ACADEMIC World Academic Union

ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 11, No. 1, 2016, *pp.* 003-015

On the Growth Estimate of Iterated Entire Functions

Ratan Kumar Dutta
Department of Mathematics, Netaji Mahavidyalaya, Arambagh, Dist.- Hooghly, Pin-712601, West Bengal, India, E-mail: ratan_3128@yahoo.com
(*Received March 30, 2014, accepted October 23, 2015)*

Abstract. In this paper we study some growth properties of iterated entire functions which improve some earlier results.

Keywords: Entire functions, growth, iteration, order, lower order, (p,q)-th order, lower (p,q)-th order.

1. Introduction

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane **C**. It is well known [1], {[15], p-67, Th-1.46} that $\lim_{r\to\infty}\frac{T(r,fog)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\frac{T(r,fog)}{T(r,g)}=\infty$.

After this Singh [12], Lahiri [8], Song and Yang [14], Singh and Baloria [13], Lahiri and Sharma [9] and Datta and Biswas [3], [4] proved different results on comparative growth property of composite entire functions.

In this paper, we investigate the comparative growth of iterated entire functions in terms of its (p,q)-th order which is the generalization of previous results. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [5], [15] and [16].

The following definitions are well known.

Definition 1.1. The orde ρ_f and lower order λ_f of a meromorphic function $f(z)$ is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r} \quad \text{and} \quad \lambda_f=\liminf_{r\to\infty}\frac{\log\log M(r,f)}{\log r}.$$

Notation 1.2. [11] $\log^{[0]}x=x$, $\exp^{[0]}x=x$ and for positive integer *m*, $\log^{[m]}x=\log(\log^{[m-1]}x)$, $\exp^{[m]}x=\exp(\exp^{[m-1]}x)$.

Definition 1.3. [6] The (p,q)-th order $\rho_f(p,q)$ and lower (p,q)-th order $\lambda_f(p,q)$ of a meromorphic function $f(z)$ is define as

$$\rho_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f)}{\log^{[q]}r} \quad \text{and} \quad \lambda_f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f)}{\log^{[q]}r}.$$

If $f(z)$ is entire then

$$\rho_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}r} \quad \text{and} \quad \lambda_f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}r}.$$

where $p>q$. Clearly $\rho_f(2,1)=\rho_f$ and $\lambda_f(2,1)=\lambda_f$.

According to Lahiri and Banerjee [7] if $f(z)$ and $g(z)$ are entire functions then the iteration of $f(z)$ with respect to $g(z)$ is defined as follows:

$$f_1(z) = f(z)$$
$$f_2(z) = f(g(z)) = f(g_1(z))$$
$$f_3(z) = f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))$$
$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$
$$f_n(z) = f(g(f \ldots\ldots\ldots\ldots(f(z) \text{ or } g(z)) \ldots\ldots\ldots)) \quad \text{according as } n \text{ is odd or even},$$

and so

$$g_1(z) = g(z)$$
$$g_2(z) = g(f(z)) = g(f_1(z))$$
$$g_3(z) = g(f(g(z))) = g(f_2(z)) = g(f(g_1(z)))$$
$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$
$$g_n(z) = g(f_{n-1}(z)) = g(f(g_{n-2}(z))).$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [5] Let $f(z)$ be an entire function. For $0 \le r < R < \infty$, we have

$$T(r, f) \le \log^+ M(r, f) \le \frac{R+r}{R-r} T(R, f).$$

Lemma 2.2. [1] If $f(z)$ and $g(z)$ are any two entire functions, for all sufficiently large values of *r*,

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \le M(r, fog) \le M(M(r, g), f).$$

Lemma 2.3. [10] Let $f(z)$ and $g(z)$ be two entire functions. Then we have

$$T(r, fog) \ge \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

Lemma 2.4. Let $f(z)$ and $g(z)$ be two entire functions of non- zero finite (p,q)-th order $\rho_f(p,q)$ and $\rho_g(p,q)$ respectively, then for any $\varepsilon > 0$ and $p > q$,

$$\log^{[(n-1)p-(n-2)q]} M(r, f_n) \le \begin{cases} (\rho_f(p,q) + \varepsilon)\log^{[q]} M(r, g) + O(1) & \text{when } n \text{ is even}, \\ (\rho_g(p,q) + \varepsilon)\log^{[q]} M(r, f) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of *r*.

Proof. First suppose that *n* is even. Then from second part of Lemma 2.2 and Definition of (p,q)-th order, it follows that for all sufficiently large values of *r*,

$$M(r, f_n) \le M(M(r, g_{n-1}), f)$$
$$\text{i.e., } \log^{[p]} M(r, f_n) \le \log^{[p]} M(M(r, g_{n-1}), f)$$
$$\le (\rho_f(p,q) + \varepsilon)\log^{[q]} M(r, g_{n-1})$$
$$\text{So, } \log^{[p+1]} M(r, f_n) \le \log^{[q+1]} M(r, g(f_{n-2})) + O(1)$$
$$\text{i.e., } \log^{[p+1-q]} M(r, f_n) \le \log M(r, g(f_{n-2})) + O(1).$$

Taking repeated logarithms *p-1* times, we get

$$\log^{[2p-q]} M(r, f_n) \leq \log^{[p]} M(M(r, f_{n-2}), g) + O(1)$$
$$\leq (\rho_g(p,q)+\varepsilon)\log^{[q]} M(r, f_{n-2}) + O(1).$$
$$i.e., \log^{[2p+1-q]} M(r, f_n) \leq \log^{[q+1]} M(r, f_{n-2}) + O(1)$$
$$i.e., \log^{[2p+1-2q]} M(r, f_n) \leq \log M(r, f_{n-2}) + O(1).$$

Again taking repeated logarithms *p-1* times, we get

$$\log^{[3p-2q]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r, g_{n-3}) + O(1).$$

Finally, after taking repeated logarithms *(n-4)p* times more, we have for all sufficiently large values of *r*,

$$\log^{[(n-1)p-(n-2)q]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r, g) + O(1).$$

Similarly if *n* is odd then for all sufficiently large values of *r*,

$$\log^{[(n-1)p-(n-2)q]} M(r, f_n) \leq (\rho_g(p,q)+\varepsilon)\log^{[q]} M(r, f) + O(1).$$

This proves the lemma.

