b_{14} & a_{12} \\ b_{24} & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \overline{x_1}|_{\propto=w}$$

$$\underline{x_2}\Big|_{\propto=w} = \frac{-a_{21}b_{12} + a_{11}b_{22}}{(a_{11}a_{22} - a_{12}a_{21})} = \frac{\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \leq \frac{-a_{21}b_{14} + a_{11}b_{24}}{(a_{11}a_{22} - a_{12}a_{21})} = \frac{\begin{vmatrix} a_{11} & b_{14} \\ a_{21} & b_{24} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \overline{x_2}|_{\propto=w}$$

Then the strong solution exists.

Example3.1: Consider the 2×2 FLS

$$\widetilde{x_1} + \widetilde{x_2} = \widetilde{b_1}$$
$$2\widetilde{x_1} + 3\widetilde{x_2} = \widetilde{b_2}$$

where $\widetilde{b_1} = (4,5,6)$, $\widetilde{b_2} = (11,12,13)$ are TFNs

$$\left(\widetilde{b_1}\right)_\propto = \left[\underline{b_1}, \overline{b_1}\right] = [4+\propto, 6-\propto], \left(\widetilde{b_2}\right)_\propto = \left[\underline{b_2}, \overline{b_2}\right] = [11+\propto, 13-\propto]$$

Let the $\propto$-cut of the solution be $(\widetilde{x_1})_\propto = \left[\underline{x_1}, \overline{x_1}\right]$, $(\widetilde{x_2})_\propto = \left[\underline{x_2}, \overline{x_2}\right]$

The 4×4 system is

$1.\underline{x_1} + 1.\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = 4+\propto$

$2.\underline{x_1} + 3.\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = 11+\propto$

$0.\underline{x_1} + 0.\underline{x_2} + 1.(-\overline{x_1}) + 1.(-\overline{x_2}) = -(6-\propto)$

$0.\underline{x_1} + 0.\underline{x_2} + 2.(-\overline{x_1}) + 3.(-\overline{x_2}) = -(13-\propto)$

or,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} 4+\propto \\ 11+\propto \\ \propto -6 \\ \propto -13 \end{pmatrix}$$

here $S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$ where $S_1 = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$

Since $S_1 - S_2 = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ and $S_1 + S_2 = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ are non-singular matrices then by the Theorem3.1.1, S is nonsingular.

Therefore the above crisp system has a unique solution.

Now

$$S^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}^{-1} = \begin{pmatrix} {S_1}^{-1} & 0 \\ 0 & {S_1}^{-1} \end{pmatrix}.$$

Now

$${S_1}^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}.$$

So

$$S^{-1} = \begin{pmatrix} 3 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

Therefore the solution is

$$x = S^{-1}b.$$

or,

$$\begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 4+\propto \\ 11+\propto \\ \propto -6 \\ \propto -13 \end{pmatrix}.$$

$$\therefore \underline{x_1} = 3(4+\propto) - 1(11+\propto) = 1 + 2\propto$$
$$\underline{x_2} = -2(4+\propto) + 1(11+\propto) = 3-\propto$$
$$\overline{x_1} = -3(\propto -6) + 1(\propto -13) = 5 - 2\propto$$
$$\overline{x_2} = +2(\propto -6) - 1(\propto -13) = 1+\propto$$

Here we see that $\widetilde{x_1} = (1,3,5)$ is a strong solution and $\widetilde{x_2} = (1,2,3)$ is a weak solution and both solutions are TFNs.

Example3.2: consider the 2×2 FLS

$$\widetilde{x_1} - \widetilde{x_2} = \widetilde{b_1}$$
$$\widetilde{x_1} + 2\widetilde{x_2} = \widetilde{b_2}$$

where $\widetilde{b_1} = (1,2,3)$, $\widetilde{b_2} = (7,8,9)$

$$(\widetilde{b_1})_\propto = [\underline{b_1}, \overline{b_1}] = [1+\propto, 3-\propto], (\widetilde{b_2})_\propto = [\underline{b_2}, \overline{b_2}] = [7+\propto, 9-\propto]$$

Let the $\propto$-cut of the solution be $(\widetilde{x_1})_\propto = [\underline{x_1}, \overline{x_1}]$, $(\widetilde{x_2})_\propto = [\underline{x_2}, \overline{x_2}]$

The 4×4 system is

$1.\underline{x_1} + 0.\underline{x_2} + 0.(-\overline{x_1}) + 1.(-\overline{x_2}) = 1+\propto$

$1.\underline{x_1} + 2.\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = 7+\propto$

$0.\underline{x_1} + 1.\underline{x_2} + 1.(-\overline{x_1}) + 0.(-\overline{x_2}) = -(3-\propto)$

