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Haar Wavelet Method for the Numerical Solution of Benjamin–Bona–Mahony Equations

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Abstract. In this paper, we proposed an efficient numerical method based on uniform Haar wavelet for the numerical solutions oflinear and nonlinear Benjamin–Bona–Mahony (BBM) Equations. Such types of problems arise in various fields of science and engineering. In present study more accurate solutions have been obtained by Haar wavelet decomposition with multiresolution analysis. Three test problems are considered to check theefficiency and accuracy of the proposed method.An extensiveamount of error analysis has been carried out to obtain the convergence of the method.The numerical results are found in good agreement with exact and finite difference method (FDM), which shows that the solution using Haar wavelet method (HWM) is more effective and accurate and manageable for these equations.

Keywords: Haar wavelet method;Benjamin–Bona–MahonyEquation;Finitedifference method;Numerical simulation; Error analysis.

1. Introduction

Most scientific problems arise in real-world physical problems such as plasma physics, fluid mechanics, solid state physics and in many branches of chemistry [1]. We know that except a limited number of these problems, most of them do not have analytical solutions. The importance of obtaining the approximate solutions of nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover approximate solutions. Therefore, these nonlinear equations should be solved using numerical methods i.e. variational iterationmethod (VIM) [4] and homotopy-perturbation method (HPM)[5]. The BBM equation was introduced by Benjamin-Bona-Mahony, as an improvement of the Ko-rtewegde Vries equation (KdV equation). It describes the model for propagation of long waves which incorporates nonlinear and dissipative effects. It is used in the analysis of the surface waves of long wavelength in liquids, hydro magnetic waves in cold plasma, acoustic-gravity waves in compressible fluids, and acoustic waves in harmonic crystals. Many mathematicians paid their attention to the dynamics of the BBM equation.The main mathematical difference between KdV and BBM models can be most readily appreciated by comparing the dispersion relation for the respective linearized equations. It can be easily seen that these relations are comparable only for small wave numbers and they generate drastically different responses to short waves. Recently, various authors have been proposed different methods to solving the different type of BBM equations [6, 13].

In numerical analysis, Wavelets are used as appropriate tools at various places to provide good mathematical model for scientific phenomena, which are usually modeled through linear or nonlinear differential equations.Haar wavelet method is one of them because of Haar functions appearing very attractive in many applications.The previous work in wavelet analysis via Haar wavelets was led by Chen and Hsiao [2], who first derived a Haar operational matrix for the integrals of the Haar function and put the application for the Haar analysis into the dynamic systems. In order to take the advantages of the local property, many authors researched the Haar wavelet to solve the differential and integral equations [7-12].

The objective of the present work is to apply the Haar wavelet method(HWM) for the numerical solution of different types of Benjamin-Bona-Mahony equations and obtained results are compared with the classical FDM and exact solution. The present method is illustrated by some of the Benjamin-Bona-Mahony equations.

The present paper is organized as follows; in section 2, Haar wavelets and its generalized operational matrix of integration are given. Haar Wavelet Method for solving Benjamin-Bona-Mahony equationsis presented in section 3. Section 4 deals with the numerical Experiment, results and error analysis of the illustrative problems. Finally, conclusion of the proposed work is discussed in section 5.

2. Haar wavelets and Operational matrix of integration

We used the simplest wavelet function i.e Haar wavelet. We establish an operational matrix for integration via Haar wavelets.

The scaling function $h_1(x)$ for the family of the Haar wavelets is defined as

$$h_1(x) = \begin{cases} 1, & for \ x \in [0,1) \\ 0, & Otherwise \end{cases} \quad (2.1)$$

The Haar wavelet family for $x \in [0,1)$ is defined as

$$h_i(x) = \begin{cases} 1, & for \ x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1, & for \ x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0, & Otherwise \end{cases} \quad (2.2)$$

In the above definition the integer $m = 2^l$, $l = 1, 2, ..., J$, indicates the level of resolution and integer $k = 0, 1, 2, ..., m-1$ is the translation parameter. Maximum level of resolution is J. The index i in Eq. (2.2) is calculated using $i = m + k + 1$. In case of minimal values $m = 1,\ k = 0,$ then $i = 2$. The maximal value of $i$is $N = 2^{J+1}$.

