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Bernoulli Wavelet Based Numerical Method for Solving Fredholm Integral Equations of the Second Kind

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Abstract. In this paper, a Bernoulli wavelet based numerical method for the solution of Fredholm integral equations of the second kind is proposed. The method is based upon Bernoulli wavelet approximations. The Bernoulli wavelet (BW) is first presented and the resulting Bernoulli wavelet matrices are utilized to reduce the Fredholm integral equations into algebraic equations. Solving these equations using MATLAB to obtain Bernoulli coefficients. The numerical results of the proposed method through the illustrative examples is presented in comparison with the exact and existing methods (Haar wavelet method (HWM) [13], Hermite cubic splines (HCS) [11]) of solution from the literature are shown in tables and figures, which show that the validity and applicability of the technique with higher accuracy even for the smaller values of N.

Keywords: Bernoulli Wavelet, Haar wavelet, Hermite cubic splines, Bernoulli Polynomials, Bernoulli numbers, Fredholm Integral equations.

1. Introduction

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms. Since from 1991 the various types of wavelet method have been applied for numerical solution of different kinds of integral equation, a detailed survey on these papers can be found in [1-6].

Consider the Fredholm integral equation of the second kind:

$$y(t) = f(t) + \int_0^1 K(t,s)\, y(s) ds \quad 0 \le t, s \le 1, \tag{1}$$

where $f(t)$ and the kernels $k(t,s)$ are assumed to be in $L^2(R)$ on the interval $0 \le t, s \le 1.$ We assume that Eq.(1) has a unique solution *y* to be determined. Integral equations find its applications in various fields of science and engineering. There are several numerical methods for approximating the solution of Fredholm integral equations are known and many different basic functions have been used. Such as Galerkin methods for the constructions of orthonormal wavelet bases approached by Liang et.al [7], Maleknejad et al. [8] used the continuous Legendre wavelets, a combination of Hybrid taylor and block-pulse functions [9], Rationalized haar wavelet [10], Hermite Cubic splines [11], Coifman wavelet as scaling functions [12]. Lepik et al. [13] applied the Haar Wavelets, Yousefi et al. [14] have introduced a new CAS wavelet, Babolian et al. [15] derived the operational matrix for the product of two triangular orthogonal functions, Muthuvalu et al. [16] applied Half-sweep arithmetic mean method with composite trapezoidal scheme for the solution of Fredholm integral equations. Keshavarz et al. [17] applied Bernoulli wavelet operational matrix for the approximate solution of fractional order differential equations. In this paper, we introduced the numerical method based on Bernoulli wavelets approximations for solving Fredholm integral equations.

The article is organized as follows: In Section 2, the basic formulation of Bernoulli wavelets and the function approximation is presented. Section 3 is devoted to the method of solution. In section 4, we report our numerical findings and demonstrated the accuracy of the proposed scheme by considering illustrative examples. Conclusion of the proposed method is discussed in section 5.

2. Bernoulli Wavelets and Function Approximation

Bernoulli wavelets are $b_{n,m}(t) = b(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = n-1, n = 1, 2, 3, ..., 2^{k-1}$, k is any positive integer, m is the order of Bernoulli polynomials and t is the normalized time. Then it can be defined [17] on the interval [0, 1) as follows,

$$\mathrm{b}_{m.n}(\mathrm{t}) = \begin{cases} 2^{\frac{k-1}{2}}\tilde{\beta}_m(2^{k-1}t-\hat{n}), \frac{n}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0, else \end{cases} \tag{2}$$

with

$$\tilde{\beta}_m(t) = \begin{cases} 1, & m = 0, \\ \dfrac{1}{\sqrt{\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}\beta_m(t), & m > 0, \end{cases}$$

where m = 0,1,2,...,M-1 and n = 1,2,...,2^{k-1}. The coefficient $\dfrac{1}{\sqrt{\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for normality, $2^{-(k-1)}$ is the dilation parameter, $\hat{n}2^{-(k-1)}$ is the translation parameter and $\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i$ are the well-known Bernoulli polynomials of order m. Where $\alpha_i, i = 0,1,...,m$ are Bernoulli numbers. These numbers are a sequence of signed rational numbers which arise in the series expansion of trigonometric functions and can be defined by the identity,

