ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 11, No. 4, 2016, *pp*.251-261

Approximate Solutions to System of Nonlinear Partial Differential Equations using Reduced Differential Transform Method

Mohamed S. Mohamed[1, 2], M. Sayed[1, 3] and TurkiaT. Al-Qarshi[2]
[1]*Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt, E-mail: m_s_mohamed2000@yahoo.com*
[2]*Mathematics Department, Faculty of Science, Taif University, Taif, Saudi Arabia.*
[3]*Department of Engineering Mathematics, Faculty of Electronic Engineering, Menofia University, Menouf, 32952, Egypt, E-mail: moh_6_11@yahoo.com*
(Received May 04, 2016, accepted August 12, 2016)

Abstract. The main objective of this paper is to use the reduced differential transform method (RDTM) for finding the analytical approximate solutions for solving systems of nonlinear partial differential equations (NPDEs). The approximate solutions obtained by RDTM is verified by comparison with the exact solutions to show that the RDTM is quite accurate, reliable and can be applied for many other nonlinear partial differential equations. The method considers the use of the appropriate initial or boundary conditions and finds the solution without any discretization, transformation, or restrictive assumptions. This method is a simple and efficient method for solving the nonlinear partial differential equations. The numerical results show that this method is a powerful tool for solving systems of NPDEs. The analysis shows that our analytical approximate solutions converge very rapidly to the exact solutions.

Keywords: reduced differential transforms method, nonlinear partial differential equations, analytic and approximate solutions.

1. Introduction

Nonlinear partial differential equations (NPDEs) are mathematical models that are used to describe complex phenomena arising in the world around us. The nonlinear equations appear in many applications of science and engineering such as fluid dynamics, plasma physics, hydrodynamics, solid state physics, optical fibers, acoustics and other disciplines [1]. On the other hand, there are many effective methods for obtaining the anaylatic approximate solutions and exact solutions of NPDEs samong of these methods are the inverse scattering method [2], Hirota's bilinear method [3], Backlund transformation [4, 5], Painlevé expansion [6] sine-cosine method [7], homogenous balance method [8], homotopy perturbation method [9-12], variation method [13, 14], Adomian decomposition method [15, 16], perturbation method [17, 18], tanh-function method [19-21], Jacobi elliptic function expansion method [22-25], F-expansion method [26, 27], (G'/G)-expansion method [28-32], exp-function method [33-35], complex transformation [36] and Riccati equations method [37, 38]. Recently Mabood [39] and Mohamed et al. [40-42] used the optimal homotopy asymptotic to study the MHD slips flow over radiating sheet with heat transfer, the flow heat transfer viscoelastic fluid in an axisymetric channel with a porous wall and for the heat transfer in hollow sphere with the Robin boundary conditions. Also, Mabood et al. [43] have discussed the analytical solutions for radiation effects on heat transfer in Blasius flow.

In the present article, we use the reduced differential to transform method (RDTM) which discussed in [44-47], to construct an appropriate solution of some highly nonlinear partial differential equations of mathematical physics. The reduced differential transforms technique is an iterative procedure for obtaining a Taylor series solution of differential equations. This method reduces the size of computational work and easily applicable to many nonlinear physical problems. In this paper, we discuss the analytic approximate solution for two systems of nonlinear wave equations, these systems can be seen in [48, 49]. In mathematical physics, they play a major role in various fields, such as plasma physics, fluid mechanics, optical fibers, solid state physics, geochemistry and so on.

$$u_t - u_{xxx} - 2v u_x - u v_x = 0, \tag{1}$$

and

$$\upsilon_t - u u_x = 0. \tag{2}$$

One of these systems is the generalized coupled Hirota Satsuma KdV system given by

$$u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3\frac{\partial}{\partial x}(vw) = 0, \tag{3}$$

$$\upsilon_t + \upsilon_{xxx} - 3u\upsilon_x = 0, \tag{4}$$

$$w_t + w_{xxx} - 3u\,w_x = 0. \tag{5}$$

The paper has been organized as follows. Notations and basic definitions are given in Section 2. In Section 3, we apply the RDTM to solve two types of NPDEs. Conclusions are given in Section 4.

2. Preliminaries and notations

In this section, we give some basic definitions and properties of the reduced differential transform method which are further used in this paper. Consider a function of three variables $u(x, y, t)$ and suppose that it can be represented as a product of three single-variable functions, i.e., $u(x, y, t) = f(x)h(y)g(t)$. Based on the properties the (2+1) of-dimensional differential transform, the function $u(x, y, t)$ can be represented as follows:

$$u(x,y,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} H(j)x^j\right)\left(\sum_{l=0}^{\infty} G(l)x^l\right) = \sum_{k=0}^{\infty} U_k(x,y)t^k \tag{6}$$

where $U_k(x, y)$ is called *t*-dimensional spectrum function of $u(x, y, t)$. The basic definitions of RDTM are introduced as follows [44-47].

