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Solution for singularly perturbed problems via cubic spline in tension

K. Aruna [1], A. S. V. Ravi Kanth [2]
[1] *Fluid Dynamics Division, School of Advanced Sciences, VIT University, Vellore, 632014, India,E-mail: k.aruna@vit.ac.in.*
[2] *Department of Mathematics, National Institute of Technology, Kurukshetra, 136119, Haryana, IndiaE-mail: asvravikanth@yahoo.com*
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Abstract. This paper concerns the solution for singularly perturbed via cubic spline in tension. The derived scheme leads to a tridiagonal system. The error analysis is proved and the method is shown to have a fourth order convergence for the particular choice of the parameters. Computational efficiency of the method is confirmed through numerical examples whose results are in good agreement with theory.

Keywords: singularly Perturbed Problems, Cubic Spline in Tension, Boundary Value Problems.

1. Introduction

In this paper, we consider the following second-order singularly perturbed boundary value problem

$$\varepsilon y''(x) = p(x)y'(x) + q(x)y(x) + r(x) \tag{1}$$

subject to the boundary conditions

$$y(0) = \alpha, y(1) = \beta \tag{2}$$

where $p(x), q(x), r(x)$ are smooth, bounded functions. It is well-known that the problem (1)-(2) exhibits boundary layer at one or both ends of the interval depending on the properties of $p(x)$ [1]. Singular perturbation problems arise very frequently in fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, geophysics and many other areas in applied science and engineering. Numerical treatment of the problem (1)-(2) has been widespread in recent years, for instance [2, 4-14].

In [4], a tension spline method for the linear singularly perturbed problems was presented which has second and fourth order convergence depending on the choice of the parameters λ_1 and λ_2 involved in the method. However, Khan and Aziz[4] claim of fourth order convergence for the problem with first derivative term lacks theoretical and computational support because of two reasons. The replacement of first derivative term with given approximations does not affect the error analysis and no numerical example is given to test the competence of the method involving first derivative term. Khan and Aziz method[4] gives fourth order convergence only for the problems with absence of first derivative term for some particular choice of parameters λ_1 and λ_2 concerned, but the order of convergence for the problems with first derivative term cannot exceed two, for any choice of parameters λ_1 and λ_2. The proposed scheme is the modified form of Khan and Aziz scheme in which a new parameter ω is introduced to obtain the desired fourth order convergence for problems with first derivative term i.e., equation of the form (1) and (2). For the particular value of ω i.e., $\omega = 0$, the proposed scheme reduces to Khan and Aziz[4] scheme. The derivation of the scheme is developed in section 2. In section 3 error analysis is discussed and it shows convergence of order four is achieved only for a particular value of parameter ω, i.e., $\omega = -\frac{1}{20\varepsilon}$ along with $\lambda_1 = \frac{1}{12}$ and $\lambda_2 = \frac{5}{12}$. Also, it is showed that for any other choice of parameters, the order of convergence is two.

2. A review of the research background

We develop a smooth approximate solution of (1) using cubic spline in tension. For this purpose we discretize the interval $[0,1]$ divided into a set of grid points $x_i = ih$, $i = 0,...,N$ with $h = \frac{1}{N}$. A function $S(x,\tau)$ of $C^2[a,b]$ which interpolates $y(x)$ at the mesh point x_i depends on a parameter τ, reduces to cubic spline in $[a,b]$ as $\tau \to 0$ is termed as parametric cubic-spline function. The spline function $S(x,\tau) = S(x)$ satisfying in $[x_i, x_{i+1}]$, the differential equation,

$$S''(x) - \tau S(x) = [S''(x_i) - \tau S(x_i)]\frac{(x_{i+1}-x)}{h} + [S''(x_{i+1}) - \tau S(x_{i+1})]\frac{(x-x_i)}{h} \quad (3)$$

