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Numerical Treatment of Lacunary Spline Function of Fractional Order

Faraidun K. Hamasalh [1], Pshtiwan O. Muhammed[2]
[1,2]Faculty of Science and Science Education
School of Science Education, Sulaimani Univ., Sulaimani, Iraq
[1]E-mail: faraidun.hamasalh@univsul.edu.iq.com
[2]E-mail: pshtiwansangawi@gmail.com
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Abstract. In this paper, effects of lacunary fractional derivatives and existence/ uniqueness on the fifth order spline function in fractional order have been studied. These results use in the convergence analysis of fractional spline polynomial method for such problems numerically. Theoretical results are illustrated with efficient of the absolute errors by some fractional differential examples.

Keywords: caputo derivatives, spline function, convergence analysis.

1. Introduction

Fractional calculus is a field of mathematics arise in various areas of science and engineering. In the recent past years, that grows of the traditional definitions of the integral and derivative theory, based on these requirements, the numerical analysis has become very important of fractional differential equations (see, [6, 8, 12, 13, 14]).

Interpolation polynomials are generalized of approximating solution of data interpolated functions and ordinary differential equations to an arbitrary order. These approximations have attracted considerable interest because of their ability to solving complex phenomena. In recent years the splines of fractional order have also had a significant increasing attention of the wavelet transform [5, 10]. We refer to the recent papers of describe properties fractional spline derivatives that make them useful in a variety of applications [7, 9, 11].

In this paper, we will extend the construction of fractional quintic spline polynomial on (faraidun& pshtiwan [4]). We prove the existence and uniqueness of the model of spline function in Theorem 1, also prove the convergence of the method and drive error estimates for the all derivatives gives effective for fractional derivatives of the method. Finally, the theoretical results illustrated by some numerical examples and the absolute errors be more efficient with good accuracy when compared to [11].

2. System description

We start the definitions of the fractional Taylor series and Caputo fractional derivative and integral:

Definition 2.1 (Sabatier and Agrawal, 2007, Michael and B. Thierry, 2000, [3, 10,] respectively) The Caputo fractional derivative of order $\alpha > 0$, is defined by

$$D_a^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^x \dfrac{f^{(n)}(s)}{(x-s)^{\alpha+1-n}} ds & \text{for } n-1 < \alpha < n\,, n \in \mathbb{N} \\ \dfrac{d^n}{dx^n} f(x) & \text{for } \alpha = n, \quad n \in \mathbb{N} \end{cases}$$

Definition 2.2 (Podlubny, 1999 [2])

Suppose that $D_a^{k\alpha} f(x) \in C[a,b]$ for $k = 0,1,\dots,n+1$ where $0 < \alpha \le 1$,then we have the Taylor Series expansion about $x = \tau$,

$$f(x) = \sum_{i=0}^{n} \frac{(x-\tau)^{i\alpha}}{\Gamma(i\alpha+1)} D_a^{i\alpha} f(\tau) + \frac{(D_a^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-\tau)^{(n+1)\alpha}$$

with $a \le \xi \le x$, for all $x \in (a,b]$, where $D_a^{k\alpha} = D_a^{\alpha}. D_a^{\alpha} \dots D_a^{\alpha}$ (k time).

Definition 2.3 (Kincaid and Cheney, 2002 [])

The modulus of continuity of a function f continuous on a segment [a, b], $f \in C[a,b]$ is a function $\omega(t) = \omega(f,t)$ defined for $t \in [0, b-a]$ by the relation $\omega(t) = \omega(f, t{:}\,\delta) = \max_{|t-x|\le\delta} |f(t) - f(x)|$.

3. Description of the method

We will construct the spline function of fifth order with fractional lacunary derivatives in equation (1) are developed, for the difference degree polynomial with boundary conditions is appeared in (Faraidun and Pshtiwan, 2015, Fawzy and Saxena [4, 15, 16] respectively), and $S_\Delta(x)$in the $C^5[a, b]$ that satisfy the boundary conditions as follows:

$$S(x_k) = y_k + \frac{2}{\sqrt{\pi}}(x\text{-}x_k)^{\frac{1}{2}}a_k + (x\text{-}x_k)Dy_k + \frac{4}{3\sqrt{\pi}}(x\text{-}x_k)^{\frac{3}{2}}b_k + \frac{(x\text{-}x_k)^2}{2}D^{(2)}y_k + \frac{8}{15\sqrt{\pi}}(x\text{-}x_k)^{\frac{5}{2}}c_k + \frac{(x\text{-}x_k)^3}{6}D^{(3)}y_k + \frac{16}{105\sqrt{\pi}}(x\text{-}x_k)^{\frac{7}{2}}d_k + \frac{(x\text{-}x_k)^4}{24}D^{(4)}y_k + \frac{32}{945\sqrt{\pi}}(x\text{-}x_k)^{\frac{9}{2}}D^{\left(\frac{9}{2}\right)}y_k + \frac{(x\text{-}x_k)^5}{120}D^{(5)}y_k \quad (1)$$