$0.\underline{x_1} + 0.\underline{x_2} + 1.(-\overline{x_1}) + 2.(-\overline{x_2}) = -(9-\propto)$

or,

$$\begin{pmatrix}1&0&0&1\\1&2&0&0\\0&1&1&0\\0&0&1&2\end{pmatrix}\begin{pmatrix}\underline{x_1}\\\underline{x_2}\\-\overline{x_1}\\-\overline{x_2}\end{pmatrix}=\begin{pmatrix}1+\propto\\7+\propto\\\propto-3\\\propto-9\end{pmatrix}$$

here $S=\begin{pmatrix}1&0&0&1\\1&2&0&0\\0&1&1&0\\0&0&1&2\end{pmatrix}=\begin{pmatrix}S_1&S_2\\S_2&S_1\end{pmatrix}$ where $S_1=\begin{pmatrix}1&0\\1&2\end{pmatrix}, S_2=\begin{pmatrix}0&1\\0&0\end{pmatrix}$

Since $S_1-S_2=\begin{pmatrix}1&-1\\1&2\end{pmatrix}$ and $S_1+S_2=\begin{pmatrix}1&1\\1&2\end{pmatrix}$ are non singular matrices then by the Theorem3.1.1, S is nonsingular.

Therefore the above crisp system has a unique solution.

Now

$$S^{-1}=\begin{pmatrix}1&0&0&1\\1&2&0&0\\0&1&1&0\\0&0&1&2\end{pmatrix}^{-1}=\begin{pmatrix}S_1&S_2\\S_2&S_1\end{pmatrix}^{-1}=\begin{pmatrix}B&C\\C&B\end{pmatrix}$$

where $B=\left(S_1-S_2{S_1}^{-1}S_2\right)^{-1}$ and $C=-{S_1}^{-1}S_2\left(S_1-S_2{S_1}^{-1}S_2\right)^{-1}$

Now

$${S_1}^{-1}=\begin{pmatrix}1&0\\1&2\end{pmatrix}^{-1}=\frac{1}{2}\begin{pmatrix}2&0\\-1&1\end{pmatrix}=\begin{pmatrix}1&0\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}.$$

$$\therefore S_1-S_2{S_1}^{-1}S_2=\begin{pmatrix}1&0\\1&2\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$=\begin{pmatrix}1&0\\1&2\end{pmatrix}-\begin{pmatrix}-\frac{1}{2}&\frac{1}{2}\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$=\begin{pmatrix}1&0\\1&2\end{pmatrix}-\begin{pmatrix}0&-\frac{1}{2}\\0&0\end{pmatrix}=\begin{pmatrix}1&\frac{1}{2}\\1&2\end{pmatrix}$$

$$\therefore B=\left(S_1-S_2{S_1}^{-1}S_2\right)^{-1}=\begin{pmatrix}1&\frac{1}{2}\\1&2\end{pmatrix}^{-1}=\frac{2}{3}\begin{pmatrix}2&-\frac{1}{2}\\-1&1\end{pmatrix}=\begin{pmatrix}\frac{4}{3}&-\frac{1}{3}\\-\frac{2}{3}&\frac{2}{3}\end{pmatrix}$$

and

$$C=-{S_1}^{-1}S_2\left(S_1-S_2{S_1}^{-1}S_2\right)^{-1}=-\begin{pmatrix}1&0\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}\frac{4}{3}&-\frac{1}{3}\\-\frac{2}{3}&\frac{2}{3}\end{pmatrix}$$

$$=-\begin{pmatrix}0&1\\0&-\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{4}{3}&-\frac{1}{3}\\-\frac{2}{3}&\frac{2}{3}\end{pmatrix}=\begin{pmatrix}\frac{2}{3}&-\frac{2}{3}\\-\frac{1}{3}&\frac{1}{3}\end{pmatrix}$$

So

$$S^{-1}=\begin{pmatrix}\frac{4}{3}&-\frac{1}{3}&\frac{2}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&-\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&-\frac{2}{3}&\frac{4}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{1}{3}&-\frac{2}{3}&\frac{2}{3}\end{pmatrix}$$

Therefore the solution is

$$x = S^{-1}b.$$

or,

$$\begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1+\propto \\ 7+\propto \\ \propto -3 \\ \propto -9 \end{pmatrix}$$

$$\therefore \underline{x_1} = \frac{4}{3}(1+\propto) - \frac{1}{3}(7+\propto) + \frac{2}{3}(\propto -3) - \frac{2}{3}(\propto -9) = 3+\propto$$

$$\underline{x_2} = -\frac{2}{3}(1+\propto) + \frac{2}{3}(7+\propto) - \frac{1}{3}(\propto -3) + \frac{1}{3}(\propto -9) = 2$$

$$\overline{x_1} = -\frac{2}{3}(1+\propto) + \frac{2}{3}(7+\propto) - \frac{4}{3}(\propto -3) + \frac{1}{3}(\propto -9) = 5-\propto$$

$$\overline{x_2} = \frac{1}{3}(1+\propto) - \frac{1}{3}(7+\propto) + \frac{2}{3}(\propto -3) - \frac{2}{3}(\propto -9) = 2$$

Here we see that the solution $\widetilde{x_1} = (3,4,5)$ is a TFN and $x_2 = 2$ is a crisp number.

