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Intuitionistic Fuzzy Linear and Quadratic Equations

Sanhita Banerjee, Tapan Kumar Roy
[1,2] Department of Mathematics, Indian Institute of Engineering Science and Technology
Shibpur, Howrah-711103, West Bengal, India
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Abstract. In this paper we have considered simple linear and quadratic equations in Intuitionistic Fuzzy Environment. Here coefficients of the equations are taken as Generalized Triangular Intuitionistic Fuzzy Numbers. We have used the strong and weak solution concept to solve these Intuitionistic Fuzzy Equations. Solution procedures have shown in details.

Keywords: Inuitionistic Fuzzy Linear Equations, Inuitionistic Fuzzy Quadratic Equations, Generalized Triangular Intuitionistic Fuzzy Number (GTIFN), strong and weak solutions.

1. Introduction

Atanassov[1] introduced the concept of Intuitionistic Fuzzy Sets (IFS), which is a generalization of the concept of fuzzy set. In uncertain environments, the modeled equation can be an Intuitionistic Fuzzy Equation. The solution for Intuitionistic Fuzzy Equations plays an important role in uncertain decision making.

In literature, standard analytical techniques are proposed by Buckley and Qu [7,8,11]. Buckley [6,7] considered the solution of linear fuzzy equations using Classical methods, Zadeh's extension principle and the concept of fuzzy numbers and arithmetic operations on it introduced by Zadeh [14,15]. But we see that the solutions of Zadeh's extension principle method and interval arithmetic methods do not satisfy the given fuzzy equations always. In our previous paper [19] we have used another approach (strong and weak solution concept) to solve fuzzy linear and quadratic equations.

In this paper we have solved Intuitionistic Fuzzy Linear and Quadratic Equations by using the concept of strong and weak solution and coefficients are taken as a Generalized Triangular Intuitionistic Fuzzy Numbers. We have also solved two problems following Buckley,Qu[7] using this concept in this paper.

2. Preliminary concepts

Definition-2.1: Intuitionistic Fuzzy Sets: Let $U = \{x_1, x_2, \ldots, x_n\}$ be a finite universal set. An Intuitionistic Fuzzy Set $\tilde{A}^i$ in a given universal set U is an object having the form

$$\tilde{A}^i = \{\langle x_i, \mu_{\tilde{A}^i}(x_i), \upsilon_{\tilde{A}^i}(x_i)\rangle : x_i \in U\}$$

Where the functions
$\mu_{\tilde{A}^i}: U \to [0,1]$; i.e. , $x_i \in U \to \mu_{\tilde{A}^i}(x_i) \in [0,1]$
and $\upsilon_{\tilde{A}^i}: U \to [0,1]$; i.e. , $x_i \in U \to \upsilon_{\tilde{A}^i}(x_i) \in [0,1]$
define the degree of membership and the degree of non-membership of an element $x_i \in U$, such that they satisfy the following conditions:

$$0 \le \mu_{\tilde{A}^i}(x_i) + \upsilon_{\tilde{A}^i}(x_i) \le 1, \forall x_i \in U$$

which is known as Intuitionistic Condition. The degree of acceptance $\mu_{\tilde{A}^i}(x_i)$and of non-acceptance $\upsilon_{\tilde{A}^i}(x_i)$ can be arbitrary.

Definition-2.2: $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-cuts: A set of (α, β)-cut, generated by IFS $\tilde{A}^i$, where $\alpha, \beta \in [0,1]$ are fixed numbers such that $\alpha + \beta \le 1$ is defined as

$$\tilde{A}^i{}_{\alpha,\beta} = \begin{cases} \big(x, \mu_{\tilde{A}^i}(x), \upsilon_{\tilde{A}^i}(x)\big); & x \in U \\ \mu_{\tilde{A}^i}(x) \ge \alpha, \upsilon_{\tilde{A}^i}(x) \le \beta; & \alpha, \beta \in [0,1] \end{cases},$$

where (α, β)-cut, denoted by $\tilde{A}^i{}_{\alpha,\beta}$, is defined as the crisp set of elements x which belong to $\tilde{A}^i$ at least to the degree α and which does belong to $\tilde{A}^i$ at most to the degree β.

Definition-2.3: Intuitionistic Fuzzy Number: An Intuitionistic Fuuzy Number $\tilde{A}^i$ is

i. An Intuitionistic Fuuzy Subset on the real line

ii. Normal i.e. there exists $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}^i}(x_0) = 1$(so $\upsilon_{\tilde{A}^i}(x_0) = 0$)

iii. Convex for the membership function $\mu_{\tilde{A}^i}$ i.e.

$$\mu_{\tilde{A}^i}(\lambda x_1 + (1-\lambda x_2)) \geq \min\{\mu_{\tilde{A}^i}(x_1), \mu_{\tilde{A}^i}(x_2)\}; \forall\, x_1, x_2 \in \mathbb{R}\ , \lambda \in [0,1]$$

iv. Concave for the non-membership function $\upsilon_{\tilde{A}^i}$ i.e.

$$\upsilon_{\tilde{A}^i}(\lambda x_1 + (1-\lambda x_2)) \leq \max\{\upsilon_{\tilde{A}^i}(x_1), \upsilon_{\tilde{A}^i}(x_2)\}; \forall\, x_1, x_2 \in \mathbb{R}\ , \lambda \in [0,1]$$

Definition-2.4: Generalized Triangular Intuitionistic Fuzzy Number: A Generalized Triangular Intuitionistic Fuzzy Number (GTIFN) is denoted by $\tilde{A}^i = (a_1, a_2, a_3; w, u)$ is a special Intuitionistic Fuzzy Set on a real number set $\mathbb{R}$, whose membership function and non-membership functions are defined as

$$\mu_{\tilde{A}^i}(x) = \begin{cases} 0 & x \leq a_1 \\ w\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ w\ , & x = a_2 \\ w\frac{a_3-x}{a_3-a_2}\ , & a_2 \leq x \leq a_3 \\ 0\ , & a_3 \leq x \end{cases}, \quad \upsilon_{\tilde{A}^i}(x) = \begin{cases} 1 & x \leq a_1 \\ 1-(1-u)\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ u\ , & x = a_2 \\ 1-(1-u)\frac{a_3-x}{a_3-a_2}\ , & a_2 \leq x \leq a_3 \\ 1\ , & a_3 \leq x \end{cases}$$