Lemma 2.5. Let $f(z)$ and $g(z)$ be two entire functions of non -zero finite lower (p,q)-th order $\lambda_f(p,q)$ and $\lambda_g(p,q)$ respectively, then for any $0<\varepsilon<\min\{\lambda_f(p,q), \lambda_g(p,q)\}$ and *p>q*,

$$\log^{[(n-1)p-(n-2)q]} M(r, f_n) \geq \begin{cases} (\lambda_f(p,q)-\varepsilon)\log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even,} \\ (\lambda_g(p,q)-\varepsilon)\log^{[q]} M\left(\frac{r}{2^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of *r*.

Proof. First suppose that *n* is even. Then from first part of Lemma 2.2 and using the Definition 1.3, we have for all sufficiently large values of *r* and for any $0<\varepsilon<\min\{\lambda_f(p,q), \lambda_g(p,q)\}$,

$$M(r, f_n) = M(r, f(g_{n-1}))$$
$$\geq M\left(\frac{1}{8}M\left(\frac{r}{2}, g_{n-1}\right) - |g_{n-1}(0)|, f\right)$$
$$\geq M\left(\frac{1}{16}M\left(\frac{r}{2}, g_{n-1}\right), f\right)$$
$$\therefore \quad \log^{[p]} M(r, f_n) \geq (\lambda_f(p,q)-\varepsilon)\log^{[q]}\left[\frac{1}{16}M\left(\frac{r}{2}, g_{n-1}\right)\right]$$
$$i.e., \ \log^{[p]} M(r, f_n) \geq (\lambda_f(p,q)-\varepsilon)\log^{[q]}\left[M\left(\frac{r}{2}, g_{n-1}\right)\right] + O(1)$$
$$i.e., \ \log^{[p+1]} M(r, f_n) \geq \log^{[q+1]} M\left(\frac{r}{2}, g(f_{n-2})\right) + O(1)$$
$$i.e., \ \log^{[p+1-q]} M(r, f_n) \geq \log M\left(\frac{1}{16}M\left(\frac{r}{2^2}, f_{n-2}\right), g\right) + O(1).$$

Taking repeated logarithms *p-1* times, we get

$$\log^{[2p-q]} M(r, f_n) \geq \log^{[p]} M\left(\frac{1}{16} M\left(\frac{r}{2^2}, f_{n-2}\right), g\right) + O(1)$$

$$\geq (\lambda_g(p,q) - \varepsilon) \log^{[q]}\left(\frac{1}{16} M\left(\frac{r}{2^2}, f_{n-2}\right)\right) + O(1)$$

$$i.e., \quad \log^{[2p+1-q]} M(r, f_n) \geq \log M\left(\frac{r}{2^2}, f_{n-2}\right) + O(1).$$

Again taking repeated logarithms *p-1* times, we get

$$\log^{[3p-2q]} M(r, f_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]}\left[\frac{1}{16} M\left(\frac{r}{2^3}, g_{n-3}\right)\right] + O(1)$$

$$\geq (\lambda_f(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{2^3}, g_{n-3}\right) + O(1).$$

Finally, after taking repeated logarithms *(n-4)p* times more, we have for all sufficiently large values of *r*,

$$\log^{[(n-1)p-(n-2)q]} M(r, f_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]}\left[\frac{1}{16} M\left(\frac{r}{2^{n-1}}, g\right)\right] + O(1)$$

$$i.e., \log^{[(n-1)p-(n-2)q]} M(r, f_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) + O(1).$$

Similarly if *n* is odd then for all sufficiently large values of *r*,

$$i.e., \log^{[(n-1)p-(n-2)q]} M(r, f_n) \geq (\lambda_g(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{2^{n-1}}, f\right) + O(1).$$

This proves the lemma.

Lemma 2.6. Let $f(z)$ and $g(z)$ be two non- constant entire functions, such that $0 < \rho_f(p,q) < \infty$ and $0 < \rho_g(p,q) < \infty.$ Then for all sufficiently large *r* and $\varepsilon > 0$ and for $p > q$,

$$\log^{[(n-1)p-(n-2)q-1]} M(r, f_n) \leq \begin{cases} (\rho_f(p,q) + \varepsilon) \log^{[q]} M(r, g) + O(1) & \text{when } n \text{ is even,} \\ (\rho_g(p,q) + \varepsilon) \log^{[q]} M(r, f) + O(1) & \text{when } n \text{ is odd.} \end{cases}$$

The lemma follows from Lemma 2.1 and Lemma 2.4.

Lemma 2.7. Let $f(z)$ and $g(z)$ be two entire functions such that that $0 < \lambda_f(p,q) < \infty$ and $0 < \lambda_g(p,q) < \infty.$ Then for any $\varepsilon \ (0 < \varepsilon < \min\{\lambda_f(p,q), \ \lambda_g(p,q)\})$ and $p > q$,

$$\log^{[(n-1)p-(n-2)q-1]} T(r, f_n) \geq \begin{cases} (\lambda_f(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even,} \\ (\lambda_g(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of *r*.