Example3.3: consider the 3×1 FLS

$$2\widetilde{x_1} = \widetilde{b_1}$$
$$-\widetilde{x_1} = \widetilde{b_2}$$
$$-2\widetilde{x_1} = \widetilde{b_3}$$

where $\widetilde{b_1} = (0,1,2)$, $\widetilde{b_2} = (0,2,3), \widetilde{b_3} = (3,4,5)$

$$(\widetilde{b_1})_\propto = [\underline{b_1}, \overline{b_1}] = [\propto, 2-\propto], (\widetilde{b_2})_\propto = [\underline{b_2}, \overline{b_2}] = [2\propto, 3-\propto]$$

$$(\widetilde{b_3})_\propto = [\underline{b_3}, \overline{b_3}] = [3+\propto, 5-\propto]$$

Let the $\propto$-cut of the solution be $(\widetilde{x_1})_\propto = [\underline{x_1}, \overline{x_1}]$, $(\widetilde{x_2})_\propto = [\underline{x_2}, \overline{x_2}]$

The 6×2 system is

$2.\underline{x_1} + 0.(-\overline{x_1}) = \propto$

$0.\underline{x_1} + 1.(-\overline{x_1}) = 2\propto$

$0.\underline{x_1} + 2.(-\overline{x_1}) = 3+\propto$

$0.\underline{x_1} + 2.(-\overline{x_1}) = -(2-\propto)$

$1.\underline{x_1} + 0.(-\overline{x_1}) = -(3-\propto)$

$2.\underline{x_1} + 0.(-\overline{x_1}) = -(5-\propto)$

or,

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 2 \\ 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ -\overline{x_1} \end{pmatrix} = \begin{pmatrix} \propto \\ 2\propto \\ 3+\propto \\ \propto -2 \\ \propto -3 \\ \propto -5 \end{pmatrix}$$

here

$$S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 2 \\ 1 & 0 \\ 2 & 0 \end{pmatrix}$$

It is a rectangular matrix so we calculate inverse of S in terms of generalized inverse.

Let the generalized inverse of S be S^+

Now

$$S^T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 2 \\ 1 & 0 \\ 2 & 0 \end{pmatrix}^T = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 0 \end{pmatrix}$$

Now

$$S^T S = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 2 \\ 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

So

$$(S^T S)^{-1} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}$$

$$\therefore S^+ = (S^T S)^{-1} S^T = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{9} & 0 & 0 & 0 & \frac{1}{9} & \frac{2}{9} \\ 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} & 0 & 0 \end{pmatrix}$$

Therefore the solution is

$$x = S^+ b.$$

or,

$$\begin{pmatrix} \underline{x_1} \\ -\overline{x_1} \end{pmatrix} = \begin{pmatrix} \frac{2}{9} & 0 & 0 & 0 & \frac{1}{9} & \frac{2}{9} \\ 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} & 0 & 0 \end{pmatrix} \begin{pmatrix} \propto \\ 2\propto \\ 3+\propto \\ \propto -2 \\ \propto -3 \\ \propto -5 \end{pmatrix}$$

$$\therefore \underline{x_1} = \frac{2\propto}{9} + \frac{1}{9}(\propto -3) + \frac{2}{9}(\propto -5) = -\frac{13}{9} + \frac{5\propto}{9}$$

$$\overline{x_1} = -\frac{2\propto}{9} - \frac{2}{9}(3+\propto) + \frac{2}{9}(\propto -2) = -\frac{2}{9} - \frac{2\propto}{3}$$

Here we see that the solution $\widetilde{x_1} = \left(-\frac{13}{9}, -\frac{8}{9}, -\frac{2}{9}\right)$ is a TFN which is a strong solution.

Example3.4: consider the 1×2 fuzzy linear system

$$\widetilde{x_1} - \widetilde{x_2} = \widetilde{b_1}$$

where

$$\widetilde{b_1} = (0,1,2)$$

$$\left(\widetilde{b_1}\right)_\propto = \left[\underline{b_1}, \overline{b_1}\right] = [\propto, 2-\propto]$$

Let the $\propto$-cut of the solution be $(\widetilde{x_1})_\propto = \left[\underline{x_1}, \overline{x_1}\right]$, $(\widetilde{x_2})_\propto = \left[\underline{x_2}, \overline{x_2}\right]$

The 4×2 system is

$1.\underline{x_1} + 0.\underline{x_2} + 0.(-\overline{x_1}) + 1.(-\overline{x_2}) = \propto$