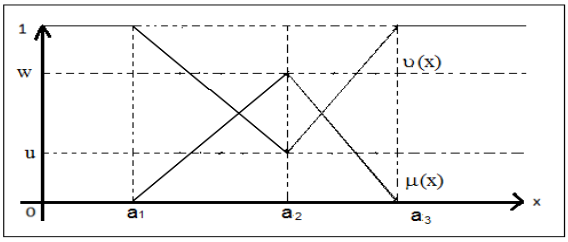


Fig-2.1: Rough sketch of membership and non-membership functions of the above GTFN

where w and u represent the maximum degree of membership and minimum degree of non-membership respectively such that they satisfy the condition

$$0 \leq w \leq 1, 0 \leq u \leq 1 \text{ and } 0 \leq w + u \leq 1$$

Note: GTIFN $(a_1, a_2, a_3; w, u) \xrightarrow{w=1-u}$ GTFN $(a_1, a_2, a_3; w) \xrightarrow{w=1,\ u=0}$ TFN (a_1, a_2, a_3)

Definition 2.5: Equality of two GTIFNs: Two GTIFNs $\tilde{A}^i = (a_1, a_2, a_3; w_1, u_1)$ and $\tilde{B}^i = (b_1, b_2, b_3; w_2, u_2)$ are equal when $a_1 = b_1, a_2 = b_2,, a_3 = b_3$ and $w_1 = w_2, u_1 = u_2$.

Definition2.6: Let $\tilde{A}^i$ =$(a_1, a_2, a_3; w_a, u_a)$ and $\tilde{B}^i$ =$(b_1, b_2, b_3; w_b, u_b)$ be two positive GTIFNs.
Let $w = min(w_a, w_b)$ and $u = max(u_a, u_b)$where $0 < w, u \leq 1$and $0 < w + u \leq 1$

(i) The addition of two GTIFNs $\tilde{A}^i, \tilde{B}^i$ is another GTIFN

$$\tilde{C}^i = (a_1 + b_1, a_2 + b_2, a_3 + b_3; w, u)$$

(ii) The subtraction of two GTIFNs $\tilde{A}^i, \tilde{B}^i$is another GTIFN

$$\tilde{C}^i = (a_1 - b_3, a_2 - b_2, a_3 - b_1; w, u)$$

(iii) The multiplication of two GTIFNs $\tilde{A}^i, \tilde{B}^i$ is a Generalized Triangular Shaped Intuitionistic Fuzzy Number (GTsIFN) $\tilde{C}^i \approx (a_1 b_1, a_2 b_2, a_3 b_3; w, u)$

(iv) The division of two GTIFNs $\tilde{A}^i, \tilde{B}^i$ is a GTsIFN $\tilde{C}^i \approx \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}; w, u\right)$

Definition 2.7: A Generalized Intuitionistic Fuzzy Number is completely determined by the pair $\langle [X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)], [X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)] \rangle$ of functions $X_{\mu_L}(\alpha), X_{\mu_R}(\alpha), X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)$,
$0 \leq \alpha \leq w, u \leq \beta \leq 1$ and $0 \leq \alpha + \beta \leq 1$ which satisfy the following requirements:

1. $X_{\mu_L}(\alpha)$ is a bounded monotonic increasing left continuous function over $[0, w]$.

$$\text{i.e.} \frac{d}{d\alpha}[X_{\mu_L}(\alpha)] > 0$$

2. $X_{\mu_R}(\alpha)$ is a bounded monotonic decreasing left continuous function over $[0, w]$.

$$\text{i.e.} \frac{d}{d\alpha}[X_{\mu_R}(\alpha)] < 0$$

3. $X_{\upsilon_L}(\beta)$ is a bounded monotonic decreasing left continuous function over $[u,1]$.
   i.e. $\frac{d}{d\beta}\left[X_{\upsilon_L}(\beta)\right] < 0$
4. $X_{\upsilon_R}(\beta)$ is a bounded monotonic increasing left continuous function over $[u,1]$.
   i.e. $\frac{d}{d\beta}\left[X_{\upsilon_R}(\beta)\right] > 0$
5. $X_{\mu_L}(w) \le X_{\mu_R}(w)$ and $X_{\upsilon_L}(u) \le X_{\upsilon_R}(u)$

Definition 2.8: Intuitionistic Fuzzy Linear Equation (IFLE):
These are the equations in which coefficients and unknowns are Intuitionistic Fuzzy Numbers, and formulas are constructed by operations of Intuitionistic Fuzzy Arithmetic. These equations are of three very simple types: $\tilde{A}^i\tilde{X}^i = \tilde{B}^i$, $\tilde{X}^i + \tilde{A}^i = \tilde{B}^i$ and $\tilde{A}^i\tilde{X}^i + \tilde{B}^i = \tilde{C}^i$ where $\tilde{A}^i, \tilde{B}^i, \tilde{C}^i$ are Intuitionistic Fuzzy Numbers, and $\tilde{X}^i$ is an unknown Intuitionistic Fuzzy Number for which either of the equations is to be satisfied.
Definition 2.9: Intuitionistic Fuzzy Quadratic Equation (IFQE):
Intuitionistic Fuzzy Quadratic Equations are of three very simple types: $\tilde{A}^i{\tilde{X}^{i}}^2 = \tilde{B}^i$, ${\tilde{X}^{i}}^2 + \tilde{A}^i = \tilde{B}^i$ and $\tilde{A}^i{\tilde{X}^{i}}^2 + \tilde{B}^i\tilde{X}^i = \tilde{C}^i$ where $\tilde{A}^i, \tilde{B}^i, \tilde{C}^i$ are Intuitionistic Fuzzy Numbers, and $\tilde{X}^i$ is an unknown Intuitionistic Fuzzy Number for which either of the equations is to be satisfied.

Definition 2.10: Strong and Weak solution of an Intuitionistic Fuzzy Equation:

Let the solution of any Intuitionistic Fuzzy Equation be $\tilde{X}^i$ and its (α,β) -cut be $X^i_{\alpha,\beta} = \langle [X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)], [X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)] \rangle$.If $X_{\mu_L}(\alpha) \le X_{\mu_R}(\alpha)$ and $X_{\upsilon_L}(\beta) \le X_{\upsilon_R}(\beta)$ then $\tilde{X}^i$ is called strong solution of the given Intuitionistic Fuzzy Equation otherwise $\tilde{X}^i$ is called weak solution.

3. Solution methods for Intuitionistic Fuzzy Equations

In this section we have discussed the solution method of Intuitionistic Fuzzy Linear and Quadratic Equations by considering coefficients as GTIFNs.