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The first few Bernoulli numbers are

$$\alpha_0 = 1, \alpha_1 = \frac{-1}{2}, \alpha_2 = \frac{1}{6}, \alpha_4 = \frac{-1}{30}, \alpha_6 = \frac{1}{42}, \alpha_8 = \frac{-1}{30}, \alpha_{10} = \frac{5}{66}, \ldots$$

With $\alpha_{2i+1} = 0, i = 1, 2, 3, \ldots$

The first few Bernoulli Polynomials are

$$\beta_0(t) = 1,\ \beta_1(t) = t - \frac{1}{2},\ \beta_2(t) = t^2 - t + \frac{1}{6},$$

$$\beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \beta_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30},$$

$$\beta_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t,$$

$$\beta_6(t) = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2 + \frac{1}{42}, \ldots$$

The six basis functions are given by:

$$\left.\begin{aligned} b_{10}(t) &= \sqrt{2} \\ b_{11}(t) &= \sqrt{6}\,(4t-1) \\ b_{12}(t) &= \sqrt{10}\,(24t^2 - 12t + 1) \end{aligned}\right\}; 0 \le t < \frac{1}{2},$$

$$\left.\begin{aligned} b_{20}(t) &= \sqrt{2} \\ b_{21}(t) &= \sqrt{6}\,(4t-3) \\ b_{22}(t) &= \sqrt{10}\,(24t^2 - 36t + 13) \end{aligned}\right\}; \frac{1}{2} \le t < 1$$

For k = 2 implies n = 1, 2 and M=3 implies m = 0, 1, 2 then Eq.(2) gives the Bernoulli wavelet matrix of order $(N = 2^{k-1}M)$ 6x6 as,

$$B_{6\times 6} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ -1.6330 & 0 & 1.6330 & 0 & 0 & 0 \\ 0.5270 & -1.5811 & 0.5270 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & -1.6330 & 0 & 1.6330 \\ 0 & 0 & 0 & 0.5270 & -1.5811 & 0.5270 \end{bmatrix}$$

For k = 2 and M = 4 of order 8x8 as,

$$B_{8\times 8} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 & 0 \\ -1.8371 & -0.6124 & 0.6124 & 1.8371 & 0 & 0 & 0 & 0 \\ 1.0870 & -1.2847 & -1.2847 & 1.0870 & 0 & 0 & 0 & 0 \\ 1.6811 & 1.2008 & -1.2008 & -1.6811 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0 & -1.8371 & -0.6124 & 0.6124 & 1.8371 \\ 0 & 0 & 0 & 0 & 1.0870 & -1.2847 & -1.2847 & 1.0870 \\ 0 & 0 & 0 & 0 & 1.6811 & 1.2008 & -1.2008 & -1.6811 \end{bmatrix}$$

A function $f(x) \in L^2[0,1]$ may be expanded as:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m} b_{n,m}(t), \tag{3}$$

where

$$c_{n,m} = (f(t), b_{n,m}(t)). \tag{4}$$

In (4), (. , .) denotes the inner product.

If the infinite series in (3) is truncated, then (3) can be rewritten as:

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m} b_{n,m}(t) = C^T B(t), \tag{5}$$

where C and $B(t)$ are $N \times 1$ matrices given by:

$$\begin{aligned} C &= [c_{10}, c_{11}, ..., c_{1,M-1}, c_{20}, ..., c_{2,M-1}, ..., c_{2^{k-1},0}, ..., c_{2^{k-1},M-1}]^T \\ &= [c_1, c_2, ..., c_{2^{k-1}M}]^T, \end{aligned} \tag{6}$$

and

$$\begin{aligned} B(t) &= [b_{10}(t), b_{11}(t), ..., b_{1,M-1}(t), b_{20}(t), ..., b_{2,M-1}(t), ..., b_{2^{k-1},0}(t), ..., b_{2^{k-1},M-1}(t)]^T \\ &= [b_1(t), b_2(t), ..., b_{2^{k-1}M}(t)]^T. \end{aligned} \tag{7}$$

Similarly a function $k(t,s) \in L^2([0,1]\times[0,1])$ may be approximated as:

$$k(t,s) \approx B^T(t) K B(s), \tag{8}$$

where K is $N \times N$ matrix, with $K_{ij} = (b_i(t), (k(t,s), b_j(s)))$.