$0.\underline{x_1} + 1.\underline{x_2} + 1.(-\overline{x_1}) + 0.(-\overline{x_2}) = \propto -2$

or,

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} \propto \\ \propto -2 \end{pmatrix}$$

here

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

It is a rectangular matrix so we calculate inverse of S in terms of generalized inverse.
Let the generalized inverse of S be S^+
Now

$$S^T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now

$$SS^T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

So

$$(SS^T)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

$$\therefore S^+ = S^T(SS^T)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

Therefore the solution is

$$x = S^+ b.$$

or,

$$\begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} \propto \\ \propto -2 \end{pmatrix}$$

$$\therefore \underline{x_1} = \frac{\propto}{2}$$

$$\underline{x_2} = \frac{1}{2}(\propto -2)$$

$$\overline{x_1} = -\frac{1}{2}(\propto -2)$$

$$\overline{x_2} = -\frac{\propto}{2}$$

Here we see that one pair of solutions $\widetilde{x_1} = \left(0, \frac{1}{2}, 1\right)$ and $\widetilde{x_2} = \left(-1, -\frac{1}{2}, 0\right)$ are TFNs which are strong solutions.

3.3 Solution method for Fully Fuzzy Linear System (FFLS):

Consider the $m \times n$ linear system

$$\begin{array}{c} \widetilde{a_{11}}\widetilde{x_1} + \widetilde{a_{12}}\widetilde{x_2} + \cdots + \widetilde{a_{1n}}\widetilde{x_n} = \widetilde{b_1} \\ \widetilde{a_{21}}\widetilde{x_1} + \widetilde{a_{22}}\widetilde{x_2} + \cdots + \widetilde{a_{2n}}\widetilde{x_n} = \widetilde{b_2} \\ \vdots \\ \widetilde{a_{m1}}\widetilde{x_1} + \widetilde{a_{m2}}\widetilde{x_2} + \cdots + \widetilde{a_{mn}}\widetilde{x_n} = \widetilde{b_m} \end{array} \qquad (3.3.1)$$

where the co-efficient matrix $\tilde{A} = (\widetilde{a_{ij}})$, $1 \le i \le$ m, $1 \le$ j $\le$ n is a fuzzy $m \times n$ matrix , $\widetilde{b_i}$,$1 \le i \le m$ are known fuzzy numbers and $\widetilde{x_j}$,$1 \le j \le n$ are unknown fuzzy numbers.

In order to solve the above fully fuzzy linear system we must solve two $(2m) \times (2n)$ crisp linear systems where the right hand side columns are the function vector $\left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_m}, \overline{b_1}, \overline{b_2}, \dots, \overline{b_m}\right)^T$

Let us now rearrange the linear system so that the unknowns are $\underline{x_j}, (-\overline{x_j})$,$1 \le j \le n$, and the right hand side columns are

$$b = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_m}, -\overline{b_1}, -\overline{b_2}, \dots, -\overline{b_m}\right)^T.$$

We get the $(2m) \times (2n)$ crisp linear system

$$S_{1,1}\, \underline{x_1} + S_{1,2}\, \underline{x_2} + \dots + S_{1,n}\, \underline{x_n} + S_{1,n+1}(-\overline{x_1})$$
$$+S_{1,n+2}(-\overline{x_2}) + \dots + S_{1,2n}(-\overline{x_n}) = \underline{b_1}$$
$$\vdots$$
$$S_{m,1}\underline{x_1} + S_{m,2}\underline{x_2} + \dots + S_{m,n}\underline{x_n} + S_{m,n+1}(-\overline{x_1})$$
$$+S_{m,n+2}(-\overline{x_2}) + \dots + S_{m,2n}(-\overline{x_n}) = \underline{b_m}$$
$$S_{m+1,1}\underline{x_1} + S_{m+1,2}\underline{x_2} + \dots + S_{m+1,n}\underline{x_n} + S_{m+1,n+1}(-\overline{x_1})$$
$$+S_{m+1,n+2}(-\overline{x_2}) + \dots + S_{m+1,2n}(-\overline{x_n}) = -\overline{b_1}$$
$$\vdots$$
$$S_{2m,1}\underline{x_1} + S_{2m,2}\underline{x_2} + \dots + S_{2m,n}\underline{x_n} + S_{2m,n+1}(-\overline{x_1})$$
$$+S_{2m,n+2}(-\overline{x_2}) + \dots + S_{2m,2n}(-\overline{x_n}) = -\overline{b_m}$$

Where $S_{i,j}$ are determined as follows:

$$\widetilde{a_{ij}} \ge 0 \Longrightarrow S_{i,j} = \underline{a_{ij}}\ , S_{i,j+n} = 0\ , S_{i+m,j} = 0, S_{i+m,j+n} = -\overline{a_{ij}}\ ,$$
$$\widetilde{a_{ij}} < 0 \Longrightarrow S_{i,j} = 0, \quad S_{i,j+n} = -\underline{a_{ij}}, \quad S_{i+m,j} = \overline{a_{ij}}, \quad S_{i+m,j+n} = 0,$$
$$1 \le i \le m, 1 \le j \le n$$