Definition 2.1 If the function $u(x, y, t)$ is analytic and differentiated continuously with respect to time t and space in the domain of interest, then let

$$U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}, \tag{7}$$

where the $t-$ dimensional spectrum function $U_k(x, y)$ is the transformed function. In this paper the lowercase $u(x, y, t)$ represents the original function while the uppercase $U_k(x, y)$ stands for the transform function.

Definition 2.2 The differential inverse transform $U_k(x, y)$ is defined as follows:

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k. \tag{8}$$

Then, combining Eqs. (7) and (8) we have:

$$u(x,y,t) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k u(x,y,t)}{\partial t^k}\right]_{t=0} t^k. \tag{9}$$

From the above definitions, it can be found that the concept of the RDTM is derived from the power series expansion. To illustrate the basic concepts of the RDTM, consider the following nonlinear partial differential equation written in an operator form:

$$L[u(x,y,t)] + R[u(x,y,t)] + N[u(x,y,t)] = g(x,y,t) \tag{10}$$

with initial condition

$$u(x,y,0) = f(x,y), \tag{11}$$

where $L = \frac{\partial}{\partial t}$, R is a linear operator which has partial derivatives, N is a nonlinear operator and $g(x, y, t)$ is an inhomogeneous term. According to the RDTM, we can construct the following iteration formula:

$$(k+1)U_{k+1}(x,y,t) = G_k(x,y) - R[U_k(x,y)] - N[U_k(x,y)] \tag{12}$$

where $U_k(x, y)$, $R[U_k(x, y)]$, $N[U_k(x, y)]$ and $G_k(x, y)$ are the transformations of the functions $u(x, y, t)$, $R[u(x, y, t)]$, $N[u(x, y, t)]$ and $g(x, y, t)$ respectively. From the initial condition (8), we write:

$$U_0(x,y) = f(x,y). \tag{13}$$

Substituting Eq. (13) into Eq. (12) and by straightforward iterative calculation, we get the following $U_k(x,y)$ values. Then, the inverse transformation of the set of $U_k(x,y), k=1,2,3,\ldots$ is giving the *n*-terms approximation solution as follows:

$$u_n(x,y,t)=\sum_{k=0}^{n} U_k(x,y)t^k. \tag{14}$$

Therefore, the exact solution of the problem is given by:

$$u(x,y,t)=\lim_{n\to\infty} u_n(x,y,t). \tag{15}$$

The fundamental mathematical operations performed by RDTM can be readily obtained and are listed in **Table 1**.

Table 1. The fundamental operations of RDTM.

Functional Form	Transformed Form
$u(x,t)$	$\frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,y,t)]_{t=0}$,
$w(x,t)=u(x,t)\pm v(x,t)$	$W_k(x)=U_k(x)\pm V_k(x)$
$w(x,t)=\alpha u(x,t)$	$W_k(x)=\alpha U_k(x,t)$ (α is a constant)
$w(x,t)=x^m t^n$	$W_k(x)=x^m\delta(k-n)$, $\delta(k)=\begin{cases}1 & k=0\\ 0 & k\neq 0\end{cases}$
$w(x,t)=x^m t^n u(x,t)$	$W_k(x)=x^m U(k-n)$
$w(x,t)=u(x,t)v(x,t)$	$W_k(x)=\sum_{r=0}^{k}U_r(x)V_{k-r}(x)=\sum_{r=0}^{k}V_r(x)W_{k-r}(x)$
$w(x,t)=\frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x)=(k+1)\ldots(k+r)W_{k+r}(x)=\frac{(k+r)!}{k!}W_{k+r}(x)$
$w(x,t)=\frac{\partial}{\partial x}u(x,t)$	$W_k(x)=\frac{\partial}{\partial x}U_k(x)$
$w(x,t)=\frac{\partial^2}{\partial x^2}u(x,t)$	$W_k(x)=\frac{\partial^2}{\partial x^2}U_k(x)$.

3. Numerical results

To demonstrate the effectiveness of the reduced differential transform method (RDTM) algorithm this discussed in the above section. We use this method to construct the analytic approximate solutions for two systems of nonlinear wave equations which have a great attention by many researchers in physics and engineering. The results have been provided by software packages such as Mathematica 9.

Example 3.1 We first consider a two components evolutionary system of a homogeneous KdV equatons of order 3 [48, 49] Eqs. (1) and (2) reads:

$$u_t-u_{xxx}-2vu_x-uv_x=0, \tag{16}$$

$$\upsilon_t-uu_x=0. \tag{17}$$

Subject to

$$u(x,0) = -\tanh(\frac{x}{\sqrt{3}}), \tag{18}$$

$$\upsilon(x,0) = -\frac{1}{6} - \frac{1}{2}\tanh^2(\frac{x}{\sqrt{3}}). \tag{19}$$

Applying the reduced differential transform to the Eq. (12), we obtain the following iteration relation:

$$(k+1)U_{k+1}(x) = \frac{\partial^3}{\partial x^3}U_k(x) + 2\sum_{r=0}^{k} V_r(x)\frac{\partial}{\partial x}U_{k-r}(x) + \sum_{r=0}^{k} U_r(x)\frac{\partial}{\partial x}V_{k-r}(x), \tag{20}$$

$$(k+1)V_{k+1}(x) = \sum_{r=0}^{k} U_r(x)\frac{\partial}{\partial x}U_{k-r}(x), \tag{21}$$