Let us define the grid points $x_j = (j - 0.5)/N,\ J = 1, 2, ..., N,$ discretize the Haar function $h_i(x)$, in this way we get Haar coefficient matrix $H(i, j) = h_i(x_j)$ which has the dimension $N \times N$ .The operational matrix of integration is obtained by integrating (2.2) is as

$$Ph_i(x) = \int_0^x h_i(x)\,dx \quad (2.3)$$

$$Qh_i(x) = \int_0^x Ph_i(x)\,dx \quad (2.4)$$

and

$$Ch_i(x) = \int_0^1 Ph_i(x)\,dx \quad (2.5)$$

These integrals can be evaluated by using equation (2.2) and they are given by

$$Ph_i(x) = \begin{cases} x - \frac{k}{m}, & for \ x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ \frac{k+1}{m} - x, & for \ x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0, & Otherwise \end{cases} \quad (2.6)$$

$$Qh_i(x) = \begin{cases} \frac{1}{2}\left(x - \frac{k}{m}\right)^2, & \text{for } x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ \frac{1}{4m^2} - \frac{1}{2}\left(\frac{k+1}{m} - x\right)^2, & \text{for } x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ \frac{1}{4m^2}, & \text{for } x \in \left[\frac{k+1}{m}, 1\right) \\ 0, & \text{Otherwise} \end{cases} \tag{2.7}$$

and

$$Ch_i(x) = \begin{cases} \frac{1}{2}\left(1 - \frac{k}{m}\right)^2, & \text{for } x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ \frac{1}{4m^2} - \frac{1}{2}\left(\frac{k+1}{m} - 1\right)^2, & \text{for } x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ \frac{1}{4m^2}, & \text{for } x \in \left[\frac{k+1}{m}, 1\right) \\ 0, & \text{Otherwise} \end{cases} \tag{2.8}$$

For instance, $J = 2 \Rightarrow$ N = 8, then from (2.2), (2.6), (2.7) & (2.8) we have

$$H(8,8) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$Ph(8,8) = \frac{1}{16}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$Qh(8,8) = \frac{1}{512}\begin{pmatrix} 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 \\ 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 \\ 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

and

$$C\,h(8,8)=\frac{1}{64}\begin{pmatrix} 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \\ 32 & 32 & 32 & 32 & 16 & 16 & 16 & 16 \\ 32 & 32 & -4 & -4 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 8 & 8 & 4 & 4 \\ 32 & -17 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 18 & -7 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 8 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

3. Haar Wavelet Method for solving Benjamin-Bona-Mahony equations

Consider the general Benjamin-Bona-Mahony equation of the form:

$$u_t+\alpha u_x+\beta u u_x-u_{xxt}=\phi(x,t) \tag{3.1}$$

with the initial conditions

$$u(x,0)=f(x),\ 0\le x<1 \tag{3.2}$$

and the boundary condition

$$u(0,t)=g_0(t),\ u(1,t)=g_1(t) \quad t>0 \tag{3.3}$$

where $\phi(x,t),\ f(x),\ g_0(t),\ g_1(t)$ are functions of independent variables and α,β are constants.

Let us assume that

$$\dot{u}''(x,t)=\sum_{i=1}^{N}a_i h_i(x) \tag{3.4}$$

where $.$ *and* $'$ are differentiation w.r.t. t and x respectively& a_i's, $i=1,2,...,N$ are Haar coefficients to be determined.

Now integrating the equation(3.4)once w.r.t. t from t_s *to* t

$$u''(x,t)-u''(x,t_s)=(t-t_s)\sum_{i=1}^{N}a_i h_i(x)$$

$$\Rightarrow u''(x,t)=(t-t_s)\sum_{i=1}^{N}a_i h_i(x)+u''(x,t_s) \tag{3.5}$$

where t_s is the initial time and $\Delta t=t-t_s$ is the time interval.

Also integrating (3.5) w.r.t. x from 0 *to* x, we get

$$u'(x,t)-u'(0,t)=\Delta t\sum_{i=1}^{N}a_i Ph_i(x)+u'(x,t_s)-u'(0,t_s)$$

$$u'(x,t)=\Delta t\sum_{i=1}^{N}a_i Ph_i(x)+u'(x,t_s)+u'(0,t)-u'(0,t_s) \tag{3.6}$$

and again integrating equation (3.6) w.r.t. x

$$u(x,t)-u(0,t)=\Delta t\sum_{i=1}^{N}a_i Qh_i(x)+u(x,t_s)-u(0,t_s)+x(u'(0,t)-u'(0,t_s)) \tag{3.7}$$

$$u(x,t)=\Delta t\sum_{i=1}^{N}a_i Qh_i(x)+u(x,t_s)+u(0,t)-u(0,t_s)+x(u'(0,t)-u'(0,t_s)) \tag{3.8}$$