On the interval $[x_k, x_{k+1}]$ where a_k, b_k, c_k and $d_k, k = 0,1,2,3,\dots,n-1$, are unknowns to be determined, with the boundary continuity conditions

$$S^{(j)}(x_k) = y_{k+1}^{(j)}, \text{ j=0, 1, 2, 3, 4, 9/2, 5.}$$

Theorem 3.1: Let $S(x) \in C^5[a,b]$, $x \in [x_k,\ x_{k+1}]$, $k = 0,1,2,3,\dots,n-1$, then there exists a unique spline function with fractional order for formula (1).

Proof: From the construct formula in equation (1), can be using the continuity conditions to obtain:

$$A_k = y_{k+1} - y_k - hDy_k - \frac{h^2}{2}D^{(2)}y_k - \frac{h^3}{6}D^{(3)}y_k - \frac{h^4}{24}D^{(4)}y_k - \frac{32}{945\sqrt{\pi}}h^{\frac{9}{2}}D^{\left(\frac{9}{2}\right)}y_k + \frac{h^5}{120}D^{(5)}y_k \quad (2)$$

where $A_k = \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}a_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}b_k + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}c_k + \frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}d_k$, $h = x - x_k$,

$$B_k = Dy_{k+1} - Dy_k - hD^{(2)}y_k - \frac{h^2}{2}D^{(3)}y_k - \frac{h^3}{6}D^{(4)}y_k - \frac{16}{105\sqrt{\pi}}h^{\frac{7}{2}}D^{\left(\frac{9}{2}\right)}y_k - \frac{h^4}{24}D^{(5)}y_k \quad (3)$$

where $B_k = \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}b_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}c_k + \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}d_k$,

$$C_k = D^{(2)}y_{k+1} - D^{(2)}y_k - hD^{(3)}y_k - \frac{h^2}{2}D^{(4)}y_k - \frac{8}{15\sqrt{\pi}}h^{\frac{5}{2}}D^{\left(\frac{9}{2}\right)}y_k - \frac{h^3}{6}D^{(5)}y_k \quad (4)$$

where $C_k = \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}c_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}d_k$,

$$D_k = D^{(3)}y_{k+1} - D^{(3)}y_k - hD^{(4)}y_k - \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}D^{\left(\frac{9}{2}\right)}y_k - \frac{h^2}{2}D^{(5)}y_k \quad (5)$$

where $D_k = \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}d_k$.

Equating these equations (2) to (5), to obtain the following:

$$a_k = \frac{\sqrt{\pi}}{1890}h^{-\frac{1}{2}}[945A_k - 630hB_k + 168h^2C_k - 16h^3D_k], \quad (6)$$

$$b_k = \frac{\sqrt{\pi}}{90}h^{-\frac{1}{2}}[45B_k - 30hC_k + 8h^2D_k], \quad (7)$$

$$c_k = \frac{\sqrt{\pi}}{6}h^{-\frac{1}{2}}[3C_k - 2hD_k], \quad (8)$$

$$d_k = \frac{\sqrt{\pi}}{2}h^{-\frac{1}{2}}D_k. \quad (9)$$

Lemma 3.2: Let $y(x_k) \in C^5[a,b]$, and the error estimate of the spline function in equation (1), for the unknown derivatives is

$$|a_k - D^{\left(\frac{1}{2}\right)}y_k| \le \frac{13\sqrt{\pi}}{15120} h^{\frac{9}{2}}\, \omega_5(h,y),$$
$$|b_k - D^{\left(\frac{3}{2}\right)}y_k| \le \frac{7\sqrt{\pi}}{6!} h^{\frac{7}{2}}\omega_5(h,y),$$
$$|c_k - D^{\left(\frac{5}{2}\right)}y_k| \le \frac{\sqrt{\pi}}{12} h^{\frac{5}{2}}\omega_5(h,y),$$
$$|d_k - D^{\left(\frac{7}{2}\right)}y_k| \le \frac{\sqrt{\pi}}{4} h^{\frac{3}{2}}\omega_5(h,y).$$

Where k=0, 1, 2,…,n-1, and $\omega_5(h,y)$is the Modulus continuity.