The integration of the product of two Bernoulli wavelet function vectors is obtained as:

$$I = \int_0^1 B(t) B^T(t) dt, \tag{9}$$

where I is an identity matrix.

And also, we approximate

$$y(t) = Y^T B(t), \tag{10}$$

where, Y is a vector of Bernoulli Coefficients and $y(t)$ is the unknown function.

3. Method of solution

In this section, we present a Bernoulli wavelet method (BWM) for solving Fredholm integral equation of the second kind of the form (1),

$$y(t) = f(t) + \int_0^1 k(t,s)\, y(s)\, ds,$$

where $f(t) \in L^2[0,1),\ k(t,s) \in L^2([0,1) \times [0,1))$ and $y(t)$ is an unknown function.

Let us approximate $f(t)$, $y(t)$, and $k(t,s)$ by using the procedure as follows:

STEP 1: $f(t) \approx C^T B(t),\ y(t) = Y^T B(t)$

STEP 2: $k(t,s) \approx B^T(t) K B(s)$

STEP 3: Substituting $f(t)$ and $k(t,s)$ in Eq.(1), we have:

$$B^T(t)Y = B^T(t)C + \int_0^1 B^T(t) K B(s) B^T(s) Y ds$$

$$B^T(t)Y = B^T(t)C + B^T(t)K\left(\int_0^1 B(s)B^T(s)\, ds\right)Y$$

$$B^T(t)Y = B^T(t)(C + KY),$$

$$then$$

$$(I - K)Y = C.$$

By solving this system we obtain the vector Y and substituting this Y in step 4.

STEP 4: $y(t) \approx Y^T B(t)$

which is the required approximate solution of Eq. (1).

4. Illustrative Examples

In this section we consider the some of the examples to demonstrate the capability of the method and error function is presented to verify the accuracy and efficiency of the following numerical results:

$$Error\ \ function = \|y_e(t_i) - y_a(t_i)\|_2 = \sqrt{\sum_{i=1}^{n}\left(y_e(t_i) - y_a(t_i)\right)^2}$$

where y_e and y_a are the exact and approximate solution respectively.

Example 1. Let us consider the Fredholm integral equation of the second kind.

$$y(t) = t^2 + \int_0^1 (t+s)\ \ y(s)\ \ ds \tag{11}$$

The solution $y(t)$ of (11) with the help of Bernoulli coefficients Y as [-1.71875, -0.17140, 0.00232, 2.56e-16, -2.28125, -0.15335, 0.00232, 3.02e-16, -2.78125, -0.13531, 0.00232, 3.50e-16, -3.21875, -0.11727, 0.00232, 4.21e-16] is obtained using the method described in section 3. The computational results for *k* = 3, M = 4 (N = 16) are compared with the exact solution $y(t) = t^2 - 5t - 17/6$ and HWM [13] are given in table 1 & for *k* = 4, M = 4 (N = 32) in Fig. 1. Error analysis is shown in table 2.

Table 1. Comparison of BWM and HWM solutions with exact solution and errors for N=16.