Using the initial conditions Eq. (18) and Eq. (19), we have:

$$U_0(x) = -\tanh\left(\frac{x}{\sqrt{3}}\right), \tag{22}$$

$$V_0(x) = -\frac{1}{6} - \frac{1}{2}\tanh^2\left(\frac{x}{\sqrt{3}}\right), \tag{23}$$

Now, substituting Eqs. (18)-(19) into Eqs. (20)-(21), we obtain the following $V_{k+1}(x,y,t)$ values successively as follows:

$$\left.\begin{aligned}
U_1(x) &= \frac{1}{\sqrt{3}}\operatorname{sech}^2\left(\frac{x}{\sqrt{3}}\right),\\
V_1(x) &= \frac{1}{\sqrt{3}}\operatorname{sech}^2\left(\frac{x}{\sqrt{3}}\right)\tanh\left(\frac{x}{\sqrt{3}}\right),\\
U_2(x) &= \frac{1}{3}\operatorname{sech}^2\left(\frac{x}{\sqrt{3}}\right)\tanh\left(\frac{x}{\sqrt{3}}\right),\\
V_2(x) &= \frac{1}{6}(-2+\cosh(\frac{2x}{\sqrt{3}}))\operatorname{sech}^4\left(\frac{x}{\sqrt{3}}\right),\\
U_3(x) &= \frac{1}{9\sqrt{3}}(-2+\cosh\left(\frac{2x}{\sqrt{3}}\right))\operatorname{sech}^2\left(\frac{x}{\sqrt{3}}\right),\\
V_3(x) &= \frac{1}{18\sqrt{3}}\operatorname{sech}^5\left(\frac{x}{\sqrt{3}}\right)\left(-11\sinh\left(\frac{x}{\sqrt{3}}\right)+\sinh\left(\sqrt{3}x\right)\right),
\end{aligned}\right\} \tag{24}$$

and so on. In the same manner, the rest of components can be obtained by using Mathematica software. Taking the inverse transformation of the set of values $[V_k(x,y,t)]_{k=0}^{n}$ gives n-terms approximation solutions. Finally the differential inverse transform of $V_k(x,y,t)$ give:

$$\begin{aligned}
u_n(x,t) &= \sum_{k=0}^{n} U_k(x,t)t^n = U_0 + U_1 t + U_2 t^2 + U_3 t^3 + U_4 t^4 + U_5 t^5 + \dots\\
&= -\tanh\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\sec h^2\left(\frac{x}{\sqrt{3}}\right)t + \frac{1}{3}\sec h^2\left(\frac{x}{\sqrt{3}}\right)\tanh\left(\frac{x}{\sqrt{3}}\right)t^2\\
&\quad + \frac{1}{9\sqrt{3}}(-2+\cosh(\frac{2x}{\sqrt{3}}))\sec h^4(\frac{x}{\sqrt{3}})t^3 + \dots,
\end{aligned} \tag{25}$$

$$v_n(x,t) = \sum_{k=0}^{n} V_k(x,t)t^n = V_0 + V_1 t + V_2 t^2 + V_3 t^3 + V_4 t^4 + V_5 t^5 + \dots$$

$$=-\frac{1}{6}-\frac{1}{2}\tanh^2\left(\frac{x}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\sec h^2\left(\frac{x}{\sqrt{3}}\right)\tanh\left(\frac{x}{\sqrt{3}}\right)t+\frac{1}{6}(-2+\cosh(\frac{2x}{\sqrt{3}}))\sec h^4\left(\frac{x}{\sqrt{3}}\right)t^2$$
$$+\frac{1}{18\sqrt{3}}\sec h^5\left(\frac{x}{\sqrt{3}}\right)\left(-11\sinh\left(\frac{x}{\sqrt{3}}\right)+\sinh\left(\sqrt{3}x\right)\right)t^3+..., \qquad (26)$$

Therefore, the exact solution of the problem is readily obtained as follows:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=-\tanh(\frac{x-t}{\sqrt{3}}) \qquad (27)$$

$$\upsilon(x,t)=\lim_{n\to\infty}\upsilon_n(x,t)=-\frac{1}{6}-\frac{1}{2}\tanh^2(\frac{x-t}{\sqrt{3}}) \qquad (28)$$

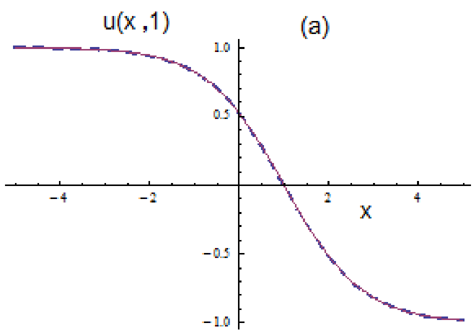
To examine the accuracy of the RDTM solution, the absolute errors of the 7-terms approximate solution is listed in **Table 2** and plotted in Fig. 1 and Fig. 2.