Proof: The Taylor series for $y(x_k)$, can be expand and subtracting with value a_k, we obtain

$$|a_k - D^{(\frac{1}{2})}y_k| = |\frac{\sqrt{\pi}}{1890} h^{\frac{-1}{2}} [945\left(\frac{h^5}{120}\omega_5(h,y)\right) - 630h\left(\frac{h^4}{24}\omega_5(h,y)\right) + 167h^2\left(\frac{h^3}{6}\omega_5(h,y)\right) - 16h^3\left(\frac{h^2}{2}\omega_5(h,y)\right)|,$$

$$\le \frac{\sqrt{\pi}}{1890} h^{-\frac{1}{2}}\left(\frac{13\sqrt{\pi}}{80} h^5\, \omega_5(h,y)\right) = \frac{13\sqrt{\pi}}{15120} h^{\frac{9}{2}}\, \omega_5(h,y),$$

and

$$|b_k - D^{(\frac{3}{2})}y_k| = |\frac{\sqrt{\pi}}{90} h^{\frac{-1}{2}} [45\left(\frac{h^4}{20}\omega_5(h,y)\right) - 30h\left(\frac{h^3}{6}\omega_5(h,y)\right) + 8h^2\left(\frac{h^2}{2}\omega_5(h,y)\right)|,$$

$$\le \frac{\sqrt{\pi}}{90} h^{-\frac{1}{2}}\left(\frac{7\sqrt{\pi}}{8} h^4\, \omega_5(h,y)\right) = \frac{7\sqrt{\pi}}{6!} h^{\frac{7}{2}}\, \omega_5(h,y),$$

Similarly, using Taylor series expansion for $y(x_k)$, with taking derivatives for equation (8) and (9), we obtain

$$|c_k - D^{\left(\frac{5}{2}\right)}y_k| \le \frac{\sqrt{\pi}}{12} h^{\frac{5}{2}}\omega_5(h,y),$$
$$|d_k - D^{(\frac{7}{2})}y_k| \le \frac{\sqrt{\pi}}{4} h^{\frac{3}{2}}\omega_5(h,y).$$

Theorem 3.2: Let $S(x_k) \in C^5[a,b]$, be the spline function defined in equation (1), then the error bound with the function $y(x_k)$, as follows:

$$\left|D^{(m\alpha)}S(x_k) - D^{(m\alpha)}y(x_k)\right| \le \delta_{m\alpha} h^{5-m\alpha}\omega_5(h,y) \tag{10}$$

where $\delta_0 = \frac{19}{180}, \delta_{\frac{1}{2}} = \frac{(355\pi+128)}{3780\sqrt{\pi}}$, $\delta_1 = \frac{58}{315}$, , $\delta_{\frac{3}{2}} = \frac{(1099\pi+768)}{7!\sqrt{\pi}}$, $\delta_2 = \frac{2}{3}$, , $\delta_{\frac{5}{2}} = \frac{(5\pi+8)}{15}, \delta_3 = 1$, , $\delta_{\frac{7}{2}} = \frac{(3\pi+16)}{12\sqrt{\pi}}$, $\delta_4 = 1$, , $\delta_{\frac{9}{2}} = \frac{2}{\sqrt{\pi}}$, , $\delta_5 = 1$, and m=0, 1,.., 5, k=0, 1, 2,…, n-1, $\alpha = \frac{1}{2}$, and $\omega_5(h,y)$is the Modulus continuity.

Proof**:** Assume that $y(x_k)$ is an arbitrary function that can be represented by Taylor expansion about the value $x = x_k$, in [3], then take it the derivatives respects to definitions (2.1) and (2.2), we obtain

$$|S(x_k) - y(x_k)| \le \frac{\sqrt{\pi}}{2} h^{-\frac{1}{2}} |a_k - D^{(\frac{1}{2})} y_k| + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} |b_k - D^{(\frac{3}{2})} y_k| + \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} |c_k - D^{(\frac{5}{2})} y_k| + \frac{16}{105\sqrt{\pi}} h^{\frac{7}{2}} |d_k - D^{(\frac{7}{2})} y_k| + \frac{h^5}{120}\ \omega_5(h,y),$$