Put $x=1$ in (3.8) and by using equation (3.3) (i.e. boundary conditions), we get

$$u\,(1,t)\ =\ \Delta t\sum_{i=1}^{N} a_i Ch_i(x) + u(1,t_s) + u\,(0,t) - u(0,t_s) + (u'(0,t) - u'(0,t_s))$$

$$\Rightarrow\ u'(0,t) - u'(0,t_s)\ =\ g_1(t) - \Delta t\sum_{i=1}^{N} a_i Ch_i(x) - g_1(t_s) - g_0(t) + g_0(t_s) \tag{3.9}$$

Substituting the equation (3.9) in (3.8), then the equation (3.8) becomes

$$u\,(x,t) = \Delta t\sum_{i=1}^{N} a_i Qh_i(x) + u(x,t_s) + g_0\,(t) - g_0\,(t_s) + x\left\{ g_1(t) - \Delta t\sum_{i=1}^{N} a_i Ch_i(x) - g_1(t_s) - g_0(t) + g_0(t_s) \right\} \tag{3.10}$$

Differentiating equation (3.10) w. r. t. t then we have

$$\dot{u}(x,t) = \sum_{i=1}^{N} a_i Qh_i(x) + \dot{g}_0\,(t) + x\left\{ \dot{g}_1\,(t) - \sum_{i=1}^{N} a_i Ch_i(x) - \dot{g}_0\,(t) \right\} \tag{3 .11}$$

Substituting equations (3.4), (3.6), (3.10) and (3.11)in equation(3.1) and by solving using Inexact Newton's method [10], we obtain the Haar wavelet coefficients $a_i\,'s$. Substituting these values of $a_i\,'s$ in (3.10), we get the HWM based numerical solution of the given problem (3.1). The error will be calculated by $L_\infty = \max\left|u(x,t)_e - u(x,t)_a\right|$, where $u(x,t)_e$ and $u(x,t)_a$ are exact and approximate solutions respectively.

4. Numerical Experiment

In this section, we apply the HWM discussed in section 3 to some of the BBM type of equations.

Problem 1. Consider the linear BBM equation of the form [3]

$$u_t - 2u_{xxt} + u_x = 0 \tag{4.1}$$

with initial condition:

$$u(x,0) = e^{-x}\ ,\ 0 \le x < 1 \tag{4.2}$$

and boundary conditions:

$$u(0,\,t) = e^{-t},\quad u(1,\,t) = e^{-1-t}\quad t > 0 \tag{4.3}$$

Let

$$\dot{u}'' = \sum_{i=1}^{N} a_i h_i(x) \tag{4.4}$$

Using (4.2)& (4.3), then the equations (3.4), (3.6), (3.10) and (3.11) becomes

$$\dot{u}''(x,t) = \sum_{i=1}^{N} a_i h_i(x) \tag{4.5}$$

$$u'(x,t)\ =\ \Delta t\sum_{i=1}^{N} a_i Ph_i(x) + e^{-x} - e^{-t} + 1 \tag{4.6}$$

$$u\,(x,t) = \Delta t\sum_{i=1}^{N} a_i Qh_i(x) + e^{-x} + e^{-t} - 1 + x\left\{ e^{-1-t} - \Delta t\sum_{i=1}^{N} a_i Ch_i(x) - e^{-1} - e^{-t} + 1 \right\} \tag{4.7}$$

$$\dot{u}(x,t) = \sum_{i=1}^{N} a_i Qh_i(x) - e^{-t} + x\left\{ -e^{-1-t} - \sum_{i=1}^{N} a_i Ch_i(x) + e^{-t} \right\} \tag{4.8}$$

Substituting (4.5), (4.6) and (4.8) in (4.1), we get

$$\sum_{i=1}^{N} a_i Qh_i(x) - e^{-t} + x\left\{-e^{-1-t} - \sum_{i=1}^{N} a_i Ch_i(x) + e^{-t}\right\} - 2\sum_{i=1}^{N} a_i h_i(x) + \left[\Delta t \sum_{i=1}^{N} a_i Ph_i(x) + e^{-x} - e^{-t} + 1\right] = 0 \quad (4.9)$$

By solving (4.9), we get the Haar wavelet coefficients a_i's using Inexact Newton's method [10]. i.e.[-0.55, -0.14, -0.09, -0.05, -0.05, -0.04, -0.03, -0.02]. Substituting these a_i's in (4.7), we obtain the HWM based numerical solution of the equation (4.1) and is compared with the FDM and exact solution $u(x,t) = e^{-x-t}$ in Table 1 for *N= 8* and Fig. 1 for N=32. The error analysis for higher values of N is given in Table 2 with $\Delta t = 1/N$.