where $S(x_i) = y_i$ and $\tau > 0$ is termed as cubic spline in tension. Solving the equation (3) and determining the arbitrary constants from the interpolatory conditions $S(x_i) = y_i$ and $S(x_{i+1}) = y_{i+1}$. After writing $\lambda = h\sqrt{\tau}$, we get

$$S(x) = \frac{h^2}{\lambda^2 \sinh\lambda}[M_{i+1}\sinh\frac{\lambda(x-x_i)}{h} + M_i \sinh\frac{\lambda(x_{i+1}-x)}{h}]$$
$$-\frac{h^2}{\lambda^2}[\frac{(x-x_i)}{h}(M_{i+1} - \frac{\lambda^2}{h^2}y_{i+1}) + \frac{(x_{i+1}-x)}{h}(M_i - \frac{\lambda^2}{h^2}y_i)] \quad (4)$$

Differentiating equation (4) and using continuity conditions which lead to the tridiagonal system

$$h^2(\lambda_1 M_{i-1} + 2\lambda_2 M_i + \lambda_1 M_{i+1}) = y_{i+1} - 2y_i + y_{i-1} \quad i = 1(1)N-1 \quad (5)$$

where $\lambda_1 = \frac{1}{\lambda^2}(1 - \frac{\lambda}{\sinh\lambda})$, $\lambda_2 = \frac{1}{\lambda^2}(\lambda\coth\lambda - 1)$, $M_i = S''(x_i)$. The condition (3) ensures the continuity of the first order derivatives of the spline $S(x,\tau)$ at interior nodes. We write (1) in the form $\varepsilon M_i = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i)$ and substituting into equation (5), and using the following approximations for first order derivatives of y:

$$y'_{i-1} \cong \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} \quad (6)$$

$$y'_{i+1} \cong \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} \quad (7)$$

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2h}, \; y'_i \cong \tilde{y}'_i + h\omega(\tilde{f}_{i+1} - \tilde{f}_{i-1})$$

$$y'_i \cong \frac{1 + 2h^2\omega q_{i+1} + h\omega(3p_{i+1} + p_{i-1})}{2h}y_{i+1} - 2\omega(p_{i+1} + p_{i-1})y_i$$
$$+\frac{-1 - 2h^2\omega q_{i-1} + h\omega(3p_{i-1} + p_{i+1})}{2h}y_{i-1} + h\omega(r_{i+1} - r_{i-1}) \quad (8)$$

We get the following three term recurrence relation, which gives the approximation $y_1, y_2, ..., y_{N-1}$ of the solution $y(x)$ at the points $x_1, x_2, ..., x_{N-1}$

$$(-\frac{3}{2}h\lambda_1 p_{i-1} + h^2\lambda_1 q_{i-1} - h\lambda_2 p_i(1 + 2h^2\omega q_{i-1} - h\omega(p_{i+1} + 3p_{i-1})) + \frac{1}{2}\lambda_1 hp_{i+1} - \varepsilon)y_{i-1}$$
$$(2\lambda_1 hp_{i-1} - 4h^2\lambda_2\omega p_i(p_{i+1} + p_{i-1}) + 2h^2\lambda_2 q_i - 2h\lambda_1 p_{i+1} + 2\varepsilon)y_i$$
$$(-\frac{1}{2}h\lambda_1 p_{i-1} + h^2\lambda_1 q_{i+1} + h\lambda_2 p_i(1 + 2h^2\omega q_{i+1} + h\omega(3p_{i+1} + p_{i-1})) + \frac{3}{2}\lambda_1 hp_{i+1} - \varepsilon)y_{+1}$$
$$= -h^2((\lambda_1 - 2\lambda_2 h\omega p_i)r_{i-1} + 2\lambda_2 r_i + (\lambda_1 + 2\lambda_2 h\omega p_i)r_{i+1}), \qquad i = 1,...,N-1 \quad (9)$$

Using (9) with (2), we get the approximate solution of $y(x)$ at the grid points x_i.

Remark 1: For $\omega = 0$, the present scheme reduces to Khan and Aziz [4] method.