Using Lemma 1, we obtain

$$|S(x_k) - y(x_k)| \le \frac{\sqrt{\pi}}{2} h^{-\frac{1}{2}} \left(\frac{13\sqrt{\pi}}{15120} h^{\frac{9}{2}}\ \omega_5(h,y) \right) + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} \left(\frac{7\sqrt{\pi}}{6!} h^{\frac{7}{2}} \omega_5(h,y) \right) + \frac{8}{15\sqrt{\pi}} h^{\frac{5}{2}} \left(\frac{\sqrt{\pi}}{12} h^{\frac{5}{2}} \omega_5(h,y) \right) + \frac{16}{105\sqrt{\pi}} h^{\frac{7}{2}} \left(\frac{\sqrt{\pi}}{4} h^{\frac{3}{2}} \omega_5(h,y) \right),$$

$$|S(x_k) - y(x_k)| \le \frac{19}{180} h^5 \omega_5(h,y),$$

and take the derivative when $m\alpha = \frac{1}{2}$, we obtain

$$\left| D^{\left(\frac{1}{2}\right)} S(x_k) - D^{\left(\frac{1}{2}\right)}\ y(x_k) \right| \le |a_k - D^{(\frac{1}{2})} y_k| + h |b_k - D^{(\frac{3}{2})} y_k| + \frac{h^2}{2} |c_k - D^{(\frac{5}{2})} y_k| + \frac{h^3}{6} |d_k - D^{(\frac{7}{2})} y_k| + \frac{32\ h^{9/2}}{945\ \sqrt{\pi}}\ \omega_5(h,y)$$

$$\left| D^{\left(\frac{1}{2}\right)} S(x_k) - D^{\left(\frac{1}{2}\right)}\ y(x_k) \right| \le (71) \frac{\sqrt{\pi}}{756} h^{\frac{9}{2}}\ \omega_5(h,y) + \frac{32\ h^{9/2}}{945\ \sqrt{\pi}}\ \omega_5(h,y),$$

$$\left| D^{\left(\frac{1}{2}\right)} S(x_k) - D^{\left(\frac{1}{2}\right)}\ y(x_k) \right| \le \frac{h^{\frac{9}{2}}}{3780\ \sqrt{\pi}}\ (355\pi + 128)\ \omega_5(h,y),$$

also take the derivative when $m\alpha = 1$ and using Lemma 1,we obtain

$$|DS(x_k) - Dy(x_k)| \le \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} |b_k - D^{(\frac{1}{2})} y_k| + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} |c_k - D^{(\frac{3}{2})} y_k| + \frac{5}{105\sqrt{\pi}} h^{\frac{5}{2}} |c_k - D^{(\frac{5}{2})} y_k| + \frac{h^4}{24}\ \omega_5(h,y),$$

$$|DS(x_k) - Dy(x_k)| \le \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} \left(\frac{7\sqrt{\pi}}{6!} h^{\frac{7}{2}} \omega_5(h,y) \right) + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} \left(\frac{\sqrt{\pi}}{12} h^{\frac{5}{2}} \omega_5(h,y) \right) + \frac{5}{105\sqrt{\pi}} h^{\frac{5}{2}} \left(\frac{\sqrt{\pi}}{4} h^{\frac{3}{2}} \omega_5(h,y) \right) + \frac{h^4}{24}\ \omega_5(h,y),$$

$$|DS(x_k) - Dy(x_k)| \le \frac{58}{315} h^4\ \omega_5(h,y),$$

Similarly, we can take the derivatives for $m\alpha = \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5$, we obtain the following

$$\left| D^{(m\alpha)} S(x_k) - D^{(m\alpha)} y(x_k) \right| \le \delta_{m\alpha} h^{5-m\alpha} \omega_5(h,y)$$

$\delta_{\frac{3}{2}} = \frac{(1099\pi+768)}{7!\sqrt{\pi}}$, $\delta_2 = \frac{2}{3}$, , $\delta_{\frac{5}{2}} = \frac{(5\pi+8)}{15}$, $\delta_3 = 1$, , $\delta_{\frac{7}{2}} = \frac{(3\pi+16)}{12\sqrt{\pi}}$, $\delta_4 = 1$, , $\delta_{\frac{9}{2}} = \frac{2}{\sqrt{\pi}}$, , $\delta_5 = 1$. This completes the proof.