Using matrix notation we get

$$Sx = b \tag{3.3.2}$$

Where

$$S = (S_{i,j}),\ \ 1 \le i \le 2m, 1 \le j \le 2\text{n}\ ,$$
$$x = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}, -\overline{x_1}, -\overline{x_2}, \dots, -\overline{x_n}\right)^T$$

and

$$b = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_m}, -\overline{b_1}, -\overline{b_2}, \dots, -\overline{b_m}\right)^T.$$

Case3.3.1: when m=n then we see that S is a square matrix
i.e.

$$S = (S_{i,j}),\ \ 1 \le i,j \le 2\text{n}$$

and

$$x = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}, -\overline{x_1}, -\overline{x_2}, \dots, -\overline{x_n}\right)^T$$

and

$$b = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_n}, -\overline{b_1}, -\overline{b_2}, \dots, -\overline{b_n}\right)^T$$

So we can write

$$\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}\begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} \underline{b} \\ \overline{b} \end{pmatrix} \tag{3.3.3}$$

Further if all $\widetilde{a_{ij}}, 1 \le i,j \le n$ are positive then we get

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix}$$
$$S^{-1} = \begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix}^{-1} = \begin{pmatrix} {S_1}^{-1} & 0 \\ 0 & {S_4}^{-1} \end{pmatrix} \text{ when } S_2 = S_3 = 0.$$

Theorem3.3.1: (Friedman et al 1998): The matrix $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix}$ is non singular if and only if the matrices S_1 and S_4 both are nonsingular.

Theorem3.3.2: The necessary and sufficient conditions for the existence of a strong solution:

Let $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix}$ be a nonsingular matrix. The system (3.3.3) has a strong solution if and only if

$$S_1^{-1}\underline{b} - S_4^{-1}\overline{b} \le 0$$

Proof: let us define $\underline{x} = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}\right)^T$, $\overline{x} = (\overline{x_1}, \overline{x_2}, \dots, \overline{x_n})^T$ and

$$\underline{b} = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_n}\right)^T, \overline{b} = \left(\overline{b_1}, \overline{b_2}, \dots, \overline{b_n}\right)^T$$

from the system (3.3.3) we obtain

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix} \begin{pmatrix} \underline{x} \\ -\overline{x} \end{pmatrix} = \begin{pmatrix} \underline{b} \\ -\overline{b} \end{pmatrix}$$

By the Theorem3.3.1, the matrices S_1, S_4 are non singular.

Hence $S_1\underline{x} = \underline{b} \Longrightarrow \underline{x} = S_1^{-1}\underline{b}$ and $-S_4\overline{x} = -\overline{b} \Longrightarrow -\overline{x} = -S_4^{-1}\overline{b}$

Adding the above two equations we get

$$\underline{x} - \overline{x} = S_1^{-1}\underline{b} - S_4^{-1}\overline{b}$$

If the system (3.3.3) has a strong solution then by the Definition3.1.1, we have $\underline{x} - \overline{x} \le 0$

Hence $S_1^{-1}\underline{b} - S_4^{-1}\overline{b} \le 0$ holds.

Conversely if the above inequality holds then from the relation $\left(\underline{x} - \overline{x}\right) = S_1^{-1}\underline{b} - S_4^{-1}\overline{b}$ we get $\underline{x} - \overline{x} \le 0$.

Therefore the solution is

$$x = S^{-1}b$$

i.e.

$$\begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} S_1^{-1} & O \\ O & S_4^{-1} \end{pmatrix} \begin{pmatrix} \underline{b} \\ \overline{b} \end{pmatrix}$$

where

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix}, x = \begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix}, b = \begin{pmatrix} \underline{b} \\ \overline{b} \end{pmatrix}$$

and

$$\underline{x} = \left(\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}\right)^T, \overline{x} = (-\overline{x_1}, -\overline{x_2}, \dots, -\overline{x_n})^T$$

and

$$\underline{b} = \left(\underline{b_1}, \underline{b_2}, \dots, \underline{b_n}\right)^T, \overline{b} = \left(-\overline{b_1}, -\overline{b_2}, \dots, -\overline{b_n}\right)^T$$

or, $\underline{x} = S_1^{-1}\underline{b}$ and $\overline{x} = S_4^{-1}\overline{b}$

Case3.3.2: When $m \neq n$ then S is not square matrix. It is a rectangular matrix so we calculate inverse of S in terms of generalized inverse.