Remark 2: For $\lambda_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{3}$ and $\omega = 0$, the present scheme reduces to the Kadalbajoo and Bawa's [6] second order method for uniform mesh.

3. General overview of tracking objects proposed method

From (6), (7) and (8) we get

$$e'_{i-1} = y'(x_{i-1}) - y'_{i-1} = \frac{h^2}{3} y'''(x_i) - \frac{h^3}{12} y^{iv}(x_i) + \frac{h^4}{30} y^{v}(\zeta^{(i)}), \qquad x_{i-1} < \zeta^{(i)} < x_{i+1} \tag{10}$$

$$e'_{i+1} = y'(x_{i+1}) - y'_{i+1} = \frac{h^2}{3} y'''(x_i) + \frac{h^3}{12} y^{iv}(x_i) + \frac{h^4}{30} y^{v}(\psi^{(i)}), \qquad x_{i-1} < \psi^{(i)} < x_{i+1} \tag{11}$$

$$e'_{i} = y'(x_{i}) - y'_{i} = -h^2(\frac{1}{6} + 2\omega\varepsilon) y'''(x_i) - h^4(\frac{1}{120} + \frac{\omega\varepsilon}{3}) y^{v}(\xi^{(i)}), \quad x_{i-1} < \xi^{(i)} < x_{i+1} \tag{12}$$

substituting $\varepsilon M_i = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i)$ in (5), we obtain

$$\varepsilon(y_{i-1} - 2y_i + y_{i+1}) = h^2(\lambda_1(p_{i-1}y'_{i-1} + q_{i-1}y_{i-1} + r_{i-1}) + 2\lambda_2(p_i y'_i + q_i y_i + r_i) + \lambda_1(p_{i+1}y'_{i+1} + q_{i+1}y_{i+1} + r_{i+1})) \tag{13}$$

using exact solution in (13), we have

$$\varepsilon(y(x_{i-1}) - 2y(x_i) + y(x_{i+1})) = h^2(\lambda_1(p_{i-1}y'(x_{i-1}) + q_{i-1}y(x_{i-1}) + r_{i-1}) + 2\lambda_2(p_i y'(x_i) + q_i y(x_i) + r_i) + \lambda_1(p_{i+1}y'(x_{i+1}) + q_{i+1}y(x_{i+1}) + r_{i+1})) + T(h) \tag{14}$$

where

$$T(h) = \frac{\varepsilon h^4}{12}(-1 + 2\lambda_1) y^{iv}(\eta^{(i)}) + \frac{\varepsilon h^6}{360}(-1 + 30\lambda_1) y^{iv}(\eta^{(i)}), \qquad x_{i-1} < \eta^{(i)} < x_{i+1} \tag{15}$$

For any choice of λ_1 and λ_2 whose sum is $\frac{1}{2}$. Subtracting (13) and (14) and substituting $e_i = y(x_i) - y_i$, we get

$$(\varepsilon - h^2\lambda_1 q_{i-1})e_{i-1} - 2(\varepsilon + h^2\lambda_2 q_i)e_i + (\varepsilon - h^2\lambda_1 q_{i+1})e_{i+1} = h^2(\lambda_1 p_{i-1}e'_{i-1} + 2\lambda_2 p_i e'_i + \lambda_1 p_{i+1}e'_{i+1}) + T(h) \tag{16}$$

Using (10)-(12), we get

$$(\varepsilon - h^2\lambda_1 q_{i-1})e_{i-1} - 2(\varepsilon + h^2\lambda_2 q_i)e_i + (\varepsilon - h^2\lambda_1 q_{i+1})e_{i+1} = [\frac{h^4\lambda_1}{3}(p_{i-1} + p_{i+1}) - 2h^4\lambda_2 p_i(\frac{1}{6} + 2\omega\varepsilon)] y'''(x_i) + \frac{h^5\lambda_1}{12}(p_{i+1} - p_{i-1}) y^{iv}(x_i) + \frac{h^6\lambda_1}{30}(p_{i-1}y^{v}(\zeta^{(i)}) + p_{i+1}y^{v}(\psi^{(i)})) - 2h^6\lambda_2 p_i(\frac{1}{120} + \frac{\omega\varepsilon}{3}) y^{v}(\xi^{(i)}) + T(h) \tag{17}$$