Lemma 3.2: Let S(x) be the spline function in equation (1), and y(x) be a unique solution of any fractional differential equations, satisfying $\lim_{m\alpha\to\infty} S^{(m\alpha)}(x_k) = \beta$ and $\lim_{m\alpha\to\infty} y^{(m\alpha)}(x_k) = \gamma$, then

$$\lim_{m\alpha\to\infty} \left| D^{(m\alpha)} S(x_k) - D^{(m\alpha)} y(x_k) \right| = 0.$$

Proof**:** From equation (1), $S(x_k)$ is the spline function of finite dimension which is tend to zero, when take it the high derivatives, and $y(x_k)$ has a unique solution $y \in C^n[a,b]$, using (theorem 4.1, in Arvet Pedas [7]), we have

$$\max_{0\le k\le n} \left| D^{(m\alpha)} S(x_k) - D^{(m\alpha)} y(x_k) \right| = \left\| D^{(m\alpha)} S(x_k) - D^{(m\alpha)} y(x_k) \right\|_\infty \to 0 \quad (11)$$

as $m\alpha \to \infty$, with knots, $x \in [x_k, x_{k+1}], k = 0, 1, 2, 3, \dots, n-1$. Therefore, by taking Maximum for equation (10) in Theorem 2, we have for any step size h

$$\lim_{m\alpha\to\infty}\left|D^{(m\alpha)}S(x_k) - D^{(m\alpha)}y(x_k)\right| = 0.$$

4. Numerical illustration

In this section, we present the results of two fractional differential examples on applying the fifth order fractional spline method.

Problem 1: Consider the fractional differential equation form [7]

$$y''(t) + D^{\left(\frac{3}{2}\right)}y(t) + y(t) = \frac{15}{4}\sqrt{t} + \frac{15}{8}\sqrt{\pi t} + t^2\sqrt{t}\,, y(0) = y'(0) = 0, 0 \le t \le 1$$

This is a Cauchy problem of the Bagley-Torvik equation, and $y(t) = t^2\sqrt{t}$.

Table 1. Maximum absolute errors for example 1

h	$\lvert y(t) - s(t)\rvert$	$\left\lvert y^{\frac{1}{2}}(t) - s^{\frac{1}{2}}(t)\right\rvert$	$\lvert y'(t) - s'(t)\rvert$	$\left\lvert y^{\frac{3}{2}}(t) - s^{\frac{3}{2}}(t)\right\rvert$	$\lvert y''(t) - s''(t)\rvert$
0.1	3.3380e-09	9.7509e-08	1.4557e-06	3.1951e-05	7.9057e-04
0.01	1.0556e-16	3.0835e-14	4.6032e-12	1.0104e-09	2.5000e-07
0.001	3.3380e-24	9.7509e-21	1.4557e-17	3.1951e-14	7.9057e-11

Problem 2: Consider the following nonlinear differential equation see [12]

$$D^4(t) + D^{\left(\frac{7}{2}\right)}y(t) + y^3(t) = t^9\,, y(0) = y'(0) = y''(0) = 0, \qquad y'''(0) = 6.$$

The exact solution is $y(t) = t^3$.

Table 2. Maximum absolute errors for example 2

h	$\lvert y(t) - s(t)\rvert$	$\left\lvert y^{\frac{1}{2}}(t) - s^{\frac{1}{2}}(t)\right\rvert$	$\lvert y'(t) - s'(t)\rvert$	$\left\lvert y^{\frac{3}{2}}(t) - s^{\frac{3}{2}}(t)\right\rvert$	$\lvert y''(t) - s''(t)\rvert$
0.1	3.3380e-09	3.3502e-08	5.5238e-07	1.3722e-05	4.0000e-04
0.01	1.0556e-16	3.3502e-15	5.5238e-13	1.3722e-10	4.0000e-08
0.001	3.3380e-24	3.3502e-22	5.5238e-19	1.3722e-15	4.0000e-12

Table 3. Maximum absolute errors derivatives for example 3

h	$\lvert y(t) - s(t)\rvert$	$\left\lvert y^{\frac{1}{2}}(t) - s^{\frac{1}{2}}(t)\right\rvert$	$\lvert y'(t) - s'(t)\rvert$	$\left\lvert y^{\frac{3}{2}}(t) - s^{\frac{3}{2}}(t)\right\rvert$	$\lvert y''(t) - s''(t)\rvert$
0.1	3.3380e-09	-4.2043e-10	-9.0775e-09	-3.0833e-07	-1.3053e-05
1/8	1.7795e-08	-3.1115e-09	-5.3674e-08	-1.4560e-06	-4.9194e-05
1/16	9.8306e-11	-6.1623e-12	-2.1318e-10	-1.1607e-08	-7.8821e-07
1/32	5.4307e-13	-1.2077e-14	-8.3617e-13	-9.1134e-11	-1.2392e-08
1/64	3.0001e-15	-2.3609e-17	-3.2696e-15	-7.1287e-13	-1.9393e-10