Let the generalized inverse of S be S^+

$$\text{Where } S^+ = \begin{cases} (S^TS)^{-1}S^T & \text{if S has full column rank i.e. rank}(S) = n < m \\ S^T(SS^T)^{-1} & \text{if S has full row rank i.e. rank}(S) = m < n \end{cases}$$

Therefore the solution is

$$x = S^+b.$$

3.4 Solution for 2 × 2 FFLS

Consider the FFLS

$$\tilde{A}\tilde{X} = \tilde{B} \tag{3.4.1}$$

where $\tilde{A} = \begin{pmatrix} \widetilde{a_{11}} & \widetilde{a_{12}} \\ \widetilde{a_{21}} & \widetilde{a_{22}} \end{pmatrix}$, $\tilde{X} = \begin{pmatrix} \widetilde{x_1} \\ \widetilde{x_2} \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} \widetilde{b_1} \\ \widetilde{b_2} \end{pmatrix}$

Here $\tilde{A}$ and $\tilde{B}$ is given and we have to solve the system for $\tilde{X}$.

$\widetilde{a_{ij}}, i,j = 1,2$ and $\widetilde{b_1}, \widetilde{b_2}$ are TFNs and $\widetilde{x_1}, \widetilde{x_2}$ are unknown fuzzy numbers.

Let $\widetilde{a_{ij}}(\propto) = \left[\underline{a_{ij}}, \overline{a_{ij}}\right]$ for $1 \le i,j \le 2$

$$(\widetilde{x_1})_\propto = \left[\underline{x_1}, \overline{x_1}\right], (\widetilde{x_2})_\propto = \left[\underline{x_2}, \overline{x_2}\right]$$
$$(\widetilde{b_1})_\propto = \left[\underline{b_1}, \overline{b_1}\right], (\widetilde{b_2})_\propto = \left[\underline{b_2}, \overline{b_2}\right]$$

So system (3.4.1) becomes

$$\begin{pmatrix} \left[\underline{a_{11}}, \overline{a_{11}}\right] & \left[\underline{a_{12}}, \overline{a_{12}}\right] \\ \left[\underline{a_{21}}, \overline{a_{21}}\right] & \left[\underline{a_{22}}, \overline{a_{22}}\right] \end{pmatrix} \begin{pmatrix} \left[\underline{x_1}, \overline{x_1}\right] \\ \left[\underline{x_2}, \overline{x_2}\right] \end{pmatrix} = \begin{pmatrix} \left[\underline{b_1}, \overline{b_1}\right] \\ \left[\underline{b_2}, \overline{b_2}\right] \end{pmatrix}$$

or,

$$\left[\underline{a_{11}}, \overline{a_{11}}\right]\left[\underline{x_1}, \overline{x_1}\right] + \left[\underline{a_{12}}, \overline{a_{12}}\right]\left[\underline{x_2}, \overline{x_2}\right] = \left[\underline{b_1}, \overline{b_1}\right]$$
$$\left[\underline{a_{21}}, \overline{a_{21}}\right]\left[\underline{x_1}, \overline{x_1}\right] + \left[\underline{a_{22}}, \overline{a_{22}}\right]\left[\underline{x_2}, \overline{x_2}\right] = \left[\underline{b_2}, \overline{b_2}\right]$$

Assuming all a_{ij} are positive real numbers we get,
or,

$$\underline{a_{11}}\,\underline{x_1} + \underline{a_{12}}\,\underline{x_2} = \underline{b_1}$$
$$\underline{a_{21}}\,\underline{x_1} + \underline{a_{22}}\,\underline{x_2} = \underline{b_2}$$
$$\overline{a_{11}}\,\overline{x_1} + \overline{a_{12}}\,\overline{x_2} = \overline{b_1}$$
$$\overline{a_{21}}\,\overline{x_1} + \overline{a_{22}}\,\overline{x_2} = \overline{b_2}$$

or,

$$\underline{a_{11}}\,\underline{x_1} + \underline{a_{12}}\,\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = \underline{b_1}$$
$$\underline{a_{21}}\,\underline{x_1} + \underline{a_{22}}\,\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = \underline{b_2}$$
$$0.\underline{x_1} + 0.\underline{x_2} + \overline{a_{11}}(-\overline{x_1}) + \overline{a_{12}}(-\overline{x_2}) = -\overline{b_1}$$
$$0.\underline{x_1} + 0.\underline{x_2} + \overline{a_{21}}(-\overline{x_1}) + \overline{a_{22}}(-\overline{x_2}) = -\overline{b_2}$$

or,

$$\begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} & 0 & 0 \\ \underline{a_{21}} & \underline{a_{22}} & 0 & 0 \\ 0 & 0 & \overline{a_{11}} & \overline{a_{12}} \\ 0 & 0 & \overline{a_{21}} & \overline{a_{22}} \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} \underline{b_1} \\ \underline{b_2} \\ -\overline{b_1} \\ -\overline{b_2} \end{pmatrix}$$

or,

$$\begin{pmatrix} \underline{A} & \mathbf{0} \\ \mathbf{0} & \overline{A} \end{pmatrix} \begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} \underline{b} \\ \overline{b} \end{pmatrix}$$

