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Numerical Solution of Two-dimensional Nonlinear Volterra Integro-differential Equations by Tau Method

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Abstract. In this paper, a method is employed to approximate the solution of two-dimensional nonlinear Volterra integro-differential equations (2DNVIDEs) with supplementary conditions. First, we introduce two-dimensional Legendre polynomials, then convert 2DNVIDEs to the two-dimensional linear Volterra integro-differential equations (2DLVIDEs). Using this properties and collocation points, reduce it to the system of algebraic equations. Finally, some numerical examples are given to clarify the efficiency and accuracy of the present method.

Keywords: Two dimensional, Linear, Nonlinear, Volterra integro-differential equation, Tau method .

1. Introduction

Generally, real-world physical problems are modelled as differential, integral and integro-differential equations. Since finding the solution of these equations is too complicated, in recent years a lot of attention has been devoted by researchers to find the analytical and numerical solution of this equations. In [1] authors, have applied Legendre polynomials to solve two-dimensional Volterra integral equations. While in [2] Homotopy Perturbation and Differential transform methods have been chosen. Iterative methods have been used in [3] to solve two-dimensional nonlinear Volterra-Fredholm integro-differential equations. In [4] the authors, have applied the differential transform method for the system of two-dimensional nonlinear Volterra integro-differential equations .The Tau method has been developed for the numerical solution of two-dimensional linear Volterra integro-differential equations in [5] and Block-pulse functions have been used for solve two-dimensional Volterra integro-differential equations by the authors of [6].

On the other hand, 2DNVIDEs have interesting applications in Physics, Mechanics and applied sciences. For this reason, in this paper, we obtained numerical solution of two-dimensional nonlinear Volterra integro-differential equations with given supplementary conditions. To this end, we replace the differential and integral parts of 2DLVIDEs by Legendre polynomials and then convert it to a corresponding system of algebraic equations. In a similar manner, we transform the supplementary conditions to a algebraic system of equations. Finally, by combining these two systems of algebraic equations, we obtain a system of linear algebraic equations and solve it to obtain an approximate solution of the problem.

This paper is organized as follows. In Section 2, we describe properties of Legendre polynomials. In section 3, we explain numerical solution 2DLVIDEs by Tau method. Numerical examples are given in section 4 to evaluation of our method. Finally, conclusions are given in section 5.

2. Properties of Legendre polynomials

2.1. Definition the Legendre functions

The Legendre polynomials are defined as [7]:

$$L_0(x)=1,\qquad L_1(x)=x,\qquad x\in[-1,1],$$
$$L_i(x)=(2-\frac{1}{i})\,x\,L_{i-1}(x)-(1-\frac{1}{i})\,L_{i-2}(x)\qquad i=2,3,4,\ldots.$$

So the shifted Legendre polynomials are defined as:

$$L_0^*(x) = 1, \qquad L_1^*(x) = \frac{2x-l}{l}, \qquad x \in [0, l],$$

for $i \geq 2$ as

$$L_i^*(x) = (2-\frac{1}{i})(\frac{2x-l}{l})L_{i-1}^*(x) - (1-\frac{1}{i})L_{i-2}^*(x),$$

with the orthogonally condition as:

$$\int_0^l L_i(t)\, L_j(t)\, dt = \begin{cases} \dfrac{l}{2i+1}, & i = j, \\ 0, & otherwise. \end{cases}$$

2D shifted Legendre functions are defined on Ω ($(x,t) \in \Omega = [0, l] \times [0, T]$, l, T are finite constants) as[8]:

$$\psi_{mn}(x,t) = L_m(\frac{2}{l}x - 1)\, L_n(\frac{2}{T}t - 1), \qquad m, n = 0, 1, 2, 3, \dots. \tag{1}$$

here L_m, L_n are the well-known Legendre functions respectively of order m and n.