Proof. To prove this lemma we first suppose that *n* is even. Then from Lemma 2.1and Lemma 2.3 we get for any $\varepsilon \ (0 < \varepsilon < \min\{\lambda_f(p,q), \ \lambda_g(p,q)\})$ and for all sufficiently large values of *r*,

$$T(r, f_n) = T(r, f(g_{n-1}))$$
$$\geq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, g_{n-1}\right) + O(1), f\right)$$
$$\therefore \quad \log^{[p-1]} T(r, f_n) \geq \log^{[p]} M\left(\frac{1}{8}M\left(\frac{r}{4}, g_{n-1}\right) + O(1), f\right) + O(1)$$
$$\geq (\lambda_f(p,q) - \varepsilon)\log^{[q]}\left[\frac{1}{8}M\left(\frac{r}{4}, g_{n-1}\right) + O(1)\right] + O(1)$$
$$\geq (\lambda_f(p,q) - \varepsilon)\log^{[q]}\left[\frac{1}{9}M\left(\frac{r}{4}, g_{n-1}\right)\right] + O(1)$$
$$\geq (\lambda_f(p,q) - \varepsilon)\log^{[q]} M\left(\frac{r}{4}, g_{n-1}\right) + O(1)$$
$$\geq (\lambda_f(p,q) - \varepsilon)\log^{[q-1]} T\left(\frac{r}{4}, g_{n-1}\right) + O(1)$$
$$\geq (\lambda_f(p,q) - \varepsilon)\log^{[q-1]}\left[\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right)\right] + O(1)$$
$$i.e., \log^{[p]} T(r, f_n) \geq \log^{[q+1]} M\left(\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1)$$
$$i.e., \log^{[p-q]} T(r, f_n) \geq \log M\left(\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1)$$
$$i.e., \log^{[2p-1-q]} T(r, f_n) \geq \log^{[p]} M\left(\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1)$$
$$\geq (\lambda_g(p,q) - \varepsilon)\log^{[q]}\left[\frac{1}{8}M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1)\right] + O(1)$$
$$\geq (\lambda_g(p,q) - \varepsilon)\log^{[q]}\left[\frac{1}{9}M\left(\frac{r}{4^2}, f_{n-2}\right)\right] + O(1).$$
$$i.e., \log^{[2p-1-q]} T(r, f_n) \geq (\lambda_g(p,q) - \varepsilon)\log^{[q]} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1).$$

….. ….. …..
….. ….. …..

$$\therefore \quad \log^{[(n-1)p-(n-2)q-1]} T(r, f_n) \geq (\lambda_f(p,q) - \varepsilon)\log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) + O(1) \quad when\ n\ is\ even.$$

Similarly,

$$\log^{[(n-1)p-(n-2)q-1]} T(r, f_n) \geq (\lambda_g(p,q) - \varepsilon)\log^{[q]} M\left(\frac{r}{4^{n-1}}, f\right) + O(1) \quad when\ n\ is\ odd.$$

This proves the lemma.

3. Theorems

Theorem 3.1. Let *a, b, c, d, p, q, m* and *n* be eight positive integers with *p* > *q*, *m* > *n*, *a* > *b*, *c* > *d* and *f, g, h* and *k* are four transcendental entire functions, such that $0 < \lambda_h(a,b),\ \lambda_k(c,d) < \infty$ and

$0<\rho_f(p,q),\ \rho_g(p,q)<\infty$. Then for $\rho_g(m,n)<\lambda_k(c,d)$ and *i, j* are even, also $\rho_f(m,n)<\lambda_h(c,d)$ and *i, j* are odd,

$$(i)\ \lim_{r\to\infty}\frac{\log^{[(j-1)a-(j-2)b-1]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)p-(i-2)q-1]}T(r,f_i)}=\infty\ \ if\ \ q\geq m\ and\, b<c;$$

$$(ii)\ \lim_{r\to\infty}\frac{\log^{[(j-1)a-(j-2)b-1]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)(p-q)+m-2]}T(r,f_i)}=\infty\ \ if\ \ q<m\ and\, b<c;$$

$$(iii)\ \lim_{r\to\infty}\frac{\log^{[(j-1)(a-b)+c-2]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)p-(i-2)q-1]}T(r,f_i)}=\infty\ \ if\ \ q\geq m\ and\, b\geq c;$$

$$(iv)\ \lim_{r\to\infty}\frac{\log^{[(j-1)(a-b)+c-2]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)(p-q)+m-2]}T(r,f_i)}=\infty\ \ if\ \ q<m\ and\, b\geq c;$$

where $f_i(z)=f(g(f\ldots\ldots\ldots\ldots(f(z)\ or\ g(z))\ \ldots\ldots\ldots))$ according as *i* is odd or even and $h_j(z)=h(k(h\ldots\ldots\ldots\ldots(h(z)\ or\ k(z))\ \ldots\ldots\ldots))$ according as *j* is odd or even.

Proof. From Lemma 2.6 we have for all large *r* and $\varepsilon>0$,

$$\log^{[(i-1)p-(i-2)q-1]}M(r,f_i)\leq\begin{cases}(\rho_f(p,q)+\varepsilon)\log^{[q]}M(r,g)+O(1) & when\ i\ is\, even,\\ (\rho_g(p,q)+\varepsilon)\log^{[q]}M(r,f)+O(1) & when\ i\ is\, odd.\end{cases}\qquad(3.1)$$

Case - I. If $q\geq m$ then we have for all sufficiently large values of *r*,

$$\begin{aligned}\log^{[q]}M(r,g)&\leq\log^{[m-1]}M(r,g)\\&\leq\exp[(\rho_g(m,n)+\varepsilon)\log^{[n]}r]\\&\leq\exp[(\rho_g(m,n)+\varepsilon)\log r]\\&\leq r^{(\rho_g(m,n)+\varepsilon)}.\end{aligned}\qquad(3.2)$$

If *i* is even then from (3.1) and (3.2) it follows for all large *r* and $\varepsilon>0$,

$$\log^{[(i-1)p-(i-2)q-1]}M(r,f_i)\leq(\rho_f(p,q)+\varepsilon)r^{(\rho_g(m,n)+\varepsilon)}+O(1).\qquad(3.3)$$