t	Exact	BWM	HWM	Error(BWM)	Error(HWM)
0.03125	-2.98861	-2.98861	-2.99453	1.24 e-14	0.00592
0.09375	-3.29329	-3.29329	-3.29995	1.38 e-14	0.00666
0.15625	-3.59017	-3.59017	-3.59756	1.55 e-14	0.00739
0.21875	-3.87923	-3.87923	-3.88736	1.60 e-14	0.00813
0.28125	-4.16048	-4.16048	-4.16935	1.60 e-14	0.00886
0.34375	-4.43392	-4.43392	-4.44352	1.78 e-14	0.00960
0.40625	-4.69954	-4.69954	-4.70988	1.95 e-14	0.01033
0.46875	-4.95736	-4.95736	-4.96843	1.95 e-14	0.01107
0.53125	-5.20736	-5.20736	-5.21916	2.13 e-14	0.01180
0.59375	-5.44954	-5.44954	-5.46209	2.22 e-14	0.01254
0.65625	-5.68392	-5.68392	-5.69722	2.40 e-14	0.01327
0.71875	-5.91048	-5.91048	-5.92449	2.31 e-14	0.01401
0.78125	-6.12923	-6.12923	-6.14398	2.75 e-14	0.01474
0.84375	-6.34017	-6.34017	-6.35565	2.84 e-14	0.01548
0.90625	-6.54329	-6.54329	-6.55951	3.02 e-14	0.01621
0.96875	-6.73861	-6.73861	-6.75556	2.93 e-14	0.01695

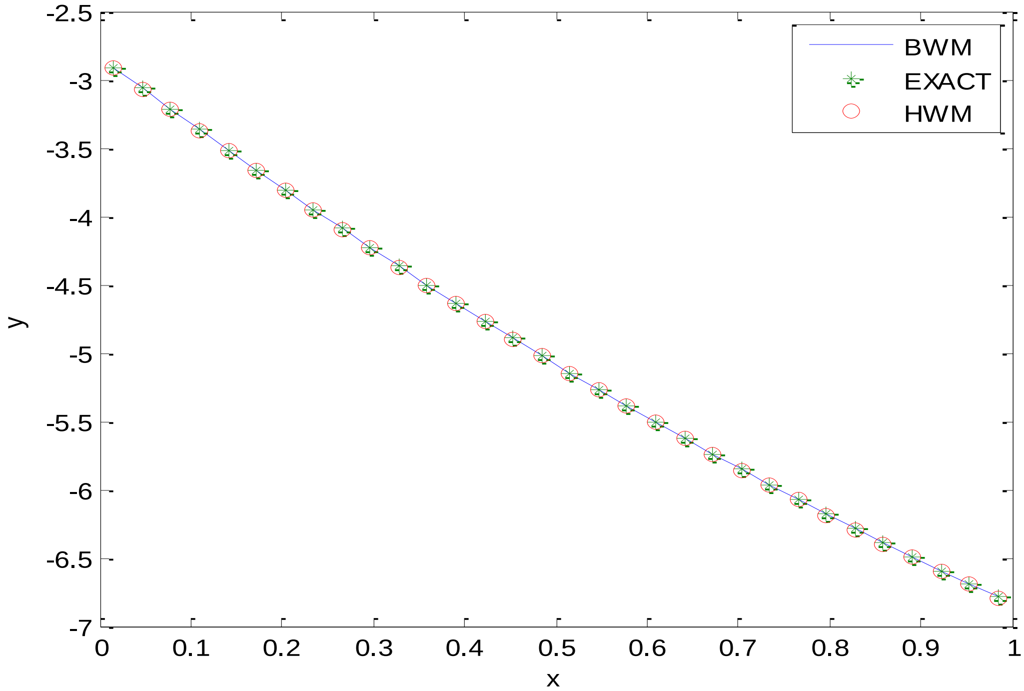

Fig. 1. Comparison of BWM and HWM solutions with exact solution for N=32.

Table 2. Error analysis of example 1.

N	$L_2(BWM)$	$L_2(HWM)$
8	2.0428 e-014	6.7150 e-002
16	3.0198 e-014	1.6953 e-002
32	4.6185 e-014	4.2716 e-003
64	1.9539 e-014	1.0728 e-003
128	4.5297 e-014	2.6887 e-004

Example 2. Next, consider

$$y(t) = \sin(2\pi t) + \int_0^1 \cos(t)\, y(s)\, ds \tag{12}$$

which has the exact solution $y(t) = \sin(2\pi t)$. We applied the Bernoulli wavelets approach and solved Eq. (12). Table 3 for k = 3 and M = 4 (N = 16) presents values of $y(t)$ with the help of the Bernoulli coefficients Y as [0.31816, 0.14413, -0.03147, -0.00772, 0.31816, -0.14413, -0.03147, 0.00772, -0.31816, -0.14413, 0.03147, 0.00772, -0.31816, 0.14413, 0.03147, -0.00772] is obtained using the present method together with the exact values and presented in Fig. 2 for k = 3 and M = 4 (N = 32). Error analysis is compared with method [11] are shown in table 4.