Table 2. The exact solution obtained in Eq. (27) is compared with the approximate solution obtained in Eq. (25).

t	x	u_5	*Exact solution*	$\lvert u_5-Exact\rvert$
	0.2	0.171478	0.171494	0.000025
	0.4	0.057637	0.057671	0.000034
0.5	0.6	-0.057712	-0.057671	0.000041
	0.8	-0.171530	-0.171494	0.000037
	1	-0.280940	-0.280915	0.000025

Table 3. The exact solution obtained in Eq. (28) is compared with the approximate solution obtained in Eq. (26).

t	x	v_5	*Exact solution*	$\lvert v_5-Exact\rvert$
	0.2	-0.181273	-0.181372	0.0000989
	0.4	-0.168271	-0.168330	0.0000582
0.5	0.6	-0.168326	-0.168330	0.0000038
	0.8	-0.181412	-0.181372	0.0000401
	1	-0.206183	-0.206123	0.0000602



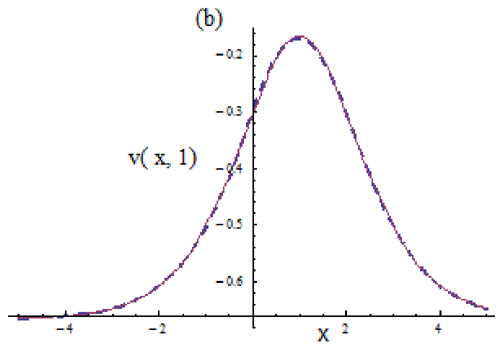


Figure 1: The exact solution (27) and (28) are compared with the approximate solution (25) and (26) at t =0.5

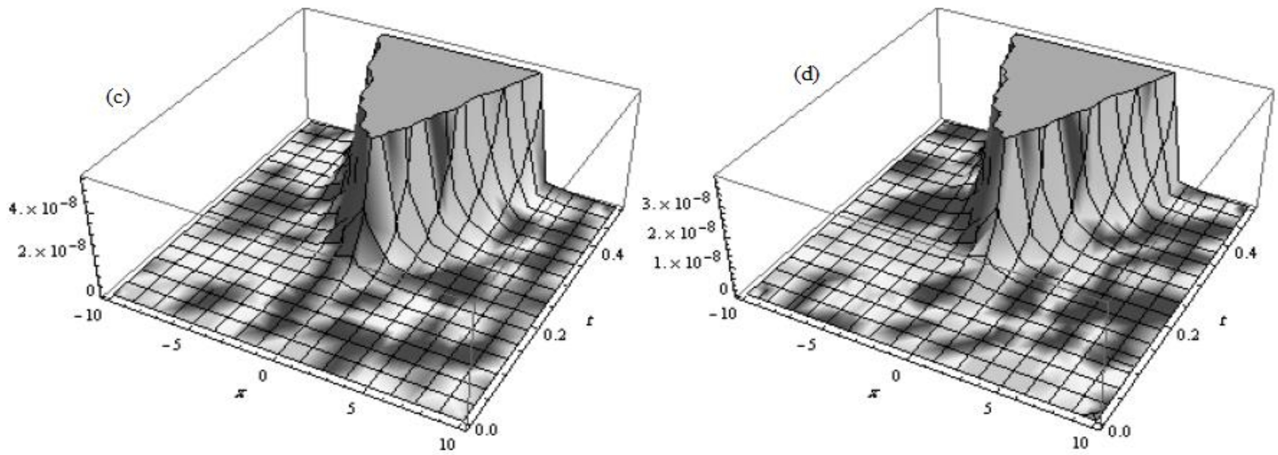


Figure 2: The absolute errors wth respect of $u_5(x,t)$ and $v_5(x,t)$ are shown respectively.

From Figs. 1-3 and **Tables 2**, **3** the approximate solution is rapidly convergence to the exact solution.

Example 3.2 Consider the generalzed coupled Hirote Satsuma KdV [48, 49]:

$$u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3\frac{\partial}{\partial x}(vw) = 0, \tag{29}$$

$$\upsilon_t + \upsilon_{xxx} - 3u\upsilon_x = 0, \tag{30}$$

$$w_t + w_{xxx} - 3u\,w_x = 0. \tag{31}$$

Subject to

$$u(x,0) = -\frac{1}{3} + 2\tanh^2 x\ , \tag{32}$$

$$v(x,0) = \tanh x, \tag{33}$$

$$w(x,0) = \frac{8}{3}\tanh \mathrm{x}. \tag{34}$$

Applying the reduced differential transform to the Eq. (25), we obtain the following iteration relation:

$$(k+1)U_{k+1} = \frac{1}{2}\frac{\partial^3 U_k}{\partial x^3} - 3\sum_{r=0}^{k} U_r \frac{\partial}{\partial x}U_{k-r} + 3\frac{\partial}{\partial x}\sum_{r=0}^{k} V_r W_{k-r}, \tag{35}$$

$$(k+1)V_{k+1} = -\frac{\partial^3 V_k}{\partial x^3} + 3\sum_{r=0}^{k} U_r \frac{\partial V_{k-r}}{\partial x}, \tag{36}$$

and

$$(k+1)W_{k+1} = -\frac{\partial^3 W_k}{\partial x^3} + 3\sum_{r=0}^{k} U_r \frac{\partial W_{k-r}}{\partial x}, \tag{37}$$