Similarly for odd *i*,

$$\log^{[(i-1)p-(i-2)q-1]}M(r,f_i)\leq(\rho_g(p,q)+\varepsilon)r^{(\rho_f(m,n)+\varepsilon)}+O(1).\qquad(3.4)$$

Case - II. If $q<m$ then we have for all sufficiently large values of *r*,

$$\begin{aligned}\log^{[q]}M(r,g)&\leq\exp^{[m-q]}\log^{[m]}M(r,g)\\&\leq\exp^{[m-q]}[(\rho_g(m,n)+\varepsilon)\log^{[n]}r]\\&\leq\exp^{[m-q]}[(\rho_g(m,n)+\varepsilon)\log r]\\ i.e.,\quad\log^{[q]}M(r,g)&\leq\exp^{[m-q-1]}\left\lfloor r^{(\rho_g(m,n)+\varepsilon)}\right\rfloor.\end{aligned}\qquad(3.5)$$

When *i* is even then from (3.1) and (3.5), it follows for all large *r* and $\varepsilon>0$,

$$\begin{aligned}&\log^{[(i-1)p-(i-2)q-1]}T(r,f_i)\leq(\rho_f(p,q)+\varepsilon)\exp^{[m-q-1]}r^{(\rho_g(m,n)+\varepsilon)}+O(1)\\ &i.e.,\ \log^{[(i-1)p-(i-2)q]}T(r,f_i)\leq\exp^{[m-q-2]}r^{(\rho_g(m,n)+\varepsilon)}+O(1)\\ &i.e.,\ \log^{[(i-1)p-(i-2)q+m-q-2]}T(r,f_i)\leq r^{(\rho_g(m,n)+\varepsilon)}+O(1)\\ &i.e.,\ \log^{[(i-1)(p-q)+m-2]}T(r,f_i)\leq r^{(\rho_g(m,n)+\varepsilon)}+O(1).\end{aligned}\qquad(3.6)$$

Similarly for odd *i*,

$$\log^{[(i-1)(p-q)+m-2]}T(r,f_i)\leq r^{(\rho_f(m,n)+\varepsilon)}+O(1).\qquad(3.7)$$

Also from Lemma 2.7 we have for all sufficiently large values of *r* and

$$0<\varepsilon<\varepsilon' = \min\left\{\frac{1}{2}(\lambda_k(c,d)-\rho_g(m,n)),\ \frac{1}{2}(\lambda_h(c,d)-\rho_f(m,n))\right\}$$

$$\log^{[(j-1)a-(j-2)b-1]} T(\exp^{[d-1]} r, h_j) \geq \begin{cases} (\lambda_h(a,b)-\varepsilon)\log^{[b]} M\left(\frac{\exp^{[d-1]} r}{4^{n-1}}, k\right)+O(1) & \text{when } j \text{ is even,} \\ (\lambda_k(a,b)-\varepsilon)\log^{[b]} M\left(\frac{\exp^{[d-1]} r}{4^{n-1}}, h\right)+O(1) & \text{when } j \text{ is odd.} \end{cases} \quad (3.8)$$

Case - III. If $b < c$ then we have for all sufficiently large values of r and arbitrary small $\varepsilon(0<\varepsilon<\varepsilon')$,

$$\begin{aligned} \log^{[b]} M\left(\frac{\exp^{[d-1]} r}{4^{n-1}}, k\right) &\geq \exp^{[c-b]} \log^{[c]} M\left(\frac{\exp^{[d-1]} r}{4^{n-1}}, k\right) \\ &\geq \exp^{[c-b]}\left[(\lambda_k(c,d)-\varepsilon)\log^{[d]}\left(\frac{\exp^{[d-1]} r}{4^{n-1}}\right)\right] \\ &\geq \exp^{[c-b]}\left[(\lambda_k(c,d)-\varepsilon)\log r\right]+O(1) \\ &\geq \exp^{[c-b-1]} r^{(\lambda_k(c,d)-\varepsilon)}+O(1). \end{aligned} \quad (3.9)$$

Therefore from (3.8), (3.9) and even j, it follows for all large r and $\varepsilon(0<\varepsilon<\varepsilon')$,

$$\log^{[(j-1)a-(j-2)b-1]} T(\exp^{[d-1]} r, h_j) \geq (\lambda_h(a,b)-\varepsilon)\exp^{[c-b-1]} r^{(\lambda_k(c,d)-\varepsilon)}+O(1). \quad (3.10)$$

Similarly for odd i,

$$\log^{[(j-1)a-(j-2)b-1]} T(\exp^{[d-1]} r, h_j) \geq (\lambda_k(a,b)-\varepsilon)\exp^{[c-b-1]} r^{(\lambda_h(c,d)-\varepsilon)}+O(1). \quad (3.11)$$

Case - IV. If $b \geq c$ then we have for all sufficiently large values of r and arbitrary small $\varepsilon(0<\varepsilon<\varepsilon')$,

$$\begin{aligned} \log^{[b]} M\left(\frac{\exp^{[d-1]} r}{4^{n-1}}, k\right) &\geq \log^{[b-c]} \log^{[c]} M\left(\frac{\exp^{[d-1]} r}{4^{n-1}}, k\right) \\ &\geq \log^{[b-c]}\left[(\lambda_k(c,d)-\varepsilon)\log^{[d]}\left(\frac{\exp^{[d-1]} r}{4^{n-1}}\right)\right] \\ &\geq \log^{[b-c]}\left[(\lambda_k(c,d)-\varepsilon)\log r\right]+O(1) \\ &\geq \log^{[b-c+1]} r^{(\lambda_k(c,d)-\varepsilon)}+O(1). \end{aligned} \quad (3.12)$$