Problem 2. Next consider the linear non-homogenous BBM equation of the form [14]

$$u_t - 2u_{xxt} = -e^{x+t} \quad (4.10)$$

With initial condition:

$$u(x,0) = e^x \;,\; 0 \le x < 1 \quad (4.11)$$

and boundary conditions:

$$u(0,\, t) = e^t, \quad u(1,\, t) = e^{1+t} \;, \quad t > 0 \quad (4.12)$$

Using (4.11)& (4.12), then the equations (3.4), (3.10) and (3.11) becomes

$$\dot{u}''(x,t) = \sum_{i=1}^{N} a_i h_i(x) \quad (4.13)$$

$$u\,(x,t) = \Delta t \sum_{i=1}^{N} a_i Qh_i(x) + e^x + e^t - 1 + x\left\{e^{1+t} - \Delta t \sum_{i=1}^{N} a_i Ch_i(x) - e^1 - e^t + 1\right\} \quad (4.14)$$

$$\dot{u}(x,t) = \sum_{i=1}^{N} a_i Qh_i(x) + e^t + x\left\{e^{1+t} - \sum_{i=1}^{N} a_i Ch_i(x) - e^t\right\} \quad (4.15)$$

Substituting (4.13and (4.15) in (4.10), we get

$$\sum_{i=1}^{N} a_i Qh_i(x) + e^t + x\left\{e^{1+t} - \sum_{i=1}^{N} a_i Ch_i(x) - e^t\right\} - 2\sum_{i=1}^{N} a_i h_i(x) + e^{x+t} = 0 \quad (4.16)$$

Equation (4.16) can be solved, we get the Haar wavelet coefficients a_i's using Inexact Newton's method [10].We obtain Haar coefficients a_i's i.e. [1.95, -0.47, -0.18, -0.30,-0.08, -0.10, -0.13, -0.17]. Substituting these a_i's in (4.14), we get the HWM based numerical solution of the equation (4.10) and is compared with theFDM &exact solution $u(x,t) = e^{x+t}$ in Table 3 for*N= 8* and Fig. 2 for *N=32*. The error analysis for higher values of *N* is given in Table 4 with $\Delta t = 1/N$.

Problem3. Nowconsider the non-linear BBM equation[3]

$$u_t - u_{xxt} + uu_x = 0 \quad (4.17)$$

with initial condition:

$$u(x,0) = x \;,\; 0 \le x < 1 \quad (4.18)$$

and boundary conditions:

$$u(0,\, t) = 0, \quad u(1,\, t) = \frac{1}{1+t} \;, \quad t > 0 \quad (4.19)$$

Using the conditions (4.18) & (4.19), the equation (4.17) can be solved as explained in Section (3) and we get Haar coefficients a_i's i.e. [0.05, -0.03, -0.01, -0.02, -0.01,-0.01, -0.01,-0.01]. Substituting thesevalues of $a_i\text{'s}$ in (3.10), we obtain the HWCM based numerical solution of the equation (4.17) and is compared with the FDM &exact solution $u(x,t)= x/(1+t)$ in Table 5 for N= 8 and Fig. 3 for N=32. The error analysis for higher values of N is given in Table 6 with $\Delta t = 1/N$.

Table 1: Comparison of FDM and HWM with Exact solutions for N=8 of the Problem 1.

x=(/16)	FDM	HWM	Exact
1	0.82942254	0.82957675	0.82902911
3	0.73277957	0.73244494	0.73161562
5	0.64756878	0.64621771	0.64564852
7	0.57245382	0.56995069	0.56978282
9	0.50625604	0.50267060	0.50283157
11	0.44793598	0.44342497	0.44374731
13	0.39657711	0.39130681	0.39160562
15	0.35137142	0.34546659	0.34559075

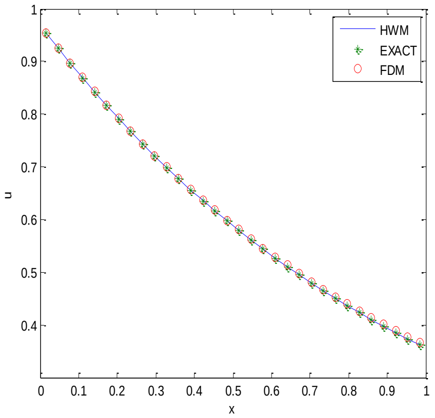

Fig. 1. Comparison of HWM and FDM with Exact solutions for *N=32* of the Problem 1.