2D shifted Legendre functions $\psi_{mn}(x,t)$ are orthogonal with each other as:

$$\int_0^T \int_0^l \psi_{ij}(x,t)\, \psi_{mn}(x,t)\, dx\, dt = \begin{cases} \dfrac{lT}{(2m+1)(2n+1)}, & i = m, \ j = n, \\ 0, & otherwise. \end{cases}$$

Suppose that $X = L^2(\Omega)$, the inner product in this space is defined by

$$< f, g > = \int_0^T \int_0^l f(x,t)\, g(x,t)\, dx\, dt, \tag{2}$$

and the norm is as follows:

$$\| f \|_2 = < f, f >^{\frac{1}{2}}.$$

Let

$$\{\psi_{00}(x,t), \psi_{01}(x,t), ..., \psi_{0N}(x,t), \dots, \psi_{N0}(x,t), \dots, \psi_{NN}(x,t)\} \subset X,$$

be the set of 2D shifted Legendre functions and

$$X_{NN} = span\{\psi_{00}(x,t), \psi_{01}(x,t), \dots, \psi_{0N}(x,t), \dots, \psi_{N0}(x,t), \psi_{N1}(x,t), \dots, \psi_{NN}(x,t)\},$$

and $f(x,t)$ be an arbitrary function in X. Since X_{NN} is a finite dimensional vector space, f has a unique best approximation $f_{NN} \in X_{NN}$ [9], such that

$$\forall g \in X_{NN}, \| f - f_{NN} \|_2 \leq \| f - g \|_2.$$

Moreover since $f_{NN} \in X_{NN}$ there exist unique coefficients $f_{00}, f_{01}, \dots, f_{NN}$ such that

$$f(x,t) \cong f_{NN}(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} f_{ij}\, \psi_{ij}(x,t) = F^T \psi(x,t) = \psi^T(x,t) F,$$

where F and $\psi(x,t)$ are $(N+1)\times(N+1)$ vectors with the form

$$F = [f_{00}, \dots, f_{0N}, f_{10}, \dots, f_{1N}, \dots, f_{N0}, \dots, f_{NN}]^T,$$

$$\psi(x,t) = [\psi_{00}(x,t), \dots, \psi_{0N}(x,t), \psi_{10}(x,t), \dots, \psi_{1N}(x,t), \dots, \psi_{N0}(x,t), \dots, \psi_{NN}(x,t)]^T,$$

2D shifted Legendre function coefficients f_{NN} are obtained by

$$f_{mn} = \frac{\langle f(x,t), \psi_{mn}(x,t) \rangle}{\| \psi_{mn}(x,t) \|_2^2} = \frac{(2m+1)(2n+1)}{lT} \int_0^T \int_0^l f(x,t)\, \psi_{mn}(x,t)\, dx\, dt.$$

Similarly, any function $k(x,t,y,z) \in L^2(\Omega \times \Omega)$, can be expanded in terms of 2D shifted Legendre functions as

$$k(x,t,y,z) \cong \psi^T(x,t)\, K\, \psi(y,z), \tag{3}$$

where

$$K = [k_{ijmn}], \qquad i, j, m, n = 0,1,2, \dots, N,$$

in which

$$k_{ijmn} = \frac{<< K(x, t, y, z), \psi_{mn}(y, z) >, \psi_{ij}(x, t) >}{\left\| \psi_{ij}(x, t) \right\|_2^2 . \left\| \psi_{mn}(y, z) \right\|_2^2}.$$

2.2. Operational matrices of integration

The integration of the vector $\psi(x,t)$ is already defined above can be approximately obtained as [8]:

$$\int_0^t \int_0^x \psi(x', t') dx' dt' \cong Q_1 \psi(x,t) = (P_1 \otimes P_2)\psi(x,t), \tag{4}$$

Where $x \in [0,l], t \in [0,T]$ and Q_1 is the $(N+1)(N+1) \times (N+1)(N+1)$ operational matrix of integration, such that P_1, P_2 are the operational matrices of 1D shifted Legendre polynomials, respectively defined on $[0,l]$ and $[0,T]$ as follows [10]:

$$P_1 = \frac{l}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{2N-1} & 0 & \frac{1}{2N-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2N+1} & 0 \end{bmatrix},$$

$$P_2 = \frac{T}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{2N-1} & 0 & \frac{1}{2N-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2N+1} & 0 \end{bmatrix},$$

and $\otimes$ denotes the kronecker product defined for two arbitrary matrices A and B as [11]:

$$A \otimes B = (a_{ij} B).$$

Analogously, we write

$$\int_0^x \psi(x', t)\, dx' \cong Q_2\, \psi(x, t), \tag{5}$$

$$\int_0^t \psi(x, t')\, dt' \cong Q_3\, \psi(x, t), \tag{6}$$

where Q_2 and Q_3 are $(N+1)(N+1) \times (N+1)(N+1)$ matrices of the form as:

$$Q_2=\frac{l}{2}\begin{bmatrix} I & I & 0 & \cdots & 0 & 0 & 0 \\ -\frac{I}{3} & 0 & \frac{I}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-I}{2N-1} & 0 & \frac{I}{2N-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-I}{2N+1} & 0 \end{bmatrix},$$

$$Q_3=\begin{bmatrix} P_2 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ 0 & 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_2 \end{bmatrix},$$

where I and 0 are the identity and zero matrix of order N+1, respectively.

3. Numerical solution

Consider a 2DNVIDE of the form:

$$u_{xx}(x,t)+u_{xt}(x,t)+u_{tt}(x,t)+u(x,t) \tag{7}$$
$$+\int_0^t\int_0^x K(x,t,y,z)F(y,z,u(y,z))\,dy\,dz = R(x,t), \qquad (x,t)\in\Omega,$$

with the supplementary conditions:

$$u(x,0)=\alpha_0, \qquad u(0,t)=\beta_0, \tag{8}$$
$$u_t(x,0)=\alpha_1, \qquad u_x(0,t)=\beta_1,$$

where $K(x,t,y,z), R(x,t), \alpha_0, \alpha_1, \beta_0, \beta_1$ are known continuous functions and u is the unknown function to be find and F is a nonlinear function in u.

In this section, we introduce a numerical method for the solution of nonlinear 2D Volterra integro-differential equations of the form (7). For this purpose, assume that $F(x,t,u(x,t))=\upsilon(x,t)$, so that Eq.(7) convert to:

$$u_{xx}(x,t)+u_{xt}(x,t)+u_{tt}(x,t)+u(x,t) \tag{9}$$
$$+\int_0^t\int_0^x K(x,t,y,z)\upsilon(y,z)\,dy\,dz = R(x,t), \qquad (x,t)\in\Omega.$$

Using the methods described in the previous section, we approximate the functions in Eq. (9) as :

$$\begin{aligned}
u_{xx}(x,t) &\cong \psi^T(x,t)A = A^T\psi(x,t),\\
u_{xt}(x,t) &\cong \psi^T(x,t)B = B^T\psi(x,t),\\
u_{tt}(x,t) &\cong \psi^T(x,t)D = D^T\psi(x,t),\\
u(x,t) &\cong \psi^T(x,t)C = C^T\psi(x,t),\\
u(x,0) &\cong \psi^T(x,t)E_1 = E_1^T\psi(x,t),\\
\int_0^t u_t(x,0)\,dt' &\cong \psi^T(x,t)E_2 = E_2^T\psi(x,t),\\
u(0,t) &\cong \psi^T(x,t)E_3 = E_3^T\psi(x,t),\\
\int_0^x u_x(0,t)\,dx' &\cong \psi^T(x,t)E_4 = E_4^T\psi(x,t),\\
u_t(x,0) &\cong \psi^T(x,t)E_5 = E_5^T\psi(x,t),
\end{aligned} \tag{10}$$

$$u_t(0,t) \cong \psi^T(x,t)E_6 = E_6^T\psi(x,t),$$
$$\upsilon(x,t) \cong \psi^T(x,t)G = G^T\psi(x,t).$$

We have:

$$\int_0^t\int_0^t u_{tt}(x,t')\,dt'dt' = \int_0^t (u_t(x,t') - u_t(x,0))\,dt' = u(x,t) - u(x,0) - \int_0^t u_t(x,0)\,dt', \quad (11)$$

on the other hand, using equation above, we derive:

$$\int_0^t\int_0^t u_{tt}(x,t')\,dt'dt' = \int_0^t\int_0^t D^T\psi(x,t')\,dt' \cong D^TQ_3^2\psi(x,t), \quad (12)$$

from Eqs. (10-12), we have:

$$D^TQ_3^2\,\psi(x,t) = C^T\psi(x,t) - E_1^T\psi(x,t) - E_2^T\psi(x,t),$$

therefore

$$D^TQ_3^2 = C^T - E_1^T - E_2^T. \quad (13)$$

Similary:

$$\int_0^x\int_0^x u_{xx}(x',t)\,dx'dx' = \int_0^x (u_x(x',t) - u_x(0,t))\,dx' = u(x,t) - u(0,t) - \int_0^x u_x(0,t)dx', \quad (14)$$

by using Eqs. (5) and (10), we obtain:

$$\int_0^x\int_0^x u_{xx}(x',t)\,dx'dx' = \int_0^x\int_0^x A^T\psi(x',t)\,dx' \cong A^TQ_2^2\,\psi(x,t), \quad (15)$$

hence from Eqs. (10), (14) and (15), we have:

$$A^TQ_2^2\,\psi(x,t) = C^T\psi(x,t) - E_3^T\psi(x,t) - E_4^T\psi(x,t),$$

thus:

$$A^TQ_2^2 = C^T - E_3^T - E_4^T. \quad (16)$$

We have

$$\int_0^t u_{tt}(x,t')\,dt' = u_t(x,t) - u_t(x,0), \quad (17)$$

and from Eqs. (10)

$$\int_0^t u_{tt}(x,t')dt' = \int_0^t D^T\psi(x,t')dt' \cong D^TQ_3\psi(x,t), \quad (18)$$

using Eqs. (10), (17) and (18), we get :

$$u_t(x,t) = D^TQ_3\psi(x,t) + u_t(x,0) = D^TQ_3\psi(x,t) + E_5{}^T\psi(x,t). \quad (19)$$

Analogously:

$$\int_0^x u_{xt}(x',t)\,dx' = u_t(x,t) - u_t(0,t), \quad (20)$$

and from Eqs. (10)

$$\int_0^x u_{xt}(x',t)\,dx' = \int_0^x B^T\psi(x',t)\,dx' \cong B^TQ_2\psi(x,t), \quad (21)$$

from Eqs. (10) and Eqs. (19-21) we obtain:

$$B^TQ_2\psi(x,t) = D^TQ_3\psi(x,t) + E_5{}^T\psi(x,t) - E_6{}^T\psi(x,t),$$

therefore we get :

$$B^TQ_2 = D^TQ_3 + E_5{}^T - E_6{}^T. \quad (22)$$

If we approximate the functions $K(x,t,y,z)$, $R(x,t)$ in the form:

$$R(x,t) \cong F^T\psi(x,t) = \psi^T(x,t)F, \quad (23)$$

$$K(x,t,y,z) \cong \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} k_{ijmn}\,\psi_{ij}(x,t)\psi_{mn}(y,z),$$

then, we obtain:

$$\int_0^t\int_0^x K(x,t,y,z)\upsilon(y,z)\,dy\,dz = \psi^T(x,t)\Pi_1 = \Pi_1^T\psi(x,t), \quad (24)$$

where

$$\Pi_1 = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} k_{ijmn}\, P^1_{ijmn},$$

and

$$P^1_{ijmn} = \begin{bmatrix} \sum_{p=0}^{N}\sum_{l=0}^{N}\sum_{s=0}^{N}\sum_{h=0}^{N} g_{pl}\eta^q_{sh}\beta_{00} \\ \sum_{p=0}^{N}\sum_{l=0}^{N}\sum_{s=0}^{N}\sum_{h=0}^{N} g_{pl}\eta^q_{sh}\beta_{01} \\ \vdots \\ \sum_{p=0}^{N}\sum_{l=0}^{N}\sum_{s=0}^{N}\sum_{h=0}^{N} g_{pl}\eta^q_{sh}\beta_{NN} \end{bmatrix},$$

$$\alpha_{rk} = \frac{(2r+1)(2k+1)}{lT}\Big(\int_0^l L_m(x)\, L_p(x)\, L_r(x)\, dx\Big)\Big(\int_0^T L_n(z)\, L_l(z)\, L_k(z)\, dz\Big),$$

$$\beta_{rk} = \frac{(2r+1)(2k+1)}{lT}\Big(\int_0^l L_i(x)\, L_s(x)\, L_r(x)\, dx\Big)\Big(\int_0^T L_j(t)\, L_h(t)\, L_k(t)\, dt\Big),$$

and

$$\eta = \alpha Q, \qquad Q = P_1 \otimes P_2,$$

such that P_1, P_2 are the operational matrices of 1D shifted Legendre polynomials that introduce in section (2.2).