Therefore when j is even then from (3.8) and (3.12), it follows for all large r and $\varepsilon(0<\varepsilon<\varepsilon')$,

$$\log^{[(j-1)a-(j-2)b-1]} T(\exp^{[d-1]} r, h_j) \geq (\lambda_h(a,b)-\varepsilon)\log^{[b-c+1]} r^{(\lambda_k(c,d)-\varepsilon)}+O(1)$$

$$i.e.,\ \log^{[(j-1)a-(j-2)b]} T(\exp^{[d-1]} r, h_j) \geq \log^{[b-c+2]} r^{(\lambda_k(c,d)-\varepsilon)}+O(1)$$

$$i.e.,\ \log^{[(j-1)a-(j-2)b-b+c-2} T(\exp^{[d-1]} r, h_j) \geq r^{(\lambda_k(c,d)-\varepsilon)}+O(1)$$

$$i.e.,\ \log^{[(j-1)(a-b)+c-2]} T(\exp^{[d-1]} r, h_j) \geq r^{(\lambda_k(c,d)-\varepsilon)}+O(1). \quad (3.13)$$

Similarly for odd j,

$$\log^{[(j-1)(a-b)+c-2]} T(\exp^{[d-1]} r, h_j) \geq r^{(\lambda_h(c,d)-\varepsilon)}+O(1) \quad (3.14)$$

Now combining (3.3) of Case I and (3.10) of Case III it follows for all sufficiently large values of r that

$$\frac{\log^{[(j-1)a-(j-2)b-1]} T(\exp^{[d-1]} r, h_j)}{\log^{[(i-1)p-(i-2)q-1]} T(r, f_i)} \geq \frac{(\lambda_h(a,b)-\varepsilon)\exp^{[c-b-1]} r^{(\lambda_k(c,d)-\varepsilon)}+O(1)}{(\rho_f(p,q)+\varepsilon)r^{(\rho_g(m,n)+\varepsilon)}+O(1)}.$$

Since $\rho_g(m,n)+\varepsilon<\lambda_k(c,d)-\varepsilon$ so we have

$$\lim_{r\to\infty} \frac{\log^{[(j-1)a-(j-2)b-1]} T(\exp^{[d-1]} r, h_j)}{\log^{[(i-1)p-(i-2)q-1]} T(r, f_i)} = \infty.$$

Similarly from (3.4) and (3.11) we have for $\rho_f(m,n)+\varepsilon<\lambda_h(c,d)-\varepsilon$

$$\lim_{r\to\infty}\frac{\log^{[(j-1)a-(j-2)b-1]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)p-(i-2)q-1]}T(r,f_i)}=\infty.$$

This is the first part of the theorem.
Again combining (3.6) of Case II and (3.10) of Case III it follows for all sufficiently large values of r that

$$\frac{\log^{[(j-1)a-(j-2)b-1]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)(p-q)+m-2]}T(r,f_i)}\geq\frac{(\lambda_h(a,b)-\varepsilon)\exp^{[c-b-1]}r^{(\lambda_k(c,d)-\varepsilon)}+O(1)}{r^{(\rho_g(m,n)+\varepsilon)}+O(1)}.$$

Since $\rho_g(m,n)+\varepsilon<\lambda_k(c,d)-\varepsilon$ so we have

$$\lim_{r\to\infty}\frac{\log^{[(j-1)a-(j-2)b-1]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)(p-q)+m-2]}T(r,f_i)}=\infty.$$

Similarly from (3.7) and (3.11) we have for $\rho_f(m,n)+\varepsilon<\lambda_h(c,d)-\varepsilon$,

$$\lim_{r\to\infty}\frac{\log^{[(j-1)a-(j-2)b-1]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)(p-q)+m-2]}T(r,f_i)}=\infty.$$

This proved the second part of the theorem.
Now combining (3.3) of Case I and (3.13) of Case IV it follows for all sufficiently large values of r,

$$\frac{\log^{[(j-1)(a-b)+c-2]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)p-(i-2)q-1]}M(r,f_i)}\geq\frac{r^{(\lambda_k(c,d)-\varepsilon)}+O(1)}{(\rho_f(p,q)+\varepsilon)r^{(\rho_g(m,n)+\varepsilon)}+O(1)}.$$

Since $\rho_g(m,n)+\varepsilon<\lambda_k(c,d)-\varepsilon$ so we have

$$\lim_{r\to\infty}\frac{\log^{[(j-1)(a-b)+c-2]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)p-(i-2)q-1]}M(r,f_i)}=\infty.$$

Similarly from (3.4) and (3.14) we have for $\rho_f(m,n)+\varepsilon<\lambda_h(c,d)-\varepsilon$

$$\lim_{r\to\infty}\frac{\log^{[(j-1)(a-b)+c-2]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)p-(i-2)q-1]}M(r,f_i)}=\infty.$$

This is the third part of the theorem.
Again combining (3.6) of Case II and (3.13) of Case IV it follows for all sufficiently large values of r,

$$\frac{\log^{[(j-1)(a-b)+c-2]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)(p-q)+m-2]}T(r,f_i)}\geq\frac{r^{(\lambda_k(c,d)-\varepsilon)}+O(1)}{r^{(\rho_g(m,n)+\varepsilon)}+O(1)}.$$

Since $\rho_g(m,n)+\varepsilon<\lambda_k(c,d)-\varepsilon$ so we have

$$\lim_{r\to\infty}\frac{\log^{[(j-1)(a-b)+c-2]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)(p-q)+m-2]}T(r,f_i)}=\infty.$$

Similarly from (3.7) and (3.14) we have for $\rho_f(m,n)+\varepsilon<\lambda_h(c,d)-\varepsilon$

$$\lim_{r\to\infty}\frac{\log^{[(j-1)(a-b)+c-2]}T(\exp^{[d-1]}r,h_j)}{\log^{[(i-1)(p-q)+m-2]}T(r,f_i)}=\infty.$$

This proved the fourth part of the theorem.

Remark 3.2. The conditions $\lambda_h(a,b),\ \lambda_k(c,d)>0$ and $\rho_f(p,q),\ \rho_g(p,q)<\infty$ also $\rho_g(m,n)<\lambda_k(c,d)$ are necessary for Theorem 3.1, which are shown by the following examples.