Let

$$p_{i+1} = p_i + hp'_i + \frac{h^2}{2} p''_i(\chi^{(i)}) \tag{18}$$

$$p_{i-1} = p_i - hp'_i + \frac{h^2}{2} p''_i(\gamma^{(i)}) \tag{19}$$

where $x_{i-1} < \chi^{(i)} < x_{i+1}$, $x_{i-1} < \gamma^{(i)} < x_{i+1}$. Using (18),(19) and (15) in (16), we get

$$(\varepsilon - h^2\lambda_1 q_{i-1})e_{i-1} - 2(\varepsilon + h^2\lambda_2 q_i)e_i + (\varepsilon - h^2\lambda_1 q_{i+1})e_{i+1} = T_{io}(h) \tag{20}$$

where

$$T_{io}=h^4(\frac{2\lambda_1}{3}-2\lambda_2(\frac{1}{6}+2\omega\varepsilon))p_i y'''(x_i)+\frac{h^4\varepsilon}{12}(1-12\lambda_1)y^{iv}(\mu^{(i)})+O(h^6) \quad (21)$$

It can be seen easily that $T_{io}(h)=O(h^4)$ for any choice of $\lambda_1+\lambda_2=\frac{1}{2}$ and for any value of ω and $T_{io}(h)=O(h^6)$ for $\lambda_1=\frac{1}{12}$, $\lambda_2=\frac{5}{12}$ and $\omega=-\frac{1}{20\varepsilon}$. Let $J=trid[\varepsilon \quad 2\varepsilon \quad \varepsilon]$ and $D=trid[\lambda_1 \quad 2\lambda_2 \quad \lambda_1]$ are $N-1\times N-1$ tridiagonal matrices and $Q=[q_1,q_2,...,q_{N-1}]^T$ and $E=[e_1,e_2,...,e_{N-1}]^T$ are $N-1$ component vectors. So, equation (20) can be written in matrix vector form as $AE=T_{io}$ where

$$A=J-h^2DQ \quad (22)$$

Following[3], it can be shown that, for sufficiently small h

$$\|E\|=\|A^{-1}T_{io}\| \Rightarrow \|E\|\le\|A^{-1}\|\|T_{io}\| \quad (23)$$

Therefore, $\|E\|=O(h^2)$ for any choice of $\lambda_1+\lambda_2=\frac{1}{2}$ and $\|E\|=O(h^4)$ for $\lambda_1=\frac{1}{12}$, $\lambda_2=\frac{5}{12}$ and $\omega=-\frac{1}{20\varepsilon}$. Thus we summarize the following.

Theorem: Let $y(x)\in C^2[a,b]$, then our method provides a second order convergent approximation for solution $y(x)$ of the boundary value problem (1)-(2) for arbitrary choice of ω with $\lambda_1+\lambda_2=\frac{1}{2}$ and a fourth order convergent solution for $\lambda_1=\frac{1}{12}$, $\lambda_2=\frac{5}{12}$ and $\omega=-\frac{1}{20\varepsilon}$.

4. An educational process

In this section, we present the numerical simulation to demonstrate the applicability of the scheme by considering two examples. Maximum absolute errors (i.e., $\max|y(x_i)-y_i|$) at nodal points are computed for different values of ε and N.