Since from Eq. (10), we have

$$\upsilon(y,z) \cong \sum_{i=0}^{N}\sum_{j=0}^{N} g_{ij}\psi_{ij}(y,z) = \psi^T(y,z)G,$$

therefore:

$$\int_0^t\int_0^x \psi_{ij}(x,t)\psi_{mn}(y,z)\sum_{p=0}^{N}\sum_{l=0}^{N} C_{pl}\psi_{pl}(y,z)\, dy\, dz = \sum_{p=0}^{N}\sum_{l=0}^{N} C_{pl}\psi_{ij}(x,t)\int_0^t\int_0^x \psi_{mn}(y,z)\psi_{pl}(y,z)\, dy\, dz,$$

on the other, we have:

$$\psi_{mn}(y,z)\psi_{pl}(y,z) = \sum_{r=0}^{N}\sum_{k=0}^{N} \alpha_{rk}\psi_{rk}(y,z) = \alpha^T\psi(y,z),$$

where

$$\alpha = [\alpha_{00}, \alpha_{01}, \cdots, \alpha_{NN}]^T,\ \psi(y,z) = [\psi_{00}, \psi_{01}, \cdots, \psi_{NN}]^T.$$

Using Eq. (4) and equation above we obtain:

$$\sum_{p=0}^{N}\sum_{l=0}^{N} C_{pl}\psi_{ij}(x,t)\int_0^t\int_0^x \psi_{mn}(y,z)\psi_{pl}(y,z)dy\, dz = \sum_{p=0}^{N}\sum_{l=0}^{N}\sum_{s=0}^{N}\sum_{h=0}^{N}\sum_{r=0}^{N}\sum_{k=0}^{N} C_{pl}\,\eta_{sh}\,\beta_{rk}\psi_{rk}(x,t)$$

$$= [\psi_{00}, \psi_{01} \cdots, \psi_{NN}] \begin{bmatrix} \sum_{p=0}^{N}\sum_{l=0}^{N}\sum_{s=0}^{N}\sum_{h=0}^{N} C_{pl}\eta^q_{sh}\beta_{00} \\ \sum_{p=0}^{N}\sum_{l=0}^{N}\sum_{s=0}^{N}\sum_{h=0}^{N} C_{pl}\eta^q_{sh}\beta_{01} \\ \vdots \\ \sum_{p=0}^{N}\sum_{l=0}^{N}\sum_{s=0}^{N}\sum_{h=0}^{N} C_{pl}\eta^q_{sh}\beta_{NN} \end{bmatrix}.$$

Substituting Eqs.(10), (23) and (24) into Eq. (9), we derive:

$$A^T\psi(x,t) + B^T\psi(x,t) + D^T\psi(x,t) + C^T\psi(x,t) + \Pi_1^T\psi(x,t) = F^T\psi(x,t), \tag{25}$$

therefore:

$$A^T + B^T + D^T + C^T + \Pi_1^T = F^T. \tag{26}$$

Using Eqs. (13), (16) and (22), we have:

$$D^T = (C^T - E_1^T - E_2^T)Q_3^{-2},$$
$$A^T = (C^T - E_3^T - E_4^T)Q_2^{-2}, \qquad (27)$$
$$B^T = (D^T Q_3 + E_5{}^T - E_6{}^T)Q_2^{-1}.$$

Substituting Eqs. (27) in Eq. (26), we get:

$$(C^T - E_3^T - E_4^T)Q_2^{-2} + ((C^T - E_1^T - E_2^T)Q_3^{-1} + E_5^T - E_6^T)Q_2^{-1} + (C^T - E_1^T - E_2^T)Q_3^{-2} + C^T + \Pi_1^T = F^T. \qquad (28)$$

which is a system of algebraic equations.