Example 3.3. Let $f=g=h=\exp z$ and $k=\exp(z^2)$. Also let $a=3;\ p=m=c=2=j$ and $q=n=b=d=1$.
Then $\rho_f(2,1)=1,\ \rho_g(2,1)=1<2=\lambda_k(2,1)$ and $\lambda_h(3,1)=0.$
Now $f_i=\exp^{[i]}z$ for all i and $h_2=\exp^{[2]}(z^2)$.

Therefore

$$3T(2r, f_i) \geq \log M(r, f_i) = \exp^{[i-1]} r$$

$$i.e., \; T(r, f_i) \geq \frac{1}{3}\exp^{[i-1]}\frac{r}{2}$$

and $\quad T(r,h_2) \leq \log M(r,h_2) = \log(\exp^{[2]}(r^2)) = \exp(r^2).$

Therefore

$$\frac{\log^{[2]} T(r,h_2)}{\log^{[i-1]} T(r, f_i)} \leq \frac{\log^{[2]}\exp(r^2)}{\log^{[i-1]}\left[\frac{1}{3}\exp^{[i-1]}\frac{r}{2}\right]} \leq \frac{4\log r}{r+O(1)} \to 0 \neq \infty \;\; as \;\; r \to \infty.$$

Example 3.4. Let *f = h = k = exp z* and *g* = exp z ^{2} . Also let *p = m = a = c = 2 =i* and *q = n = b = d = 1*. Then $\rho_f(2,1) = 1,\; \rho_g(2,1) = 2 > 1 = \lambda_k(2,1)$ and $\lambda_h(2,1) = 1.$

Now $f_2 = \exp^{[2]}(z^2)$ and $h_j = \exp^{[j]} z$ for all j.

Therefor

$$3T(2r, f_2) \geq \log M(r, f_2) = \exp(r^2)$$

$$i.e., \; T(r, f_2) \geq \frac{1}{3}\exp\frac{r^2}{4}$$

and $\quad T(r,h_j) \leq \log M(r,h_j) = \log(\exp^{[j]} r) = \exp^{[j-1]} r.$

Therefore

$$\frac{\log^{[j-1]} T(r,h_j)}{\log^{[2]} T(r, f_2)} \leq \frac{\log^{[j-1]}\exp^{[j-1]} r}{\log\left[\frac{1}{3}\exp\frac{r^2}{4}\right]} . \leq \frac{4r}{r^2+O(1)} \to 0 \neq \infty \;\; as \;\; r \to \infty.$$

If we consider *i = 3* then $f_3 = \exp^{[2]}(\exp z)^2$ and $\rho_f(2,1) = 1 = \lambda_h(2,1)$

Therefore

$$\frac{\log^{[j-1]} T(r,h_j)}{\log^{[2]} T(r, f_3)} \leq \frac{\log^{[j-1]}\exp^{[j-1]} r}{\log^{[2]}\left[\frac{1}{3}\exp\left(\exp\frac{r}{2}\right)^2\right]} \leq \frac{r}{r+O(1)} \to 1 \neq \infty \;\; as \;\; r \to \infty.$$

Example 3.5. Let $f = \exp^{[2]} z$, *g= h = exp z* and *k* = exp(z ^{2}). Also let *p = m = a = c = 2 =j* and *q = n = b = d = 1.*

Then $\rho_f(2,1) = \infty,\; \rho_g(2,1) = 1 < 2 = \lambda_k(2,1)$ and $\lambda_h(2,1) = 1.$

Now $f_i = \exp^{[3i/2]} z$ for all even i and $h_2 = \exp^{[2]}(z^2).$

Therefore

$$3T(2r, f_i) \geq \log M(r, f_i) = \exp^{[\frac{3i}{2}-1]} r$$

$$i.e., \; T(r, f_i) \geq \frac{1}{3}\exp^{[\frac{3i}{2}-1]}\frac{r}{2}$$

and $\quad T(r,h_2) \leq \log M(r,h_2) = \log(\exp^{[2]}(r^2)) = \exp(r^2).$

Therefore

$$\frac{\log T(r,h_2)}{\log^{[i-1]} T(r, f_i)} \leq \frac{\log(\exp(r^2))}{\log^{[i-1]}\left[\frac{1}{3}\exp^{[\frac{3i}{2}-1]}\frac{r}{2}\right]} \leq \frac{r^2}{\exp^{[\frac{i}{2}]}\frac{r}{2}+O(1)} \to 0 \neq \infty \;\; as \;\; r \to \infty.$$

Theorem 3.6. Let *f*, *g* be two entire functions such that $0<\lambda_f(p,q)\le\rho_f(p,q)<\infty$ and $0<\lambda_g(p,q)\le\rho_g(p,q)<\infty$ and $\lambda_g(m,n)>0$ where *p, q, m, n* are positive integer with *p* > *q* and *m* > *n*. Then for any positive integer *l*,

$$(i)\ \lim_{r\to\infty}\frac{\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)}{\log^{[p+1]}M(\exp^{[l]}r,f)}=\infty\ \ if\ \ l\le q<m;$$