We can assume the initial condition have the following form:

$$U_0(x,0) = -\frac{1}{3} + 2\tanh^2 x\ , \tag{38}$$

$$V_0(x,0) = \tanh x\,, \tag{39}$$

$$W_0(x,0) = \frac{8}{3}\tanh \mathrm{x}. \tag{40}$$

Now, substituting Eqs. (38)-(40) into Eqs. (35)-(37), we obtain the following $V_{k+1}(x,y,t)$ values successively as follows:

$$\left.\begin{aligned} U_1(x,t) &= 4\sec h^2 x \tanh x, \\ V_1(x,t) &= \sec h^2 x, \\ W_1(x,t) &= \frac{8}{3}\sec h^2 x, \\ U_2(x,t) &= 2-8\tanh^2 x-6\tanh^4 x, \\ V_2(x,t) &= \tanh^3 x-\tanh x, \\ W_2(x,t) &= \frac{8}{3}(\tanh^3 x-\tanh x), \end{aligned}\right\} \tag{41}$$

and so on. We calculate the 5$^{\text{Th}}$ iteration but write the five first terms for convence to the reader. In the same manner, the rest of components can be obtained by using Mathematica software. Taking the inverse transformation of the set of values $[V_k(x,y,t)]_{k=0}^{n}$ gives n-terms approximation solutions. Finally the differential inverse transform of $V_k(x,y,t)$ give:

$$u_n(x,t)=\sum_{k=0}^{n}U_k(x,t)t^n = U_0+U_1t+U_2t^2+U_3t^3+U_4t^4+U_5t^5+U_6t^6+U_7t^7+\ldots$$

$$=-\frac{1}{3}+2\tanh^2 x+4\sec h^2 x\tanh x+2-8\tanh^2 x-6\tanh^4 x+\ldots \tag{42}$$

$$v_n(x,t)=\sum_{k=0}^{n}V_k(x,t)t^n = V_0+V_1t+V_2t^2+V_3t^3+V_4t^4+V_5t^5+V_6t^6+V_7t^7+\ldots$$

$$=\tanh x+\sec h^2 x+\tanh^3 x-\tanh x+\ldots \tag{43}$$

$$w_n(x,t)=\sum_{k=0}^{n}W_k(x,t)t^n = W_0+W_1t+W_2t^2+W_3t^3+W_4t^4+W_5t^5+W_6t^6+W_7t^7+\ldots$$

$$=\frac{8}{3}\tanh x+\frac{8}{3}\sec h^2 x+\frac{8}{3}(\tanh^3 x-\tanh x)+\ldots \tag{44}$$

Therefore, the exact solution of the problem is readily obtained as follows:

$$u(x,t)=\lim_{n\to\infty} u_n(x,t)=-\frac{1}{3}+2\tanh^2(t+x), \tag{45}$$

$$v(x,t)=\lim_{n\to\infty} v_n(x,t)=\tanh(t+x), \tag{46}$$

$$w(x,t)=\lim_{n\to\infty} w_n(x,t)=\frac{8}{3}\tanh(\mathrm{t}+\mathrm{x}). \tag{47}$$

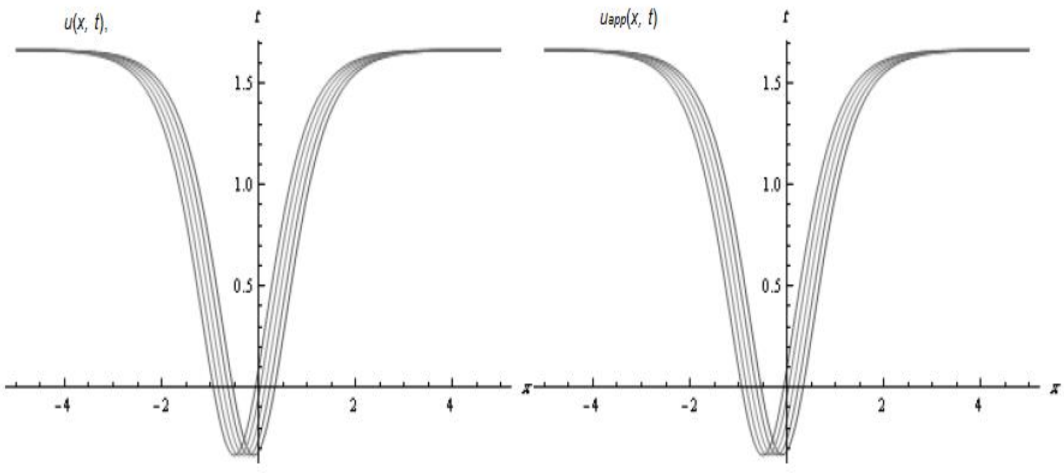


Figure 3: The exact solution (45) is compared with the approximate solution (42) for different value of $t = 0.1, 0.2, 0.3, 0.4, 0.5.$