Where

$$\underline{A} = \begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} \\ \underline{a_{21}} & \underline{a_{22}} \end{pmatrix}, \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \overline{A} = \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{21}} & \overline{a_{22}} \end{pmatrix}, \underline{x} = \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \end{pmatrix}, \overline{x} = \begin{pmatrix} -\overline{x_1} \\ -\overline{x_2} \end{pmatrix},$$
$$\underline{b} = \begin{pmatrix} \underline{b_1} \\ \underline{b_2} \end{pmatrix}, \overline{b} = \begin{pmatrix} -\overline{b_1} \\ -\overline{b_2} \end{pmatrix}$$

or,

$$\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{A}^{-1}\underline{b}$$

and

$$\overline{A}\,\overline{x} = \overline{b} \Rightarrow \overline{x} = \overline{A}^{-1}\overline{b}$$

Now if

$$\frac{\partial \underline{x}}{\partial \propto} = \frac{\partial}{\partial \propto}\left(\underline{A}^{-1}\underline{b}\right) \ge 0, \quad \frac{\partial \overline{x}}{\partial \propto} = \frac{\partial}{\partial \propto}\left(\overline{A}^{-1}\overline{b}\right) \le 0$$

and

$$\underline{A}^{-1}\underline{b}\Big|_{\propto=1} \le \overline{A}^{-1}\overline{b}\Big|_{\propto=1}$$

then we get the complete strong solutions.
Example3.5: Consider the 2×2 FFLS

$$(3,4,5,6)\widetilde{x_1} + (4,5,6,8)\widetilde{x_2} = (25,35,50,67)$$
$$(5,6,7,7)\widetilde{x_1} + (3,3,4,5)\widetilde{x_2} = (27,,32,48,55)$$
$$(\widetilde{a_{11}})_\alpha = \left[\underline{a_{11}}, \overline{a_{11}}\right] = [3+\alpha, 6-\alpha] \,,$$
$$(\widetilde{a_{12}})_\alpha = \left[\underline{a_{12}}, \overline{a_{12}}\right] = [4+\alpha, 8-2\alpha],$$
$$(\widetilde{a_{21}})_\alpha = \left[\underline{a_{21}}, \overline{a_{21}}\right] = [5+\alpha, 7],$$
$$(\widetilde{a_{22}})_\alpha = \left[\underline{a_{22}}, \overline{a_{22}}\right] = [3,5-\alpha],$$
$$\left(\widetilde{b_1}\right)_\alpha = \left[\underline{b_1}, \overline{b_1}\right] = [25+10\alpha, 67-17\alpha] \,,$$
$$\left(\widetilde{b_2}\right)_\alpha = \left[\underline{b_2}, \overline{b_2}\right] = [27+5\alpha, 55-7\alpha]$$

Let the α-cut of the solution be $(\widetilde{x_1})_\alpha = \left[\underline{x_1}, \overline{x_1}\right]$, $(\widetilde{x_2})_\alpha = \left[\underline{x_2}, \overline{x_2}\right]$

Then the 4×4 system is

$$(3+\alpha).\underline{x_1} + (4+\alpha).\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = 25+10\alpha$$
$$(5+\alpha).\underline{x_1} + 3.\underline{x_2} + 0.(-\overline{x_1}) + 0.(-\overline{x_2}) = 27+5\alpha$$
$$0.\underline{x_1} + 0.\underline{x_2} + (6-\alpha).(-\overline{x_1}) + (8-2\alpha).(-\overline{x_2}) = -(67-17\alpha)$$
$$0.\underline{x_1} + 0.\underline{x_2} + 7.(-\overline{x_1}) + (5-\alpha).(-\overline{x_2}) = -(55-7\alpha)$$

or,

$$\begin{pmatrix} 3+\alpha & 4+\alpha & 0 & 0 \\ 5+\alpha & 3 & 0 & 0 \\ 0 & 0 & 6-\alpha & 8-2\alpha \\ 0 & 0 & 7 & 5-\alpha \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix} = \begin{pmatrix} 25+10\alpha \\ 27+5\alpha \\ 17\alpha-67 \\ 7\alpha-55 \end{pmatrix}$$

Here

$$S = \begin{pmatrix} 3+\alpha & 4+\alpha & 0 & 0 \\ 5+\alpha & 3 & 0 & 0 \\ 0 & 0 & 6-\alpha & 8-2\alpha \\ 0 & 0 & 7 & 5-\alpha \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_3 \end{pmatrix}$$

where

$$S_1 = \begin{pmatrix} 3+\alpha & 4+\alpha \\ 5+\alpha & 3 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}, S_3 = \begin{pmatrix} 6-\alpha & 8-2\alpha \\ 7 & 5-\alpha \end{pmatrix}$$