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)}{\log^{[p+1]}M(\exp^{[l]}r,f)}\ge\lambda_f(p,q)\lambda_g(m,n)\ \ if\ \ l\le q=m\ and\ even\, i;$$

$$(iii)\ \limsup_{r\to\infty}\frac{\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)}{\log^{[p+1]}M(\exp^{[l]}r,f)}\ge\lambda_g(p,q)\lambda_f(m,n)\ \ if\ \ l\le q=m\ and\ odd\, i;$$

$$(iv)\ \lim_{r\to\infty}\frac{\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)}{\log^{[p-q+l+1]}M(\exp^{[l]}r,f)}=\infty\ \ if\ \ q<m\ and\ q<l;$$

$$(v)\ \limsup_{r\to\infty}\frac{\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)}{\log^{[p-q+l+1]}M(\exp^{[l]}r,f)}\ge\lambda_f(p,q)\lambda_g(m,n)\ \ if\ \ l>q=m\ and\ even\, i;$$

$$(vi)\ \limsup_{r\to\infty}\frac{\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)}{\log^{[p-q+l+1]}M(\exp^{[l]}r,f)}\ge\lambda_g(p,q)\lambda_f(m,n)\ \ if\ \ l>q=m\ and\ odd\, i;$$

$$(vii)\ \lim_{r\to\infty}\frac{\log^{[(i-1)(p-q)+m-1]}M(\exp^{[n-1]}r,f_i)}{\log^{[p+1]}M(\exp^{[l]}r,f)}=\infty\ \ if\ \ q>m\ and\ q\ge l;$$

$$(viii)\ \lim_{r\to\infty}\frac{\log^{[(i-1)(p-q)+m-1]}M(\exp^{[n-1]}r,f_i)}{\log^{[p-q+l+1]}M(\exp^{[l]}r,f)}=\infty\ \ if\ l>q>m.$$

Proof. From Lemma 2.5 we have for sufficiently large values of *r* and

$$0<\varepsilon<\varepsilon'=\min\left\{\frac{1}{2}(\lambda_k(c,d)-\rho_g(m,n)),\ \frac{1}{2}(\lambda_h(c,d)-\rho_f(m,n))\right\}$$

$$\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)\ge\begin{cases}(\lambda_f(p,q)-\varepsilon)\log^{[q]}M\left(\dfrac{\exp^{[n-1]}r}{2^{n-1}},g\right)+O(1) & when\ i\ is\, even,\\ (\lambda_g(p,q)-\varepsilon)\log^{[q]}M\left(\dfrac{\exp^{[n-1]}r}{2^{n-1}},f\right)+O(1) & when\ i\ is\, odd.\end{cases}\tag{3.15}$$

Case-I. Let $q\le m$ then for sufficiently large values of *r*,

$$\begin{aligned}\log^{[q]}M\left(\frac{\exp^{[n-1]}r}{2^{n-1}},g\right)&\ge\exp^{[m-q]}\left[(\lambda_g(m,n)-\varepsilon)\log^{[n]}\left(\frac{\exp^{[n-1]}r}{2^{n-1}}\right)\right]\\&\ge\exp^{[m-q]}\left[(\lambda_g(m,n)-\varepsilon)\log r\right]+O(1)\\&\ge\exp^{[m-q-1]}r^{(\lambda_g(m,n)-\varepsilon)}+O(1).\end{aligned}\tag{3.16}$$

When *i* is even then from (3.15) and (3.16) we have for all large values of *r* and $\varepsilon(0<\varepsilon<\varepsilon')$

$$\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)\ge(\lambda_f(p,q)-\varepsilon)\exp^{[m-q-1]}r^{(\lambda_g(m,n)-\varepsilon)}+O(1).\tag{3.17}$$

Similarly for odd *i*,

$$\log^{[(i-1)p-(i-2)q]}M(\exp^{[n-1]}r,f_i)\ge(\lambda_g(p,q)-\varepsilon)\exp^{[m-q-1]}r^{(\lambda_f(m,n)-\varepsilon)}+O(1).\tag{3.18}$$

Case-II. Let *q* > *m* then for sufficiently large values of *r*,

$$\log^{[q]} M\left(\frac{\exp^{[n-1]} r}{2^{n-1}}, g\right) \geq \log^{[q-m+1]} r^{(\lambda_g(m.n)-\varepsilon)} + O(1). \quad (3.19)$$

When i is even then from (3.15) and (3.19) we have for all large values of r and $\varepsilon(0 < \varepsilon < \varepsilon^{'})$

$$\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i) \geq (\lambda_f(p,q) - \varepsilon)\log^{[q-m+1]} r^{(\lambda_g(m,n)-\varepsilon)} + O(1)$$

$$i.e., \log^{[(i-1)p-(i-2)q+1]} M(\exp^{[n-1]} r, f_i) \geq \log^{[q-m+2]} r^{(\lambda_g(m,n)-\varepsilon)} + O(1)$$

$$i.e., \log^{[(i-1)(p-q)+m-1]} M(\exp^{[n-1]} r, f_i) \geq r^{(\lambda_g(m,n)-\varepsilon)} + O(1). \quad (3.20)$$

Similarly for odd i,

$$\log^{[(i-1)(p-q)+m-1]} M(\exp^{[n-1]} r, f_i) \geq r^{(\lambda_f(m,n)-\varepsilon)} + O(1). \quad (3.21)$$

Again from the Definitions 3.1 we get for all large values of r,

$$\log^{[p]} M(\exp^{[l]} r, f) \leq (\rho_f(p,q) + \varepsilon)\log^{[q]}\left(\exp^{[l]} r\right) + O(1).$$

Case-III. If $q \geq l$ then for all sufficiently large values of r

$$\log^{[p]} M(\exp^{[l]} r, f) \leq (\rho_f(p,q) + \varepsilon)\log^{[q]}\left(\exp^{[l]} r\right)$$

$$\leq (\rho_f(p,q) + \varepsilon) r$$

$$i.e., \log^{[p+1]} M(\exp^{[l]} r, f) \leq \log r + O(1). \quad (3.22)$$

Case-IV. If $q < l$ then for all sufficiently large values of r

$$\log^{[p]} M(\exp^{[l]} r, f) \leq (\rho_f(p,q) + \varepsilon)\exp^{[l-q]} r$$

$$i.e., \log^{[p+1]} M(\exp^{[l]} r, f) \leq \exp^{[l-q-1]} r + O(1)$$

$$i.e., \log^{[p-q+l+1]} M(\exp^{[l]} r, f) \leq \log r + O(1). \quad (3.23)$$

Now combining (3.17) of Case-I and (3.22) of Case-III it follows for all sufficiently large values of r,

$$\frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} \geq \frac{(\lambda_f(p,q) - \varepsilon)\exp^{[m-q-1]} r^{(\lambda_g(m,n)-\varepsilon)} + O(1)}{\log r + O(1)}.$$