3.1 Solution procedure of $\tilde{A}^i\tilde{X}^i = \tilde{B}^i$

Consider the equation $\tilde{A}^i\tilde{X}^i = \tilde{B}^i$ …………(3.1.1)

where $\tilde{A}^i = (a_1, a_2, a_3; w_a, u_a), \tilde{B}^i = (b_1, b_2, b_3; w_b, u_b)$ are GTIFNs.
Also the (α,β)-cuts of $\tilde{A}^i, \tilde{B}^i$ are

$$A^i_{\alpha,\beta} = \langle A^i_\alpha, A^i_\beta \rangle = \langle [A_{\mu_L}(\alpha), A_{\mu_R}(\alpha)], [A_{\upsilon_L}(\beta), A_{\upsilon_R}(\beta)] \rangle$$
$$B^i_{\alpha,\beta} = \langle B^i_\alpha, B^i_\beta \rangle = \langle [B_{\mu_L}(\alpha), B_{\mu_R}(\alpha)], [B_{\upsilon_L}(\beta), B_{\upsilon_R}(\beta)] \rangle$$

where $[A_{\mu_L}(\alpha), A_{\mu_R}(\alpha)] = \left[a_1 + \frac{\alpha}{w_a}(a_2 - a_1), a_3 - \frac{\alpha}{w_a}(a_3 - a_2)\right]$

$$[B_{\mu_L}(\alpha), B_{\mu_R}(\alpha)] = \left[b_1 + \frac{\alpha}{w_b}(b_2 - b_1), b_3 - \frac{\alpha}{w_b}(b_3 - b_2)\right]$$
$$[A_{\upsilon_L}(\beta), A_{\upsilon_R}(\beta)] = \left[a_1 + \frac{(1-\beta)}{(1-u_a)}(a_2 - a_1), a_3 - \frac{(1-\beta)}{(1-u_a)}(a_3 - a_2)\right]$$
$$[B_{\upsilon_L}(\beta), B_{\upsilon_R}(\beta)] = \left[b_1 + \frac{(1-\beta)}{(1-u_b)}(b_2 - b_1), b_3 - \frac{(1-\beta)}{(1-u_b)}(b_3 - b_2)\right]$$

Let $w = \min(w_a, w_b)$ and $u = \max(u_a, u_b)$

Let $X^i_{\alpha,\beta} = \langle [X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)], [X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)] \rangle$ be the (α,β)-cut of the strong solution.

$\therefore$ The equation (3.1.1) becomes $A^i_{\alpha,\beta}X^i_{\alpha,\beta} = B^i_{\alpha,\beta}$ (3.1.2)

or,

$$\langle [A_{\mu_L}(\alpha), A_{\mu_R}(\alpha)], [A_{\upsilon_L}(\beta), A_{\upsilon_R}(\beta)] \rangle \langle [X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)], [X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)] \rangle = \langle [B_{\mu_L}(\alpha), B_{\mu_R}(\alpha)], [B_{\upsilon_L}(\beta), B_{\upsilon_R}(\beta)] \rangle$$

or,

$$[A_{\mu_L}(\alpha), A_{\mu_R}(\alpha)][X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)] = [B_{\mu_L}(\alpha), B_{\mu_R}(\alpha)] \quad (3.1.3)$$
$$\text{and } [A_{\upsilon_L}(\beta), A_{\upsilon_R}(\beta)][X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)] = [B_{\upsilon_L}(\beta), B_{\upsilon_R}(\beta)] \quad (3.1.4)$$

Now if $\tilde{A}^i, \tilde{B}^i$ are both positive GTIFNs, then $\tilde{X}^i$ is positive and (3.1.3) becomes

$$\left[a_1+\frac{\alpha}{w}(a_2-a_1),a_3-\frac{\alpha}{w}(a_3-a_2)\right]\left[X_{\mu_L}(\alpha),X_{\mu_R}(\alpha)\right]=\left[b_1+\frac{\alpha}{w}(b_2-b_1),b_3-\frac{\alpha}{w}(b_3-b_2)\right]$$
$$\Rightarrow\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}X_{\mu_L}(\alpha)=b_1+\frac{\alpha}{w}(b_2-b_1)$$

and,

$$\left\{a_3-\frac{\alpha}{w}(a_3-a_2)\right\}X_{\mu_R}(\alpha)=b_3-\frac{\alpha}{w}(b_3-b_2)$$

or,

$$X_{\mu_L}(\alpha)=\frac{\left\{b_1+\frac{\alpha}{w}(b_2-b_1)\right\}}{\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}},\quad X_{\mu_R}(\alpha)=\frac{\left\{b_3-\frac{\alpha}{w}(b_3-b_2)\right\}}{\left\{a_3-\frac{\alpha}{w}(a_3-a_2)\right\}}$$

and from (3.1.4) we get,

$$\left[a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1),a_3-\frac{(1-\beta)}{(1-u)}(a_3-a_2)\right]\left[X_{\upsilon_L}(\beta),X_{\upsilon_R}(\beta)\right]$$
$$=\left[b_1+\frac{(1-\beta)}{(1-u)}(b_2-b_1),b_3-\frac{(1-\beta)}{(1-u)}(b_3-b_2)\right]$$
$$\Rightarrow\left\{a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1)\right\}X_{\upsilon_L}(\beta)=b_1+\frac{(1-\beta)}{(1-u)}(b_2-b_1)$$

and,

$$\left\{a_3-\frac{(1-\beta)}{(1-u)}(a_3-a_2)\right\}X_{\upsilon_R}(\beta)=b_3-\frac{(1-\beta)}{(1-u)}(b_3-b_2)$$

or,

$$X_{\upsilon_L}(\beta)=\frac{\left\{b_1+\frac{(1-\beta)}{(1-u)}(b_2-b_1)\right\}}{\left\{a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1)\right\}},\quad X_{\upsilon_R}(\beta)=\frac{\left\{b_3-\frac{(1-\beta)}{(1-u)}(b_3-b_2)\right\}}{\left\{a_3-\frac{(1-\beta)}{(1-u)}(a_3-a_2)\right\}}$$