On the other, we have:

$$F(x,t,u(x,t)) = \upsilon(x,t) \cong G^T\psi(x,t) = \psi^T(x,t)G. \qquad (29)$$

We now Collocate Eq.(29) at $(N+1)(N+1)$ points (x_i, t_j), $(i, j = 0,1, \dots, N)$ and obtain

$$F(x_i, t_j, u(x_i, t_j)) = G^T\psi(x_i, t_j) = \psi^T(x_i, t_j)G, \qquad (30)$$

where x_i, t_j are zeros of $L_{N+1}(\frac{2}{l}x-1)$ and $L_{N+1}(\frac{2}{T}t-1)$ respectively. Eqs. (28) and (30) form a system of algebraic equations. By solving it, we obtain an approximate solution of the problem.

4. Numerical examples

In this section, some examples are presented to evaluation of the present method. Note that as mentioned previously, in all case any non-polynomials term were replace by the Legendre polynomials of degree N and all computations have been done by programming in Matlab 2012.

Example1. Consider the nonlinear 2D Volterra integro-differential equation [6]:

$$u_{tt}(x,t) + u(x,t) - \int_0^t\int_0^x (y + cosz)u^2(y,z)\,dy\,dz = f(x,t), \qquad x,t \in [0,1],$$

$$f(x,t) = \frac{1}{8}x^4 sin(t)cos(t) - \frac{1}{8}x^4 t - \frac{1}{9}x^3 sin^3(t),$$

with supplementary conditions as:

$$u(x,0) = 0, \qquad u_t(x,0) = x.$$

The exact solution is $u(x,t) = x\,sin(t)$.

We substitute $\upsilon(x,t) = u^2(x,t)$ to get a linear equation and then apply the present method on it. Table 1 shows the numerical results obtained here and the numerical results of [6] for this example.

Table 1: Numerical results for Example 1.

(x,t)	$N=2$	$N=3$	m=16[6]
(0.01,0.01)	6.14×10^{-5}	3.53×10^{-7}	5.13×10^{-7}
(0.02,0.02)	1.06×10^{-4}	6.23×10^{-6}	1.30×10^{-6}
(0.1,0.1)	3.61×10^{-5}	8.94×10^{-6}	5.56×10^{-5}
(0.2,0.2)	5.21×10^{-4}	5.86×10^{-6}	2.97×10^{-3}

Example 2. In this example we consider the following linear two-dimensional integro– differential equation:

$$u_{xx}(x,t) + u_{tt}(x,t) + \int_0^t\int_0^x u(y,z)\,dy\,dz = g(x,t),$$

$$g(x,t) = \frac{1}{6}x^3t^2 + 2t, \qquad (x,t) \in [0,1]\times[0,1],$$

with supplementary conditions as:

$$u(x,0) = 0,\;\; u_x(0,t) = 0,\;\; u_t(x,0) = x^2,\;\; u(0,t) = 0.$$

Here, the exact solution is $u(x,t) = x^2 t$. The numerical results with $N=1$ are shown in Table 2 and the exact solution of this equation is obtained with *N=2.*

Table 2: Numerical results for Example 2.

$(x,t)=(2^{-l},2^{-l})$	Present method with $N=1$
$l=1$	4.1×10^{-2}
$l=2$	5.2×10^{-3}
$l=3$	7.2×10^{-3}
$l=4$	6.8×10^{-3}
$l=5$	4.4×10^{-3}
$l=6$	2.5×10^{-3}

Example3. For the end example, consider the following equation [6, 12]

$$u_{xx}(x,t)+u_{tt}(x,t)+\int_0^t\int_0^x x^2 t\, u(y,z)dy\,dz = g(x,t), \qquad x,t\in[0,1],$$

where

$$g(x,t) = xe^t - \frac{1}{2}x^4 t + \frac{1}{2}x^4 te^t ,$$

with supplementary conditions as:

$$u(0,t)=0, \quad u_x(x,1)=e^1 .$$

The exact solution of this problem is $u(x,t)=xe^t$. The absolute values of the errors for this problem are shown in Table 3 and the absolute values of the errors for this problem of [6,12] are shown in Table 4. The absolute error function for *N= 4, 5* is plotted in Figs. 1, 2.

Table 3: Numerical results for Example 3.