If $q < m$ then

$$\liminf_{r\to\infty} \frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} = \infty$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} = \infty.$$

This is the first part of the theorem.
If $q = m$ and i is even then

$$\frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} \geq \frac{(\lambda_f(p,q) - \varepsilon)(\lambda_g(m.n) - \varepsilon)\log r + O(1)}{\log r + O(1)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{r\to\infty} \frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} \geq \lambda_f(p,q)\lambda_g(m,n).$$

Similarly for odd i;

$$\limsup_{r\to\infty} \frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} \geq \lambda_g(p,q)\lambda_f(m,n).$$

This proves the (ii) and (iii).
Again in view of (3.17) of Case-I and (3.23) of Case-IV it follows for all sufficiently large values of r,

$$\frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} \geq \frac{(\lambda_f(p,q) - \varepsilon)\exp^{[m-q-1]} r^{(\lambda_g(m,n)-\varepsilon)} + O(1)}{\log r + O(1)}.$$

If $q < m$ and $q < l$ then

$$\liminf_{r\to\infty} \frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} = \infty$$

$$i.e.,\ \lim_{r\to\infty} \frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} = \infty.$$

This is the fourth part of the theorem.
If $q = m$ and $q < l$ and i is even then

$$\frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} \geq \frac{(\lambda_f(p,q)-\varepsilon)(\lambda_g(m,n)-\varepsilon)\log r + O(1)}{\log r + O(1)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{r\to\infty} \frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} \geq \lambda_f(p,q)\lambda_g(m,n).$$

Similarly for odd i,

$$\limsup_{r\to\infty} \frac{\log^{[(i-1)p-(i-2)q]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} \geq \lambda_g(p,q)\lambda_f(m,n).$$

This proves the (v) and (vi).
Again in view of (3.20) of Case-II and (3.22) of Case-III it follows for all sufficiently large values of r,

$$\lim_{r\to\infty} \frac{\log^{[(i-1)(p-q)+m-1]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} \geq \frac{r^{(\lambda_g(m,n)-\varepsilon)} + O(1)}{\log r + O(1)}$$

$$i.e.,\ \liminf_{r\to\infty} \frac{\log^{[(i-1)(p-q)+m-1]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} = \infty$$

$$i.e.,\ \lim_{r\to\infty} \frac{\log^{[(i-1)(p-q)+m-1]} M(\exp^{[n-1]} r, f_i)}{\log^{[p+1]} M(\exp^{[l]} r, f)} = \infty.$$

This is the seventh part of the theorem.
Again in view of (3.20) of Case-II and (3.23) of Case-IV it follows for all sufficiently large values of r,

$$\lim_{r\to\infty} \frac{\log^{[(i-1)(p-q)+m-1]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} \geq \frac{r^{(\lambda_g(m.n)-\varepsilon)} + O(1)}{\log r + O(1)}$$

$$i.e.,\ \liminf_{r\to\infty} \frac{\log^{[(i-1)(p-q)+m-1]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} = \infty$$

$$i.e.,\ \lim_{r\to\infty} \frac{\log^{[(i-1)(p-q)+m-1]} M(\exp^{[n-1]} r, f_i)}{\log^{[p-q+l+1]} M(\exp^{[l]} r, f)} = \infty.$$

This is the last part of the theorem.
This proves the theorem.

4. References

[1] J. Clunie, The composition of entire and meromorphic functions, Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, 1970, 75-92.

[2] R. K. Dutta, Further growth of iterated entire functions-I, Journal of Mathematical Inequalities, 2011, 5(4): 533-550.

[3] S. K. Datta and T. Biswas, On the growth estimate of composite entire and meromorphic functions, Bulletin of Mathematical Analysis and Applications, 2010, 2(2): 1-17.

[4] S. K. Datta and T. Biswas, Some comparative growth rate of composite entire and meromorphic functions, Journal of Information and Computing Science, 2012, 7(2): 111-120.

[5] W. K. Hayman, Meromorphic Functions, Oxford University Press, 1964.

[6] O. P. JUNEJA, G. P. KAPOOR AND S. K. BAJPAI, On the (p,q)-order and lower (p,q)-order of an entire function, J. Reine Angew. Math. 282 (1976), 53-67.

[7] B. K. Lahiri and D. Banerjee, Relative fix points of entire functions, J. Indian Acad. Math., 1997, 19(1): 87-97.

[8] I. Lahiri, Growth of composite integral functions, Indian J. Pure Appl. Math., 1997, 20(9): 899-907.

[9] I. Lahiri and D. K. Sharma, Growth of composite entire and meromorphic functions, Indian J. Pure Appl. Math., 1995, 26(5): 451-458.

[10] K. Niino and C. C. Yang, Some growth relationships on factors of two composite entire functions, factorization theory of meromorphic functions and related topics, Marcel Dekker Inc. (New York and Basel). (1982) 95-99.

[11] D. Sato, On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., 69 (1963), 411-414.

[12] A. P. Singh, Growth of composite entire functions, Kodai Math. J., 8 (1985), 99-102.

[13] A. P. Singh and M. S. Baloria, On maximum modulus and maximum term of composition of entire functions, Indian J. Pure Appl. Math., 22(12) (1991), 1019-1026.

[14] G. D. Song and C. C. Yang, Further growth properties of composition of entire meromorphic functions, Indian J. Pure Appl. Math., 15(1) (1984), 67-82.

[15] G. Valiron, Lectures on the general theory of Integral functions, Chelsea Publishing Company, 1949.

[16] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers and Science Press, Beijing, 2003