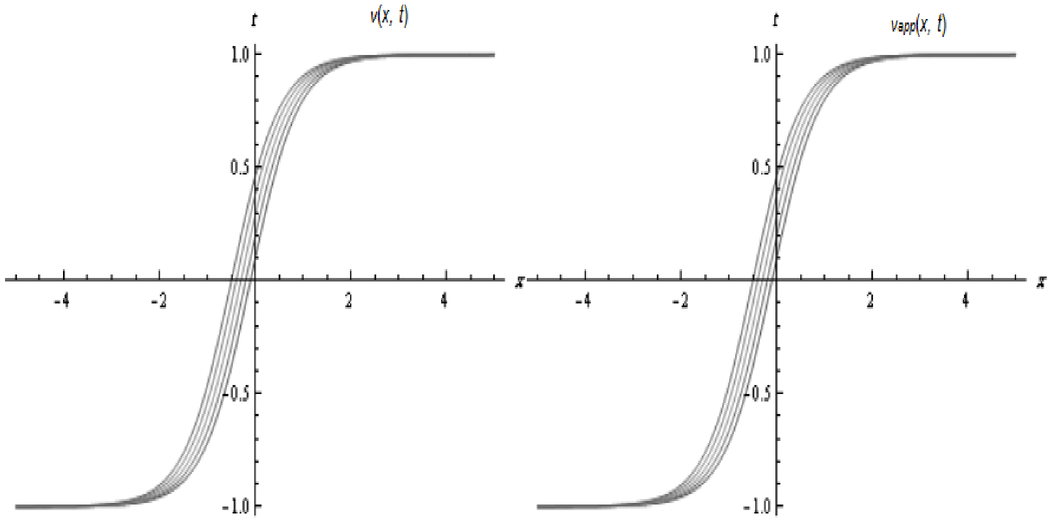


Figure 4: The exact solution (46) is compared with the approximate solution (43) for different value of $t = 0.1, 0.2, 0.3, 0.4, 0.5$.

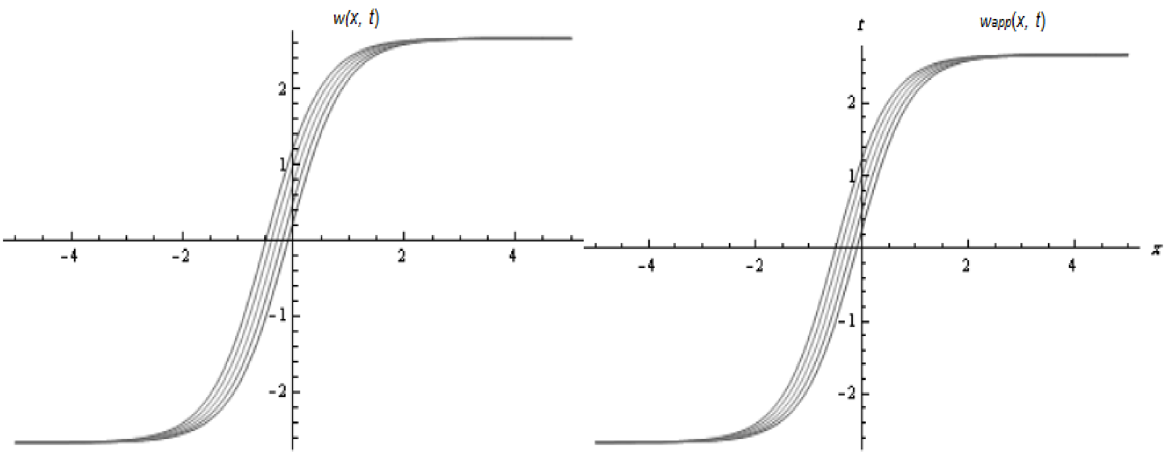


Figure 5: The exact solution (47) is compared with the approximate solution (44) for different value of $t = 0.1, 0.2, 0.3, 0.4, 0.5$.

4. Conclusions

In this work, we present new applications of the reduced differential transform method (RDTM) by handling two nonlinear physical models, namely, generalized KP hierarchy equations. This method is an alternative approach to overcome the demerit of complex calculation of differential transform method (DTM). The proposed technique, which does not require linearization, discretization or perturbation, gives the solution in the form of a convergent power series with elegantly computed components. Therefore, the solution procedure of the RDTM is simpler than other traditional methods. The main advantage of the proposed method is that it requires less amount of computation. The results show that the RDTM is a powerful mathematical tool for handling NPDEs. The approximate solutions are rabidly convergence to the exact solutions. It can be observed that the solution approach of RDTM is much simpler than differential transform method (DTM) and it needs less computational effort than DTM. In other word, RDTM is an alternative approach to overcome the demerit of complex calculation of DTM, capable of reducing the size of calculation. As a special advantage of RDTM rather than DTM, the reduced differential transform recursive equations produce exactly all the Poisson series coefficients of solutions, whereas the differential transform recursive equations produce exactly all the Taylor series coefficients of solutions. We notice that the RDTM technique is highly accurate, rapidly converge and is very easily implementable mathematical tool for the multidimensional physical problems emerging in various domains of engineering and allied sciences.