T**able 2**. Error analysis of the Problem 1.

N	L_∞ (FDM)	L_∞ (HWM)
8	5.7806E-03	8.2932E-04
16	5.4363E-03	5.4368E-04
32	3.4051E-03	3.1094E-04
64	1.8811E-03	1.6478E-04
128	9.8606E-04	8.4782E-05
256	5.0452E-04	4.2993E-05

Table 3. Comparison of FDM and HWM with Exact solutions for N=8 of the Problem 2.

x=(/16)	FDM	HWM	Exact
1	1.15741935	1.20750077	1.20623024
3	1.35750043	1.36926498	1.36683794
5	1.57881432	1.55109039	1.54883029
7	1.82434828	1.75639170	1.75505465
9	2.09744632	1.98892088	1.98873746
11	2.40185758	2.25280954	2.25353478

13	2.74179096	2.55257492	2.55358945
15	3.12197659	2.89308458	2.89359594

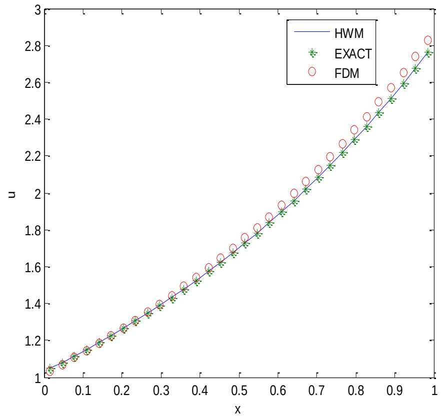

Fig. 2. Comparison of HWM and FDM with Exact solutions for N=32 of the Problem 2.

T**able 4.** Error analysis of the Problem 2.

N	L_∞ (FDM)	L_∞ (HWM)
8	2.2838E-01	2.4270E-03
16	1.2617E-01	1.4431E-03
32	6.6174E-02	7.7915E-04
64	3.3869E-02	4.0254E-04
128	1.7132E-02	2.0450E-04
256	8.6155E-03	1.0305E-04

Table 5. Comparison of FDM and HWM with Exact solutions for N=8 of the Problem 3.

x=(/16)	FDM	HWM	Exact
1	0.05887280	0.05553306	0.05555555
3	0.17667011	0.16650168	0.16666666
5	0.29462366	0.27740622	0.27777777
7	0.41284152	0.38831388	0.38888888
9	0.53143772	0.49928377	0.50000000
11	0.65053500	0.61036975	0.61111111
13	0.77026763	0.72162043	0.72222222
15	0.89078469	0.83307583	0.83333333

T**able 6.** Error analysis of the Problem 3

N	L_∞ (FDM)	L_∞ (HWM)
8	5.7451E-02	7.4135E-04
16	2.9979E-02	2.1049E-04
32	1.5306E-02	5.6012E-05
64	7.7325E-03	1.4456E-05
128	3.8862E-03	3.6730E-06
256	1.9481E-03	9.2572E-07

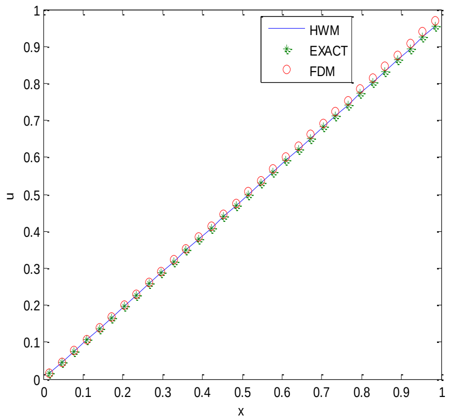

Fig. 3. Comparison of HWM and FDM with Exact solutions for N=32 of the Problem 3.

5. Conclusion

In the present study, numerical solution of Benjamin-Bona-Mahoney (BBM) equations is discussed using Haar wavelets method. The proposed method is computationally efficient and the algorithm can be easily implemented on computer, which has been justified through the illustrative problems. The numerical solutions are presented in Tables and figures, from which we observed that Haar wavelet gives better results with less computational cost: it is due to the sparsity of the transform matrix and small number of wavelet coefficients subsequently error analysis is presented, which shows that the accuracy of the solution is increased by increasing the number of grid points (i.e. N).Hence the proposed method is very effective and easy to implement for solving linear as well as non-linear Benjamin-Bona-Mahoney (BBM) equations.

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