Table 3. Comparison of BWM solutions with exact solutions for N=16.

t	Example 2		Example 3		Example 4	
	BWM	Exact	BWM	Exact	BWM	Exact
0.03125	0.19509	0.19509	0.19509	0.19509	-0.02838	-0.02838
0.09375	0.55557	0.55557	0.55557	0.55557	-0.06903	-0.06903
0.15625	0.83147	0.83147	0.83147	0.83147	-0.09064	-0.09064
0.21875	0.98078	0.98078	0.98078	0.98078	-0.09613	-0.09613
0.28125	0.98078	0.98078	0.98078	0.98078	-0.08844	-0.08844
0.34375	0.83147	0.83147	0.83147	0.83147	-0.07050	-0.07050
0.40625	0.55557	0.55557	0.55557	0.55557	-0.04523	-0.04523
0.46875	0.19509	0.19509	0.19509	0.19509	-0.01556	-0.01556
0.53125	-0.19509	-0.19509	-0.19509	-0.19509	0.01556	0.01556
0.59375	-0.55557	-0.55557	-0.55557	-0.55557	0.04522	0.04522
0.65625	-0.83147	-0.83147	-0.83147	-0.83147	0.07049	0.07049
0.71875	-0.98079	-0.98079	-0.98079	-0.98079	0.08844	0.08844
0.78125	-0.98079	-0.98079	-0.98079	-0.98079	0.09613	0.09613
0.84375	-0.83147	-0.83147	-0.83147	-0.83147	0.09063	0.09063
0.90625	-0.55557	-0.55557	-0.55557	-0.55557	0.06903	0.06903
0.96875	-0.19509	-0.19509	-0.19509	-0.19509	0.02838	0.02838