Table 4. Compare the method with [Zahra]

H	Error in Our method	Error in Ref [Zahra]
1/8	1.7795E-08	3.65E-2
1/16	9.8306E-11	1.02E-2
1/32	5.4307E-13	2.61E-3
1/64	3.0001E-15	6.56E-4
1/128	1.6573E-17	1.63E-4

Problem 3: [11] Consider the fractional boundary value problem

$$y^4(t) + D^{(\alpha)}y(t) + y(t) = g(t), \forall x \in [0, 1],$$
$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

Where, $g(t) = t(840t^2 - 120) + t^7\left(1 + \frac{7!t^{-\alpha}}{\Gamma(8-\alpha)}\right) - t^5\left(1 + \frac{5!t^{-\alpha}}{\Gamma(6-\alpha)}\right)$.

The exact solution is $y(t) = t^5(t^2 - 1)$.

The our numerical results compare to example 1 in ref.[11], where $\alpha = 0.4$, respect to the step size h, this comparison certifies that our method gives effect accuracy.

5. Conclusion

In this paper, we used quintic fractional spline model for solving fractional differential equations approximately. We obtain uniqueness fractional spline of class five with convergence analysis and the error bounds of the method. Our approach depends on the accuracy with good results, when are compared to [11], some numerical examples successfully verify the theoretical results and show that the given method is efficient.

6. References

[1] D. Kincaid and W. Cheney, Numerical Analysis, Mathematics of Scientific Computing, Third Edition, and Wadsworth group Brooks/Cole, (2002).

[2] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, (1999).

[3] J. Sabatier, O. P. Agrawal and M. J. Tenreiro, Advance in fractional calculus: Theoretical Developments and Applications in Physics and Engineering, (2007).

[4] K. H. Faraidun and O. M. Pshtiwan, Numerical solution of fractional differential equations by using fractional spline model; Journal of Information and Computing Science, 10(2)(2015) 098-105.

[5] K. H. Faraidun and O. M. Pshtiwan, Generalized quartic fractional spline interpolation with applications, Int. J. Open Problems Compt. Math., 8(1)(2015)67-80.

[6] K. Diethelm, N. J. Fordm and A. D Freed,., Yu. Luchko, Algorithms for the fractional calculus: A selection of Numerical Method, Computer methods in Applied Mechanics and Engineering, Science Direct Journal, 194(6-8)(2005) pp. 743-773.

[7] P. Arvet , T. Enn, On the convergence of spline collocation methods for solving fractional differential equations, Journal of Computational and Applied Mathematics 235(2011) 3502–3514.

[8] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional integral and derivatives. Theory and Applications, Gordon and Breach Science Publishers, Yverdon, (1993).

[9] Thomas, R. Lucas, Error bound for interpolating cubic splines under various end conditions, Siam Journal numerical Analysis, 11(3)(1974)569-584.

[10] U. Michael and B. Thierry, Fractional Splines and Wavelets, Society for industrial and applied mathematics, Siam Review, 42(1)(2000) 43-67.

[11] W. K. Zahra and S. M. Elkholy, Spline solution for fourth order fractional integro-differential equations, Journal of Fractional Calculus and Applications, 3(17)(2012)1–13,http://www.fcaj.webs.com/.

[12] Y. Yin, Solving a Nonlinear Multi-Order Fractional Differential Equation Using Legendre Pseudo-Spectral Method, Applied Mathematics, 4(2004)113-118, http://dx.doi.org/10.4236/am.2013.41020.

[13] D. Usero, Fractional Taylor Series for Caputo Fractional Derivatives. Construction of Numerical Schemes, Dpto. de Matem_atica Aplicada, Universidad Complutense de Madrid, Spain, 2008.

[14] M. Ishteva, Properties and applications of the Caputo fractional operator, Msc. Thesis, Dept. of Math., Universit•at Karlsruhe (TH), So_a, Bulgaria, (2005).

[15] T. Fawzy, (0, 1, 3) Lacunary Interpolation by G-splines, Annales Univ. Sci. Budapest, Section Mathematics, vol. XXIX (1989) 63-67.

[16] A. Saxena, (0, 1, 2, 4) Interpolation by G-splines, Acta Math. Hung. 51 (1988) 261-271.