Now $\frac{d}{d\alpha}\left[X_{\mu_L}(\alpha)\right]=\frac{a_1b_2-a_2b_1}{w\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}^2}>0\Rightarrow a_1b_2>a_2b_1$,

$$\frac{d}{d\alpha}\left[X_{\mu_R}(\alpha)\right]=\frac{a_3b_2-a_2b_3}{w\left\{a_3-\frac{\alpha}{w}(a_3-a_2)\right\}^2}<0\Rightarrow a_3b_2<a_2b_3 \quad \text{and } X_{\mu_L}(w)=X_{\mu_R}(w)$$
$$\frac{d}{d\beta}\left[X_{\upsilon_L}(\beta)\right]=\frac{a_2b_1-a_1b_2}{(1-u)\left\{a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1)\right\}^2}<0\Rightarrow a_1b_2>a_2b_1 \ ,$$
$$\frac{d}{d\beta}\left[X_{\upsilon_R}(\beta)\right]=\frac{a_2b_3-a_3b_2}{(1-u)\left\{a_3-\frac{(1-\beta)}{(1-u)}(a_3-a_2)\right\}^2}>0\Rightarrow a_3b_2<a_2b_3 \quad \text{and } X_{\upsilon_L}(u)=X_{\upsilon_R}(u)$$

Hence in this case strong solution exists if $a_1b_2>a_2b_1$, $a_3b_2<a_2b_3$.

∴ The solution of the equation $\tilde{A}^i\tilde{X}^i=\tilde{B}^i$ is given by $\tilde{X}^i$ with its (α,β)-cut

$$X^i_{\alpha,\beta}=\langle\left[X_{\mu_L}(\alpha),X_{\mu_R}(\alpha)\right],\left[X_{\upsilon_L}(\beta),X_{\upsilon_R}(\beta)\right]\rangle$$

$$\text{where }\left[X_{\mu_L}(\alpha),X_{\mu_R}(\alpha)\right]=\left\{x:\frac{\left\{b_1+\frac{\alpha}{w}(b_2-b_1)\right\}}{\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}}\le x\le\frac{\left\{b_3-\frac{\alpha}{w}(b_3-b_2)\right\}}{\left\{a_3-\frac{\alpha}{w}(a_3-a_2)\right\}}\right\} \tag{3.1.5}$$

$X_{\mu_L}(0)=\frac{b_1}{a_1},X_{\mu_L}(w)=\frac{b_2}{a_2},X_{\mu_R}(w)=\frac{b_2}{a_2},X_{\mu_R}(0)=\frac{b_3}{a_3}$

$$\text{and }\left[X_{\upsilon_L}(\beta),X_{\upsilon_R}(\beta)\right]=\left\{x:\frac{\left\{b_1+\frac{(1-\beta)}{(1-u)}(b_2-b_1)\right\}}{\left\{a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1)\right\}}\le x\le\frac{\left\{b_3-\frac{(1-\beta)}{(1-u)}(b_3-b_2)\right\}}{\left\{a_3-\frac{(1-\beta)}{(1-u)}(a_3-a_2)\right\}}\right\} \tag{3.1.6}$$

$X_{\upsilon_L}(1)=\frac{b_1}{a_1},X_{\upsilon_L}(u)=\frac{b_2}{a_2},X_{\upsilon_R}(u)=\frac{b_2}{a_2},X_{\upsilon_R}(1)=\frac{b_3}{a_3}$

Now from (3.1.5) we get

$$\frac{\left\{b_1+\frac{\alpha}{w}(b_2-b_1)\right\}}{\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}}\le x\Rightarrow w\frac{a_1x-b_1}{(b_2-b_1)-(a_2-a_1)x}\ge\alpha \text{ for } x\in\left[\frac{b_1}{a_1},\frac{b_2}{a_2}\right]$$
$$\Rightarrow\mu^L_{\tilde{X}^i}(x)\ge\alpha$$
$$\frac{d}{dx}\left[\mu^L_{\tilde{X}^i}(x)\right]=w\frac{a_1b_2-a_2b_1}{\{(b_2-b_1)-x(a_2-a_1)\}^2}>0 \quad [\because a_1b_2>a_2b_1]$$

Again,

$$x\le\frac{\left\{b_3-\frac{\alpha}{w}(b_3-b_2)\right\}}{\left\{a_3-\frac{\alpha}{w}(a_3-a_2)\right\}}\Rightarrow w\frac{b_3-a_3x}{(b_3-b_2)-(a_3-a_2)x}\ge\alpha \text{ for } x\in\left[\frac{b_2}{a_2},\frac{b_3}{a_3}\right]$$
$$\Rightarrow\mu^R_{\tilde{X}^i}(x)\ge\alpha$$
$$\frac{d}{dx}\left[\mu^R_{\tilde{X}^i}(x)\right]=w\frac{a_3b_2-a_2b_3}{\{(b_3-b_2)-(a_3-a_2)x\}^2}<0 \quad [\because a_3b_2<a_2b_3]$$
$$\mu^L_{\tilde{X}^i}\left(\frac{b_2}{a_2}\right)=\ w=\mu^R_{\tilde{X}^i}\left(\frac{b_2}{a_2}\right),\ \mu^L_{\tilde{X}^i}\left(\frac{b_1}{a_1}\right)=0\ ,\mu^R_{\tilde{X}^i}\left(\frac{b_3}{a_3}\right)=0$$
$$\mu^L_{\tilde{X}^i}\left(\frac{\frac{b_1}{a_1}+\frac{b_2}{a_2}}{2}\right)=\frac{wa_1}{a_1+a_2}<\frac{w}{2} \text{ and } \mu^R_{\tilde{X}^i}\left(\frac{\frac{b_3}{a_3}+\frac{b_2}{a_2}}{2}\right)=\frac{wa_3}{a_3+a_2}>\frac{w}{2}$$

and from (3.1.6) we get,

$$\frac{\left\{b_1+\frac{(1-\beta)}{(1-u)}(b_2-b_1)\right\}}{\left\{a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1)\right\}} \le x \Rightarrow 1-\frac{(a_1x-b_1)(1-u)}{(b_2-b_1)-(a_2-a_1)x} \le \beta \text{ for } x \in \left[\frac{b_1}{a_1},\frac{b_2}{a_2}\right]$$

$$\Rightarrow \upsilon^L_{\tilde{X}^i}(x) \le \beta$$

$$\frac{d}{dx}\left[\upsilon^L_{\tilde{X}^i}(x)\right] = (1-u)\frac{a_2b_1-a_1b_2}{\{(b_2-b_1)-x(a_2-a_1)\}^2} < 0 \quad [\because a_1b_2 > a_2b_1]$$