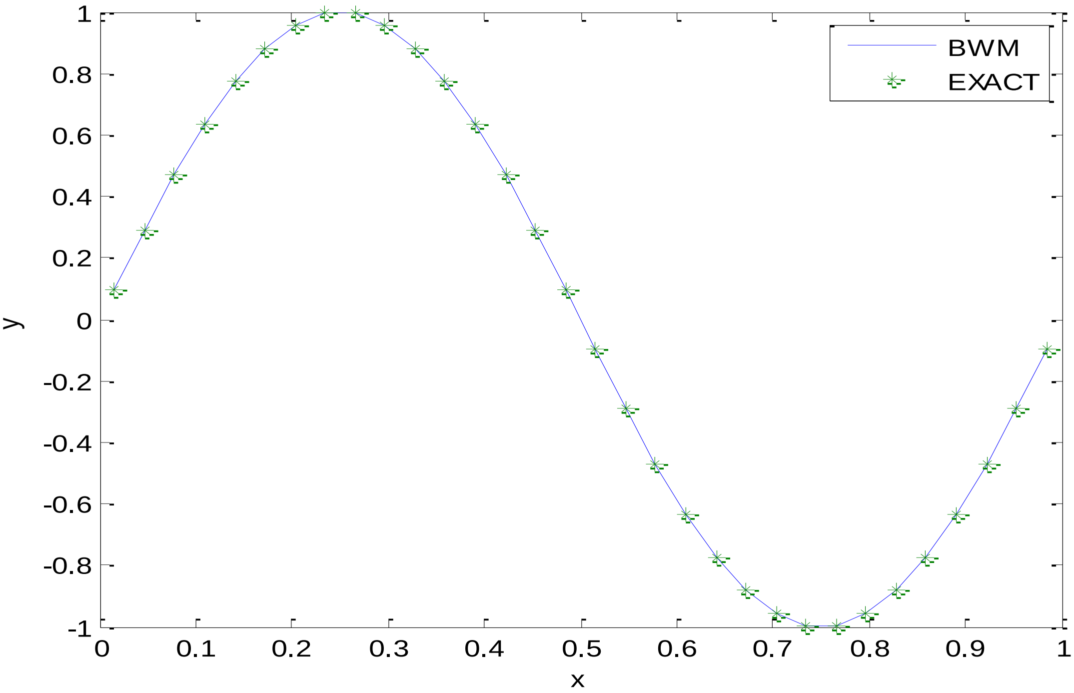

Fig. 2. Comparision of BWM with exact solution for N=32.

Example 3.Thirdly, consider

$$y(t) = \sin(2\pi t) + \int_0^1 (t^2 - t - s^2 + s)\, y(s)\, ds \tag{13}$$

has the exact solution $y(t)=\sin(2\pi t)$. By using the method presented in section 3 and solved Eq. (13). The solution $y(t)$ with the Bernoulli coefficients Y as [0.31816, 0.14413, -0.03147, -0.00772, 0.31816, -0.14413, -0.03147, 0.00772, -0.31816, -0.14413, 0.03147, 0.00772, -0.31816, 0.14413, 0.03147, -0.00772] is obtained and compared with the exact solution and method [11] given in table 3 for *k* = 3 and M = 4 (N = 16) and Fig. 3 for *k* = 3 and M = 4 (N = 32). Error analysis is shown in table 4.

Example 4. Finally, consider

$$y(t)=-2t^3+3t^2-t+\int_0^1 (t^2-t-s^2+s)\,y(s)\,ds \tag{14}$$

The approximate solution $y(t)$ of (14) with the help of Bernoulli coefficients Y as [-0.03515, -0.01353, 0.00524, -0.00053, -0.02734, 0.01353, 0.00174, -0.00053, 0.02734, 0.01353, -0.00174, -0.00053, 0.03515, -0.01353, -0.00524, -0.00053] is obtained using the proposed method. The numerical values are compared with the exact solution $y(t)=-2t^3+3t^2-t$ and method [11] are shown in table 3 for *k* = 3 and M = 4 (N = 16) and in Fig. 4 for *k* = 3 and M = 4 (N = 32). Error analysis is presented in table 4.

Table 4. Error analysis of BWM with the method [11].

N	Example 2		Example 3		Example 4	
	$L_2(BWM)$	$L_2(Method[11])$	$L_2(BWM)$	$L_2(Method[11])$	$L_2(BWM)$	$L_2(Method[11])$
4	6.6613 e-16	2.8414 e-02	1.1102 e-16	2.8414 e-02	0	1.3363 e-10
8	4.4408 e-16	2.3887 e-03	2.7755 e-16	2.3887 e-03	4.1633 e-17	3.7916 e-10
16	4.1633 e-16	2.0979 e-04	2.2204 e-16	2.1099 e-04	4.1633 e-17	3.2698 e-10
32	1.3322 e-15	1.2048 e-04	4.4408 e-16	2.0078 e-04	5.5511 e-17	4.8366 e-10