(x,t)	$N=4$	$N=5$
(0.01, 0.01)	6.37×10^{-7}	2.37×10^{-8}
(0.02, 0.02)	6.24×10^{-7}	1.76×10^{-8}
(0.1, 0.1)	2.10×10^{-6}	9.54×10^{-8}
(0.2, 0.2)	2.01×10^{-6}	7.67×10^{-8}

Table 4: Numerical results [6, 12] for Example 3.

(x,t)	$\lvert e_{5,5}(x,t)\rvert$[12]	$\lvert e_{6,6}(x,t)\rvert$[12]	$m=32$[6]	$m=64$[6]
(0.01, 0.01)	1.66×10^{-7}	1.66×10^{-7}	2.23×10^{-8}	2.12×10^{-8}
(0.02, 0.02)	1.33×10^{-6}	1.33×10^{-6}	7.98×10^{-8}	7.74×10^{-8}
(0.1, 0.1)	1.67×10^{-4}	1.66×10^{-4}	6.80×10^{-6}	4.71×10^{-6}
(0.2, 0.2)	1.34×10^{-3}	1.33×10^{-3}	2.33×10^{-4}	2.23×10^{-4}

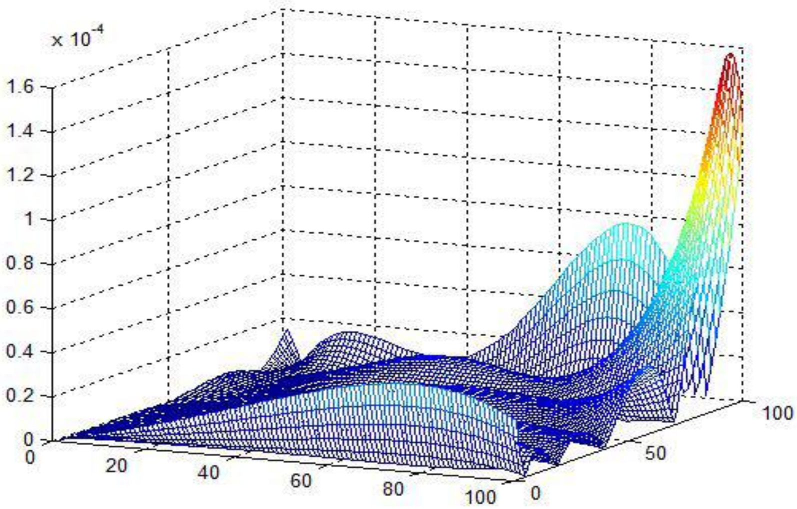

Fig. 1: Plot of the function $e_N(x,t)$ with N = 4 for Example 3.

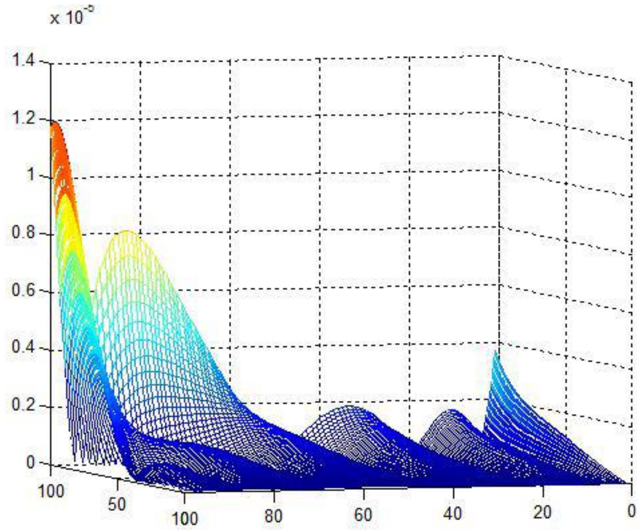

Fig. 2: Plot of the function $e_N(x,t)$ with N = 5 for Example 3.

5. Conclusion

We introduced a new method for solving two-dimensional nonlinear Volterra integro-differential equations, based on expanding the solution in terms of two dimensional Legendre polynomials. As the numerical results have shown, comparing the present method and other numerical methods, shows that the present method is more accurate. Also, it will be possible to investigate the numerical solution of two dimensional linear and nonlinear Fredholm integro- differential equations.

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