Again,

$$x \le \frac{\left\{b_3-\frac{(1-\beta)}{(1-u)}(b_3-b_2)\right\}}{\left\{a_3-\frac{(1-\beta)}{(1-u)}(a_3-a_2)\right\}} \Rightarrow 1-\frac{b_3-a_3x}{(b_3-b_2)-(a_3-a_2)x} \le \beta \text{ for } x \in \left[\frac{b_2}{a_2},\frac{b_3}{a_3}\right]$$

$$\Rightarrow \upsilon^R_{\tilde{X}^i}(x) \le \beta$$

$$\frac{d}{dx}\left[\upsilon^R_{\tilde{X}^i}(x)\right] = (1-u)\frac{a_2b_3-a_3b_2}{\{(b_3-b_2)-(a_3-a_2)x\}^2} > 0 \quad [\because a_3b_2 < a_2b_3]$$

$$\upsilon^L_{\tilde{X}^i}\left(\frac{b_2}{a_2}\right) = u = \upsilon^R_{\tilde{X}^i}\left(\frac{b_2}{a_2}\right), \upsilon^L_{\tilde{X}^i}\left(\frac{b_1}{a_1}\right) = 1 \;, \upsilon^R_{\tilde{X}^i}\left(\frac{b_3}{a_3}\right) = 1$$

$$\upsilon^L_{\tilde{X}^i}\left(\frac{\frac{b_1}{a_1}+\frac{b_2}{a_2}}{2}\right) = \frac{wa_1}{a_1+a_2} > \frac{1+u}{2} \text{ and } \upsilon^R_{\tilde{X}^i}\left(\frac{\frac{b_3}{a_3}+\frac{b_2}{a_2}}{2}\right) = \frac{wa_3}{a_3+a_2} < \frac{1+u}{2}$$

∴The solution $\tilde{X}^i \approx \left(\frac{b_1}{a_1},\frac{b_2}{a_2},\frac{b_3}{a_3};w,u\right)$ is a GTsIFN with membership function

$$\mu_{\tilde{X}^i}(x) = \begin{cases} 0 & x \le \frac{b_1}{a_1} \\ w\frac{a_1x-b_1}{(b_2-b_1)-(a_2-a_1)x}, & \frac{b_1}{a_1} \le x \le \frac{b_2}{a_2} \\ w \;, & x = \frac{b_2}{a_2} \\ w\frac{b_3-a_3x}{(b_3-b_2)-(a_3-a_2)x} \;, & \frac{b_2}{a_2} \le x \le \frac{b_3}{a_3} \\ 0 \;, & \frac{b_3}{a_3} \le x \end{cases}$$

and non membership function

$$\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1\,, & x \le \frac{b_1}{a_1} \\ 1-\frac{(a_1x-b_1)(1-u)}{(b_2-b_1)-(a_2-a_1)x}, & \frac{b_1}{a_1} \le x \le \frac{b_2}{a_2} \\ u \;, & x = \frac{b_2}{a_2} \\ 1-\frac{(b_3-a_3x)(1-u)}{(b_3-b_2)-(a_3-a_2)x} \;, & \frac{b_2}{a_2} \le x \le \frac{b_3}{a_3} \\ 1\,, & \frac{b_3}{a_3} \le x \end{cases}$$

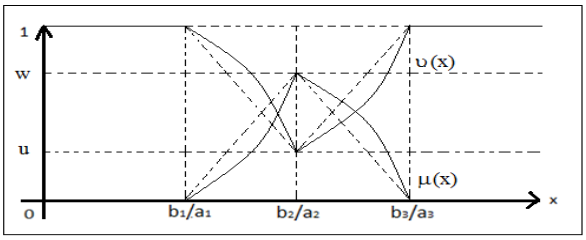
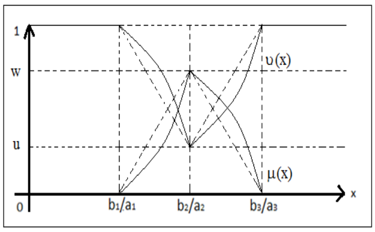
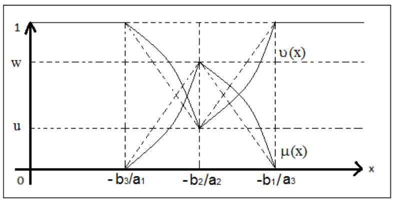


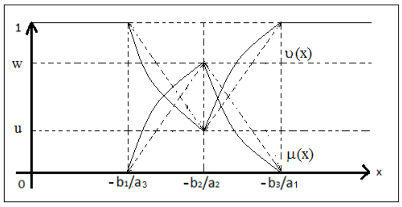
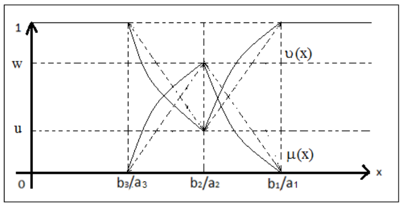
Fig3.1-:-Rough sketch of membership and non-membership function of $\tilde{X}^i$

We have solved the equation $\tilde{A}^i\tilde{X}^i = \tilde{B}^i$ for all other cases and the results are given in table-3.1.

Table-3.1:-Strong solution of $\tilde{A}^i\tilde{X}^i = \tilde{B}^i$ for different $\tilde{A}^i, \tilde{B}^i$

Nature of $\tilde{A}^i$ and $\tilde{B}^i$	Conditions for existence of strong	Membership function $\mu_{\tilde{X}^i}(x)$ and non-membership function $\upsilon_{\tilde{X}^i}(x)$	Rough sketch of membership and non-membership function of $\tilde{X}^i$