5. Acknowledgement

The authors would like to express their gratitude to the Editor and Referees for their encouragement and constructive comments in revising the paper.

6. References

[1] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, 1994.

[2] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge (1991).

[3] R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, Physical Review Letters, 27 (1971), pp. 1192-1194.

[4] M. R. Miura, Bäcklund Transformation, Springer-Verlag, Berlin (1978).

[5] C. Rogers and W. F. Shadwick, Backlund Transformations,Academic Press, New York (1982).

[6] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, Journal of Mathematical Physics, 24 (1983), pp. 522-526.

[7] D. S. Wang, Y. J. Ren and H. Q. Zhang, Further extended sinh-cosh and sin-cos methods andnew non traveling wave solutions of the (2+1)-dimensional dispersive long wave equations, Applied Mathematics E-Notes, 5 (2005), pp. 157-163.

[8] M. Wang, Y. Zhou and Z. Li, Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Physics Letters A, 216 (1-5) (1996), pp. 67-75.

[9] J. H. He, Variational principle for some nonlinear partial differential equations with variable coefficients, Chaos, Solitons & Fractals, 19 (2004), pp. 847-851.

[10] J. H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation, 151(1) (2004), pp. 287-292.

[11] M. S. Mohamed, S. M. Abo-Dahab and A. M. Abd-Alla, Homotopy analysis method for harmonic waves propagation in nonlinear thermoelasticity with magnetic field and rotation, Journal of Computational and Theoretical Nanoscience, 11(5) (2014), pp. 1354-1361.

[12] E. M. E. Zayed, T. A. Nofal and K. A. Gepreel, The travelling wave solutions for non-linear initialvalue problems using the homotopy perturbation method, Applicable Analysis, 88(4) (2009), pp. 617-634.

[13] H. M. Liu, Generalized variational principles for ion acoustic plasma waves by He's semi-inverse method, Chaos, Solitons & Fractals, 23(2) (2005), pp. 573-576.

[14] H. M. Liu, Variational approach to nonlinear electrochemical system, International Journal of Nonlinear Sciences and Numerical Simulation, 5 (1) (2004), pp. 95-96.

[15] E. Babolian, J. Biazar, and A. R. Vahidi, The decomposition method applied to systems of Fredholm integral equationsof the second kind, Applied Mathematics and Computation, 148(2) (2004), pp. 443-452.

[16] A. M. Abd-Alla, S. M. Abo-Dahab, T. A. Al-Thamali and S. R. Mahmoud, Influence of the rotation and gravity field on Stonely waves in a non-homogeneous orthotropic elastic medium, Journal of Computational and Theoretical Nanoscience, 10 (2) (2013), pp. 297-305.

[17] M. Sayed and A. A. Mousa, Vibration, stability, and resonance of angle-ply composite laminated rectangular thin plate under multi-excitations, Mathematical Problem in Engineering, Volume 2013, Article ID 418374, 26 pages, (2013).

[18] Y. S. Hamed, M. Sayed, D.-X. Cao and W. Zhang, Nonlinear study of the dynamic behavior of a string-beam coupled system under combined excitations, Acta Mechanica Sinica 27(6) (2011), pp. 1034-1051.

[19] H. A. Abdusalam, On an improved complex tanh-function method, International Journal of Nonlinear Sciences and Numerical Simulation, 6(2) (2005), pp. 99 – 106.

[20] E. M. E. Zayed and K. A. Gepreel, The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics, Journal of Mathematical Physics, 50 (2009), pp. 013502-013513.

[21] D. Lü, Jacobi elliptic function solutions for two variant Boussinesq equations, Chaos, Solitons & Fractals, 24(5) (2005), pp. 1373-1385.

[22] Y. Chen and Q. Wang, Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic functions solutions to (1+1) dimensional dispersive long wave equation, Chaos, Solitons and Fractals, 24(3) (2005), pp. 745-757.

[23] S. L. Zhang, B. Wu and S. Y. Lou, Painlevé analysis and special solutions of generalized Broer-Kaup equations, Physics Letters A, 300(1) (2002), pp. 40-48.

[24] E. M. E. Zayed, H. A. Zedan and K. A. Gepreel, Group analysis and modified Extended tanh-function to find the invariant solutions and soliton solutions for nonlinear Euler equations, International Journal of Nonlinear Sciences and Numerical Simulation, 5(3) (2004), pp. 221-234.

[25] M. A. Abdou, The extended tanh-method and its applications for solving nonlinear physical models, Applied Mathematics and Computation, 190(1) (2007), pp. 988-996.

[26] M. Wang and X. Li, Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations, Physics Letters A, 343(1-3) (2005), pp. 48-54.

[27] S. Zhang and T. Xia, Symbolic computation and new families of exact non-travelling wave solutions to (3+1)-dimensional Kadomtsev-Petviashvili equation, Applied Mathematics and Computation, 181(1) (2006), pp. 319-331.