Example 1: Consider the following homogeneous singular perturbation problem

$$-\varepsilon y''(x)+y'(x)+(1+\varepsilon)y(x)=0 \quad (24)$$

Subject to the boundary conditions

$$y(0)=1+e^{\frac{-(1+\varepsilon)}{\varepsilon}},\ y(1)=1+\frac{1}{e} \quad (25)$$

The exact solution is given by

$$y(x)=e^{\frac{(1+\varepsilon)(x-1)}{\varepsilon}}+e^{-x}. \quad (26)$$

In Table 1, we have compared the maximum absolute errors for different values of λ_1,λ_2 obtained by the present method and the fitted finite difference method [13]. The Maximum absolute errors and order of convergence obtained by the proposed method for different values of N,λ_1 and λ_2 are presented in Table 2. The estimated Maximum absolute errors and ε - uniform errors E^N using the proposed method shown in Table 3.

Example 2: Consider the following homogeneous singular perturbation problem

$$-\varepsilon y''(x)+(1+x)^2y'(x)+2(1+x)y(x)=\frac{1}{2}e^{-\frac{x}{2}}[(1+x)(3-x)+\frac{\varepsilon}{2}] \quad (27)$$

Subject to the boundary conditions

$$y(0)=0,\ y(1)=e^{-\frac{1}{2}}-e^{-\frac{7}{3\varepsilon}} \quad (28)$$

The exact solution is given by

$$y(x) = e^{-\frac{x}{2}} - e^{-\frac{x(x^2+3x+3}{3\varepsilon}} \tag{29}$$

We have compared the maximum errors and the order of convergence obtained by the present method and the Khan and Aziz method [4] in Table 4-5.

Table 1: Comparison of maximum absolute errors for N = 128

ε	Present Method	Present Method	Method in [13]
	$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$	$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$	
$\frac{1}{8}$	$4.0233e-05$	$1.1740e-08$	$4.6749e-02$
$\frac{1}{16}$	$1.2692e-04$	$1.5615e-07$	$2.3131e-02$
$\frac{1}{32}$	$4.4468e-04$	$2.2548e-06$	$1.1498e-02$
$\frac{1}{64}$	$1.6589e-03$	$3.4455e-05$	$5.6808e-03$
$\frac{1}{128}$	$6.4347e-03$	$5.5905e-04$	$2.7248e-03$

Table 2: Maximum absolute errors and order of convergence for Example 1 using present method

ε	N=64	Order	N=128	Order	N=256	Order
$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$						
2^{-3}	$1.6101e-04$	2.00	$4.0233e-05$	2.00	$1.0062e-05$	2.00
2^{-4}	$5.0826e-04$	2.00	$1.2692e-04$	2.00	$3.1773e-05$	2.00
2^{-5}	$1.7842e-03$	2.00	$4.4468e-04$	2.00	$1.1108e-04$	2.00
2^{-6}	$6.6949e-03$	2.01	$1.6589e-03$	2.00	$4.1375e-04$	2.00
2^{-7}	$1.8415e-02$	1.51	$6.4347e-03$	2.00	$1.5971e-03$	2.00
$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$						
2^{-3}	$1.8801e-07$	4.00	$1.1742e-08$	3.98	$7.3401e-10$	3.98
2^{-4}	$2.5062e-06$	4.00	$1.5615e-07$	3.99	$9.7705e-09$	3.99
2^{-5}	$3.6505e-05$	4.02	$2.2548e-06$	4.00	$1.4051e-07$	4.00
2^{-6}	$5.7654e-04$	4.06	$3.4455e-05$	4.00	$2.1289e-06$	4.00
2^{-7}	$7.7223e-03$	3.78	$5.5905e-04$	4.02	$3.3432e-05$	4.02

Table 3: Maximum absolute errors and ε - uniform errors E^N for Example 1 using present method