	solution $\tilde{X}^i$		
$\tilde{A}^i > 0$, $\tilde{B}^i > 0$	$\frac{b_1}{a_1} < \frac{b_2}{a_2} < \frac{b_3}{a_3}$	$\mu_{\tilde{X}^i}(x) = \begin{cases} 0 & x \le \frac{b_1}{a_1} \\ w\frac{a_1x-b_1}{(b_2-b_1)-(a_2-a_1)x}, & \frac{b_1}{a_1} \le x \le \frac{b_2}{a_2} \\ w, & x = \frac{b_2}{a_2} \\ w\frac{b_3-a_3x}{(b_3-b_2)-(a_3-a_2)x}, & \frac{b_2}{a_2} \le x \le \frac{b_3}{a_3} \\ 0, & \frac{b_3}{a_3} \le x \end{cases}$ $\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1, & x \le \frac{b_1}{a_1} \\ 1-\frac{(a_1x-b_1)(1-u)}{(b_2-b_1)-(a_2-a_1)x}, & \frac{b_1}{a_1} \le x \le \frac{b_2}{a_2} \\ u, & x = \frac{b_2}{a_2} \\ 1-\frac{(b_3-a_3x)(1-u)}{(b_3-b_2)-(a_3-a_2)x}, & \frac{b_2}{a_2} \le x \le \frac{b_3}{a_3} \\ 1, & \frac{b_3}{a_3} \le x \end{cases}$	
$\tilde{A}^i > 0$, $\tilde{B}^i < 0$	$-\frac{b_3}{a_1} < -\frac{b_2}{a_2} < -\frac{b_1}{a_3}$	$\mu_{\tilde{X}^i}(x) = \begin{cases} 0, & x \le -\frac{b_3}{a_1} \\ w\frac{b_3+a_1x}{(b_3-b_2)-(a_2-a_1)x}, & -\frac{b_3}{a_1} \le x \le -\frac{b_2}{a_2} \\ w, & x = -\frac{b_2}{a_2} \\ w\frac{-a_3x-b_1}{(b_2-b_1)-(a_3-a_2)x}, & -\frac{b_2}{a_2} \le x \le -\frac{b_1}{a_3} \\ 0, & -\frac{b_1}{a_3} \le x \end{cases}$ $\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1, & x \le -\frac{b_3}{a_1} \\ 1-\frac{(b_3+a_1x)(1-u)}{(b_3-b_2)-(a_2-a_1)x}, & -\frac{b_3}{a_1} \le x \le -\frac{b_2}{a_2} \\ u, & x = -\frac{b_2}{a_2} \\ 1-\frac{(-a_3x-b_1)(1-u)}{(b_2-b_1)-(a_3-a_2)x}, & -\frac{b_2}{a_2} \le x \le -\frac{b_1}{a_3} \\ 1, & -\frac{b_1}{a_3} \le x \end{cases}$	

$\tilde{A}^i < 0$, $\tilde{B}^i > 0$	$-\frac{b_1}{a_3} < -\frac{b_2}{a_2} < -\frac{b_3}{a_1}$	$\mu_{\tilde{X}^i}(x) = \begin{cases} 0\ , & x \le -\frac{b_1}{a_3} \\ w\frac{-a_3x - b_1}{(b_2 - b_1) - (a_3 - a_2)x}\ , & -\frac{b_1}{a_3} \le x \le -\frac{b_2}{a_2} \\ w\ , & x = -\frac{b_2}{a_2} \\ w\frac{b_3 + a_1x}{(b_3 - b_2) - (a_2 - a_1)x}\ , & -\frac{b_2}{a_2} \le x \le -\frac{b_3}{a_1} \\ 0\ , & -\frac{b_3}{a_1} \le x \end{cases}$ $\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1\ , & x \le -\frac{b_1}{a_3} \\ 1 - \frac{(-a_3x - b_1)(1-u)}{(b_2 - b_1) - (a_3 - a_2)x}\ , & -\frac{b_1}{a_3} \le x \le -\frac{b_2}{a_2} \\ u\ , & x = -\frac{b_2}{a_2} \\ 1 - \frac{(b_3 + a_1x)(1-u)}{(b_3 - b_2) - (a_2 - a_1)x}\ , & -\frac{b_2}{a_2} \le x \le -\frac{b_3}{a_1} \\ 1\ , & -\frac{b_3}{a_1} \le x \end{cases}$	
$\tilde{A}^i < 0$, $\tilde{B}^i < 0$	$\frac{b_3}{a_3} < \frac{b_2}{a_2} < \frac{b_1}{a_1}$	$\mu_{\tilde{X}^i}(x) = \begin{cases} 0\ , & x \le \frac{b_3}{a_3} \\ w\frac{a_3x - b_3}{(a_3 - a_2)x - (b_3 - b_2)}\ , & \frac{b_3}{a_3} \le x \le \frac{b_2}{a_2} \\ w\ , & x = \frac{b_2}{a_2} \\ w\frac{b_1 - a_1x}{(a_2 - a_1)x - (b_2 - b_1)}\ , & \frac{b_2}{a_2} \le x \le \frac{b_1}{a_1} \\ 0\ , & \frac{b_1}{a_1} \le x \end{cases}$ $\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1\ , & x \le \frac{b_3}{a_3} \\ 1 - \frac{(a_3x - b_3)(1-u)}{(a_3 - a_2)x - (b_3 - b_2)}\ , & \frac{b_3}{a_3} \le x \le \frac{b_2}{a_2} \\ u\ , & x = \frac{b_2}{a_2} \\ 1 - \frac{(b_1 - a_1x)(1-u)}{(a_2 - a_1)x - (b_2 - b_1)}\ , & \frac{b_2}{a_2} \le x \le \frac{b_1}{a_1} \\ 0\ , & \frac{b_1}{a_1} \le x \end{cases}$	

From the above table we see that in each case if the given conditions are satisfied then we get the **strong solution** otherwise we get the **weak solution**.

3.2 Solution procedure of $\tilde{X}^i + \tilde{A}^i = \tilde{B}^i$

Consider the equation

$$\tilde{X}^i + \tilde{A}^i = \tilde{B}^i \tag{3.2.1}$$

where $\tilde{A}^i = (a_1, a_2, a_3; w_a, u_a), \tilde{B}^i = (b_1, b_2, b_3; w_b, u_b)$ are positive GTIFNs.

We have

$$A^i_{\alpha,\beta} = \langle\left[a_1 + \frac{\alpha}{w_a}(a_2 - a_1), a_3 - \frac{\alpha}{w_a}(a_3 - a_2)\right], \left[a_1 + \frac{(1-\beta)}{(1-u_a)}(a_2 - a_1), a_3 - \frac{(1-\beta)}{(1-u_a)}(a_3 - a_2)\right]\rangle$$

$$B^i_{\alpha,\beta} = \langle\left[b_1 + \frac{\alpha}{w_b}(b_2 - b_1), b_3 - \frac{\alpha}{w_b}(b_3 - b_2)\right], \left[b_1 + \frac{(1-\beta)}{(1-u_b)}(b_2 - b_1), b_3 - \frac{(1-\beta)}{(1-u_b)}(b_3 - b_2)\right]\rangle$$

Let $w = \min(w_a, w_b)$ and $u = \max(u_a, u_b)$

Let $X^i_{\alpha,\beta} = \langle[X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)], [X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)]\rangle$ be the (α,β)-cut of the strong solution.