[28] S. K. Elagan, M. Sayed and Y. S. Hamed, An innovative solutions for the generalized FitzHugh-Nagumo equation by using the generalized (G'/G)-expansion method, Applied Mathematics, 2 (2011), pp. 470-474.

[29] Y. S. Hamed, M. Sayed, S. K. Elagan and E. R. Elzahar, The Improved (G'/G)-expansion method for solving (3+1)–dimensional potential-YTSF equation, Journal of modern Methods in Numerical Mathematics, 2(1-2) (2011), pp. 32–38.

[30] Y. S. Hamed, M. Sayed and S. K. Elagan, A new exact solution for the generalized combined KdV and mKdV equation using the generalized (G'/G)-expansion method, International Journal of Applied Mathematics and Computation, 3(3) (2011), pp. 200–205.

[31] T. A. Nofal, M. Sayed, Y. S. Hamed and S. K. Elagan, The improved (G'/G)-expansion method for solving the fifth-order KdV-equation, Annals of Fuzzy Mathematics and Informatics, 3(1) (2012), pp. 9- 17.

[32] T. Elghareb, S. K. Elagan, Y. S. Hamed and M. Sayed, An exact solutions for the generalized fractional Kolmogrove-Petrovskii Piskunov equation by using the generalized (G'/G)-expansion method, International Journal of Basic & Applied Sciences 13(1) (2013), pp. 19-22.

[33] J. H. He and X. H. Wu, Exp-function method for nonlinear wave equations, Chaos, Solitons & Fractals, 30(3) (2006), pp. 700-708.

[34] S. Zhang, Application of Exp-function method to high-dimensional nonlinear evolution equation, Chaos, Solitons & Fractals, 38(1) (2008), pp. 270-276.

[35] S. Zhang, Application of Exp-function method to Riccati equation and new exact solutions with three arbitrary functions of Broer-Kaup-Kupershmidt equations, Physics Letters A, 372(11) (2008), pp. 1873-1880.

[36] M. Saad, S. K. Elagan, Y. S. Hamed and M. Sayed, Using a complex transformation to get an exact solution for fractional generalized coupled MKDV and KDV equations, International Journal of Basic & Applied Sciences 13(1) (2013), pp. 23-25.

[37] S. K. Elagan and M. Sayed, On the exact solutions of fractional differentional equations using improved Riccati equations method, International Journal of Scientific & Engineering Research, 5(10) (2014), pp. 1149-1156.

[38] M. Saad, S. K. Elagan and M. Sayed, An extended Riccati sub-equation method to establish new exact solutions for fraction nonlinear differentional difference lattice equations, International Journal of Scientific & Engineering Research, 5(10) (2014), pp. 1162-1166.

[39] F. Mabood and W. A. Khan, Combined analytical-numerical solution for MHD viscous flow over a stretching sheet, Journal of Computational Engineering, (2014), Article ID 634328, 7 pages http://dx.doi.org/10.1155/2014/634328.

[40] M. S. Mohamed, S. K. Elagan, K. A. Gepreel and A. Elosimy, Optimal q-homotopy analysis method for nonlinear fractional gas dynamics equation, Jökull Journal, 64(9) (2014), pp. 317-326.

[41] Y. A. Elmaboud, Kh. S. Mekheimer and M. S. Mohamed, Series solution of a natural convection flow for a Carreau fluid in a vertical channel with peristalsis, Journal of Hydrodynamics, 27(6) (2015), pp. 969-979.

[42] K. A. Gepreel, S. M. Abo-Dahab and M. S. Mohamed, Analytical approximate solutions for nonlinear differential-difference equations arising in nanotechnology, Journal of Computational and Theoretical Nanoscience, 12(6) (2015), pp. 1040-1044.

[43] F. Mabood, W. A. Khan and A. I. Ismail, Optimal homotopy asymptotic method for heat transfer in hollow sphere with robin boundary conditions, Heat Transfer-Asian Research, 43(2) (2014), pp. 124-133.

[44] Y. Keskin and G. Oturanç, Reduced differential transform method for partial differential equations, International

Journal of Nonlinear Sciences and Numerical Simulation, 10(6) (2009), pp. 741-750.

[45] Y. Keskin and G. Oturanç, The reduced differential transform method: a new approach to factional partial differential equations, Nonlinear Science Letters A, 1(2) (2010), pp. 207-217.

[46] Y. Keskin and G. Oturanç, Reduced differential transform method for generalized KdV equations, Mathematical and Computational Applications, 15 (3) (2010), pp. 382-393.

[47] Y. Keskin, Ph. D Thesis, Selcuk University, 2010 (in Turkish).

[48] M. V. Foursov and M. M. Maza, On computer-assisted classifications of coupled integrable equations, Journal of Sympolic Computation, 33(5) (2002), pp. 647-660.

[49] E. Fan, Soliton solutions for a generalized Hirota-Satsuma coupled KdV equation and a coupled MKdV equation, Physics Letters A, 282 (1-2) (2001), pp. 18-22.