ε	N=64	Order	N=128	Order	N=256	Order
			$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$			
2^{-3}	$1.6101e-04$	2.00	$4.0233e-05$	2.00	$1.0062e-05$	2.00
2^{-4}	$5.0826e-04$	2.00	$1.2692e-04$	2.00	$3.1773e-05$	2.00
2^{-5}	$1.7842e-03$	2.00	$4.4468e-04$	2.00	$1.1108e-04$	2.00
2^{-6}	$6.6949e-03$	2.01	$1.6589e-03$	2.00	$4.1375e-04$	2.00
2^{-7}	$1.8415e-02$	1.51	$6.4347e-03$	2.00	$1.5971e-03$	2.00
			$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$			
2^{-3}	$1.8801e-07$	4.00	$1.1742e-08$	3.98	$7.3401e-10$	3.98
2^{-4}	$2.5062e-06$	4.00	$1.5615e-07$	3.99	$9.7705e-09$	3.99
2^{-5}	$3.6505e-05$	4.02	$2.2548e-06$	4.00	$1.4051e-07$	4.00
2^{-6}	$5.7654e-04$	4.06	$3.4455e-05$	4.00	$2.1289e-06$	4.00
2^{-7}	$7.7223e-03$	3.78	$5.5905e-04$	4.02	$3.3432e-05$	4.02

Table 4: Maximum absolute errors and order of convergence for Example 2

ε	N=256		N=512	
	Method in [4]	Present Method	Method in [4]	Present Method
		$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$		
2^{-3}	$1.6662e-05$	$1.5943e-05$	$4.1658e-06$	$3.9854e-06$
	1.99	2.00	2.00	2.00
2^{-4}	$9.0909e-05$	$1.9052e-05$	$2.2724e-05$	$4.7598e-06$
	2.00	2.00	2.00	2.00
2^{-5}	$4.2036e-04$	$1.2099e-05$	$1.0502e-04$	$3.0471e-06$
	2.00	1.98	2.00	1.99
2^{-6}	$1.8088e-03$	$2.1377e-04$	$4.4995e-04$	$5.3615e-05$
	2.01	1.99	2.00	1.99
2^{-7}	$7.6309e-03$	$1.1907e-03$	$1.8688e-03$	$2.9880e-04$
	2.03	1.99	2.01	1.99
		$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$		
2^{-3}	$3.9344e-05$	$1.0075e-09$	$9.8351e-06$	$6.4799e-11$
	2.00	3.96	2.00	3.59
2^{-4}	$1.3675e-04$	$1.0547e-08$	$3.4176e-05$	$6.5967e-10$
	2.00	3.99	2.00	3.91

2^{-5}	$5.1161e-04$	$1.4293e-07$	$1.2773e-04$	$8.9272e-09$
	2.00	4.00	1.99	3.99
2^{-6}	$1.9915e-03$	$2.1348e-06$	$4.9527e-04$	$1.3304e-07$
	2.00	4.00	2.00	3.99
2^{-7}	$8.0071e-03$	$3.3452e-05$	$1.9598e-03$	$2.0674e-06$
	2.03	4.01	2.01	4.00

Table 5: Maximum absolute errors for second order method with $\varepsilon = 2^{-10}$ for Example 2

λ_1, λ_2	N=256	N=512	N=1024
$\frac{1}{18}, \frac{4}{9}$	$7.7257e-02$	$1.5449e-02$	$2.6922e-03$
$\frac{1}{14}, \frac{3}{7}$	$6.7011e-02$	$1.0966e-02$	$1.4663e-03$
$\frac{1}{24}, \frac{11}{24}$	$8.6035e-02$	$1.9333e-02$	$3.7610e-03$
$\frac{1}{30}, \frac{14}{30}$	$9.1221e-02$	$2.1646e-02$	$4.4005e-03$

5. Results of the positive and negative features patterns

We have presented numerical simulations for singularly perturbed boundary value problems using cubic spline in tension. It is observed from the tables that the present method is more efficient than the methods given in [4], [13]. The computational results shows that the present method is fourth order only for a particular choice of the newly introduced parameter ω, i.e., $\omega = -\frac{1}{20\varepsilon}$ along with $\lambda_1 = \frac{1}{12}$ and $\lambda_2 = \frac{5}{12}$.Also it is shown that for any other choice of the parameters, the order of convergence is two.

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