$\therefore$ The equation (3.2.1) becomes $X^i_{\alpha,\beta} + A^i_{\alpha,\beta} = B^i_{\alpha,\beta}$ (3.2.2)

Case1:- when $\tilde{A}^i > \tilde{B}^i$ then $\tilde{X}^i$ is negative.

The strong solution $\tilde{X}^i$ is a GTIFN $(b_1 - a_1, b_2 - a_2, b_3 - a_3; w, u)$ whose membership function is

$$\mu_{\tilde{X}^i}(x) = \begin{cases} 0, & x \le b_1 - a_1 \\ w\dfrac{x - (b_1 - a_1)}{(b_2 - b_1) - (a_2 - a_1)}, & b_1 - a_1 \le x \le b_2 - a_2 \\ w, & x = b_2 - a_2 \\ w\dfrac{(b_3 - a_3) - x}{(b_3 - b_2) - (a_3 - a_2)}, & b_2 - a_2 \le x \le b_3 - a_3 \\ 0, & b_3 - a_3 \le x \end{cases}$$

and non-membership function is

$$\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1, & x \le b_1 - a_1 \\ 1 - \dfrac{\{x - (b_1 - a_1)\}(1-u)}{(b_2 - b_1) - (a_2 - a_1)}, & b_1 - a_1 \le x \le b_2 - a_2 \\ u, & x = b_2 - a_2 \\ 1 - \dfrac{\{(b_3 - a_3) - x\}(1-u)}{(b_3 - b_2) - (a_3 - a_2)}, & b_2 - a_2 \le x \le b_3 - a_3 \\ 1, & b_3 - a_3 \le x \end{cases}$$

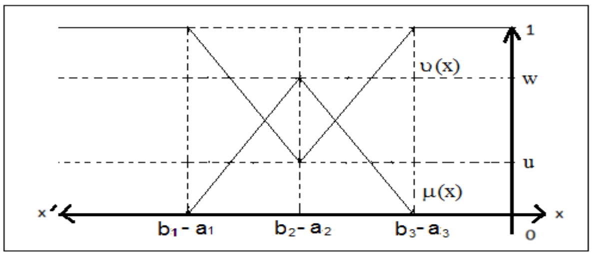


Fig-3.2:-Rough sketch of membership and non-membership function of $\tilde{X}^i$

Case2:- when $\tilde{A}^i < \tilde{B}^i$ then $\tilde{X}^i$ is positive.

The strong solution $\tilde{X}^i$ is a GTIFN $(b_1 - a_1, b_2 - a_2, b_3 - a_3; w, u)$ whose membership function is

$$\mu_{\tilde{X}^i}(x) = \begin{cases} 0, & x \le b_1 - a_1 \\ w\dfrac{x - (b_1 - a_1)}{(b_2 - b_1) - (a_2 - a_1)}, & b_1 - a_1 \le x \le b_2 - a_2 \\ w, & x = b_2 - a_2 \\ w\dfrac{(b_3 - a_3) - x}{(b_3 - b_2) - (a_3 - a_2)}, & b_2 - a_2 \le x \le b_3 - a_3 \\ 0, & b_3 - a_3 \le x \end{cases}$$

and non-membership function

$$\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1, & x \le b_1 - a_1 \\ 1 - \dfrac{\{x - (b_1 - a_1)\}(1-u)}{(b_2 - b_1) - (a_2 - a_1)}, & b_1 - a_1 \le x \le b_2 - a_2 \\ u, & x = b_2 - a_2 \\ 1 - \dfrac{\{(b_3 - a_3) - x\}(1-u)}{(b_3 - b_2) - (a_3 - a_2)}, & b_2 - a_2 \le x \le b_3 - a_3 \\ 1, & b_3 - a_3 \le x \end{cases}$$

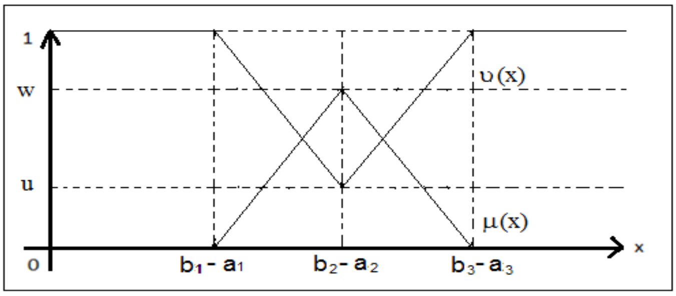


Fig-3.3:- Rough sketch of membership non-membership function of $\tilde{X}^i$

3.3 Solution procedure of $\tilde{A}^i\tilde{X}^i + \tilde{B}^i = \tilde{C}^i$

Consider the equation

$$\tilde{A}^i\tilde{X}^i + \tilde{B}^i = \tilde{C}^i \tag{3.3.1}$$

We can solve this equation in two steps:
In the first step we solve the equation $\tilde{Y}^i + \tilde{B}^i = \tilde{C}^i$ and in the second step we solve the equation $\tilde{A}^i\tilde{X}^i = \tilde{Y}^i$.

Here we discuss a problem following Buckley,Qu[7] using the strong and weak solution concept. Here datas are taken as GTIFNs.

Problem-3.1 (following [7]):- Suppose a cattle dealer has bought 1000 cows with price rupees Rs.5000/- for each cow. He decides to sell the around 25%$\left(\tilde{A}^i = (0.15,0.25,0.35;0.8,0.1)\right)$ of total cows with the profit about Rs.20%$\left(\tilde{B}^i = (0.15,0.20,0.25;0.7,0.2)\right)$ for each cow. Then he decides to sell the rest part of total cows. So, what would be the approximate selling price for each cow such that his average profit would become around Rs.30% $\left(\tilde{C}^i = (0.25,0.30,0.35;0.8,0.1)\right)$ for each cow?