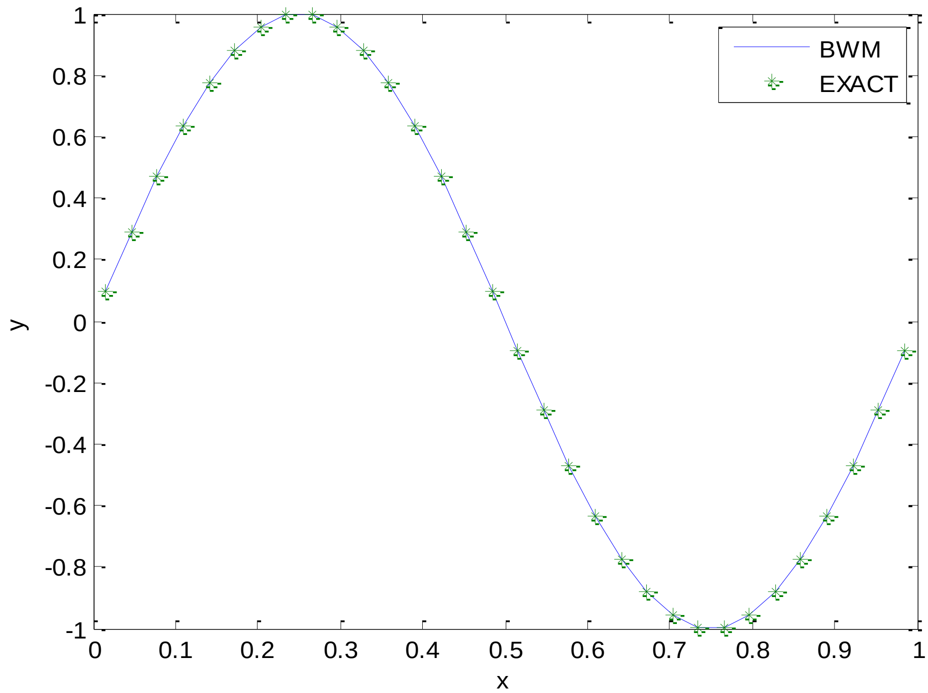

Fig. 3. Comparision of BWM with exact solution for N=32.

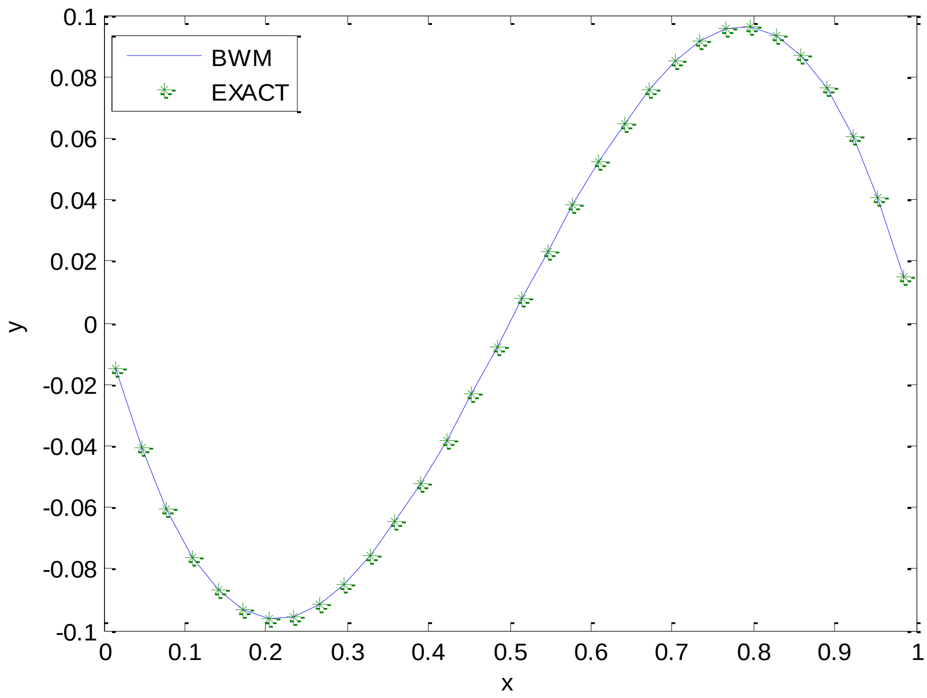

Fig. 4. Comparision of BWM with exact solution for N = 32.

5. Conclusion

The proposed work is based on Bernoulli wavelet matrix is applied for the numerical solution of Fredholm integral equations of the second kind. Our numerical findings are compared with the solutions obtained by existing methods [11, 13] and exact solutions. For instance in example 1, our results are higher accuracy with exact ones than the other method. Subsequently other examples are also same in the nature. Error analysis shows the accuracy and effectiveness of the described method. The efficiency of the present method is approached through the illustrative examples.

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