Now

$$S^{-1} = \begin{pmatrix} 3+\alpha & 4+\alpha & 0 & 0 \\ 5+\alpha & 3 & 0 & 0 \\ 0 & 0 & 6-\alpha & 8-2\alpha \\ 0 & 0 & 7 & 5-\alpha \end{pmatrix}^{-1} = \begin{pmatrix} S_1 & 0 \\ 0 & S_3 \end{pmatrix}^{-1} = \begin{pmatrix} {S_1}^{-1} & 0 \\ 0 & {S_3}^{-1} \end{pmatrix}$$

Now

$${S_1}^{-1} = \begin{pmatrix} 3+\alpha & 4+\alpha \\ 5+\alpha & 3 \end{pmatrix}^{-1} = \frac{1}{(-\alpha^2-6\alpha-11)} \begin{pmatrix} 3 & -4-\alpha \\ -5-\alpha & 3+\alpha \end{pmatrix}$$
$${S_3}^{-1} = \begin{pmatrix} 6-\alpha & 8-2\alpha \\ 7 & 5-\alpha \end{pmatrix}^{-1} = \frac{1}{(\alpha^2+3\alpha-26)} \begin{pmatrix} 5-\alpha & -8+2\alpha \\ -7 & 6-\alpha \end{pmatrix}$$

So

$$S^{-1} = \begin{pmatrix} \frac{3}{-\alpha^2-6\alpha-11} & \frac{-4-\alpha}{-\alpha^2-6\alpha-11} & 0 & 0 \\ \frac{-5-\alpha}{-\alpha^2-6\alpha-11} & \frac{3+\alpha}{-\alpha^2-6\alpha-11} & 0 & 0 \\ 0 & 0 & \frac{5-\alpha}{\alpha^2+3\alpha-26} & \frac{-8+2\alpha}{\alpha^2+3\alpha-26} \\ 0 & 0 & \frac{-7}{\alpha^2+3\alpha-26} & \frac{6-\alpha}{\alpha^2+3\alpha-26} \end{pmatrix}$$

Therefore we get the solution from

$$x = S^{-1}b \text{ Now, } x = S^{-1}b \Rightarrow \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ -\overline{x_1} \\ -\overline{x_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{-\alpha^2-6\alpha-11} & \frac{-4-\alpha}{-\alpha^2-6\alpha-11} & 0 & 0 \\ \frac{-5-\alpha}{-\alpha^2-6\alpha-11} & \frac{3+\alpha}{-\alpha^2-6\alpha-11} & 0 & 0 \\ 0 & 0 & \frac{5-\alpha}{\alpha^2+3\alpha-26} & \frac{-8+2\alpha}{\alpha^2+3\alpha-26} \\ 0 & 0 & \frac{-7}{\alpha^2+3\alpha-26} & \frac{6-\alpha}{\alpha^2+3\alpha-26} \end{pmatrix} \begin{pmatrix} 25+10\alpha \\ 27+5\alpha \\ 17\alpha-67 \\ 7\alpha-55 \end{pmatrix}$$

$$\therefore \underline{x_1} = \left(\frac{3}{-\alpha^2-6\alpha-11}\right)(25+10\alpha) + \left(\frac{-4-\alpha}{-\alpha^2-6\alpha-11}\right)(27+5\alpha) = \frac{5\alpha^2+20\alpha+33}{\alpha^2+6\alpha+11}$$

$$\underline{x_2} = \left(\frac{-5-\alpha}{-\alpha^2-6\alpha-11}\right)(25+10\alpha) + \left(\frac{3+\alpha}{-\alpha^2-6\alpha-11}\right)(27+5\alpha) = \frac{5\alpha^2+33\alpha+44}{\alpha^2+6\alpha+11}$$

$$\overline{x_1} = -\left(\frac{5-\alpha}{\alpha^2+3\alpha-26}\right)(17\alpha-67) - \left(\frac{-8+2\alpha}{\alpha^2+3\alpha-26}\right)(7\alpha-55) = \frac{3\alpha^2+14\alpha-105}{\alpha^2+3\alpha-26}$$

$$\overline{x_2} = -\left(\frac{-7}{\alpha^2+3\alpha-26}\right)(17\alpha-67) - \left(\frac{6-\alpha}{\alpha^2+3\alpha-26}\right)(7\alpha-55) = \frac{7\alpha^2+22\alpha-139}{\alpha^2+3\alpha-26}$$

Here we see that the solutions

$$\widetilde{x_1} \approx (3,3.22,4,4) \text{ and } \widetilde{x_2} \approx (4,4.56,5,5.35)$$

are Trapezoidal Shaped Fuzzy Numbers and both are strong solutions.

4. Conclusion

Various approaches have been used by the other authors to solve FLS and FFLS but those are sometimes lengthy or not efficient to compute. In this paper we have used another approach to solve both FLS and FFLS. We have also considered numerical examples and solved by using this approach. For future work, there are so many areas in Mathematical, Physical and Engineering Sciences where we can handle the problems involving linear systems by this approach.

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