Solution:- Let $\tilde{P}^i > 0$ be the unknown price of the remaining cows.
Then we have to solve

$$\left(1000\tilde{A}^i\right)\left(5000\tilde{B}^i\right) + \left(1000 - 1000\tilde{A}^i\right)\left(\tilde{P}^i - 5000\right) = (1000)(5000)\tilde{C}^i \tag{a}$$

Now

$$A^i_{\alpha,\beta} = \langle\left[0.15 + 0.10\frac{\alpha}{w}, 0.35 - 0.10\frac{\alpha}{w}\right], \left[0.15 + 0.10\frac{(1-\beta)}{(1-u)}, 0.35 - 0.10\frac{(1-\beta)}{(1-u)}\right]\rangle,$$

$$B^i_{\alpha,\beta} = \langle\left[0.15 + 0.05\frac{\alpha}{w}, 0.25 - 0.05\frac{\alpha}{w}\right], \left[0.15 + 0.05\frac{(1-\beta)}{(1-u)}, 0.25 - 0.05\frac{(1-\beta)}{(1-u)}\right]\rangle,$$

$$C^i_{\alpha,\beta} = \langle\left[0.25 + 0.05\frac{\alpha}{w}, 0.35 - 0.05\frac{\alpha}{w}\right], \left[0.25 + 0.05\frac{(1-\beta)}{(1-u)}, 0.35 - 0.05\frac{(1-\beta)}{(1-u)}\right]\rangle$$

where $w = \min(0.7,0.8,0.8) = 0.7$ and $u = \max(0.1,0.2,0.1) = 0.2$
Set $P^i_{\alpha,\beta} = \langle\left[P_{\mu_L}(\alpha), P_{\mu_R}(\alpha)\right], \left[P_{\upsilon_L}(\beta), P_{\upsilon_R}(\beta)\right]\rangle$
We 1st look at the strong solution.
Then the equation (a) becomes

$$5000\left[0.15 + 0.10\frac{\alpha}{w}, 0.35 - 0.10\frac{\alpha}{w}\right]\left[0.15 + 0.05\frac{\alpha}{w}, 0.25 - 0.05\frac{\alpha}{w}\right]$$
$$+ \left([1,1] - \left[0.15 + 0.10\frac{\alpha}{w}, 0.35 - 0.10\frac{\alpha}{w}\right]\right)\left(\left[P_{\mu_L}(\alpha), P_{\mu_R}(\alpha)\right] - [5000,5000]\right)$$
$$= 5000\left[0.25 + 0.05\frac{\alpha}{w}, 0.35 - 0.05\frac{\alpha}{w}\right]$$

and

$$5000\left[0.15 + 0.10\frac{(1-\beta)}{(1-u)}, 0.35 - 0.10\frac{(1-\beta)}{(1-u)}\right]\left[0.15 + 0.05\frac{(1-\beta)}{(1-u)}, 0.25 - 0.05\frac{(1-\beta)}{(1-u)}\right]$$
$$+ \left([1,1] - \left[0.15 + 0.10\frac{(1-\beta)}{(1-u)}, 0.35 - 0.10\frac{(1-\beta)}{(1-u)}\right]\right)\left(\left[P_{\mu_L}(\alpha), P_{\mu_R}(\alpha)\right] - [5000,5000]\right)$$
$$= 5000\left[0.25 + 0.05\frac{(1-\beta)}{(1-u)}, 0.35 - 0.05\frac{(1-\beta)}{(1-u)}\right]$$

$$\Rightarrow P_{\mu_L}(\alpha) = 5000\left\{\frac{0.8775 + 0.1275\frac{\alpha}{w} - 0.005\left(\frac{\alpha}{w}\right)^2}{\left(0.65 + 0.10\frac{\alpha}{w}\right)}\right\}, \quad P_{\mu_R}(\alpha) = 5000\left\{\frac{1.1125 - 0.1075\frac{\alpha}{w} - 0.005\left(\frac{\alpha}{w}\right)^2}{\left(0.85 - 0.10\frac{\alpha}{w}\right)}\right\}$$

$$P_{\upsilon_L}(\beta) = 5000\left\{\frac{0.8775 + 0.1275\left(\frac{1-\beta}{1-u}\right) - 0.005\left(\frac{1-\beta}{1-u}\right)^2}{\left(0.65 + 0.10\frac{1-\beta}{1-u}\right)}\right\} \text{ and } P_{\upsilon_R}(\beta) = 5000\left\{\frac{1.1125 - 0.1075\left(\frac{1-\beta}{1-u}\right) - 0.005\left(\frac{1-\beta}{1-u}\right)^2}{\left(0.85 - 0.10\frac{1-\beta}{1-u}\right)}\right\}$$

Now, $\frac{d}{d\alpha}\left[P_{\mu_L}(\alpha)\right] < 0$ and $\frac{d}{d\beta}\left[P_{\upsilon_L}(\beta)\right] > 0$

Hence the strong solution does not exist here. Now we will find the weak solution.

Set $P_{\mu_L}(\alpha) = 5000\left(\frac{1.1125-0.153571\alpha-0.0102041\alpha^2}{(0.85-0.142857\alpha)}\right)$ and $P_{\mu_R}(\alpha) = 5000\left(\frac{0.8775+0.182143\alpha-0.0102041\alpha^2}{(0.65+0.142857\alpha)}\right)$

$P_{\upsilon_L}(\beta) = 5000\left\{\frac{1.1125-0.134375(1-\beta)-0.0078125(1-\beta)^2}{0.85-0.125(1-\beta)}\right\}$, $P_{\upsilon_R}(\beta) = 5000\left\{\frac{0.8775+0.159375(1-\beta)-0.0078125(1-\beta)^2}{0.65+0.125(1-\beta)}\right\}$

∴ The (α,β)- cut of the weak solution is

$P^i_{\alpha,\beta} = \langle [P_{\mu_L}(\alpha), P_{\mu_R}(\alpha)], [P_{\upsilon_L}(\beta), P_{\upsilon_R}(\beta)] \rangle$

Where $[P_{\mu_L}(\alpha), P_{\mu_R}(\alpha)] = \left\{x: 5000\left(\frac{1.1125-0.153571\alpha-0.0102041\alpha^2}{(0.85-0.142857\alpha)}\right) \le x \le 5000\left(\frac{0.8775+0.182143\alpha-0.0102041\alpha^2}{(0.65+0.142857\alpha)}\right)\right\}$

and

$[P_{\upsilon_L}(\beta), P_{\upsilon_R}(\beta)] = \left\{x: 5000\left\{\frac{1.1125-0.134375(1-\beta)-0.0078125(1-\beta)^2}{0.85-0.125(1-\beta)}\right\} \le x \le 5000\left\{\frac{1.1125-0.134375(1-\beta)-0.0078125(1-\beta)^2}{0.85-0.125(1-\beta)}\right\}\right\}$

∴ The solution is a GTsIFN $\tilde{P}^i \approx (6544.12, 6666.67, 6750; 0.7, 0.2)$ and the rough sketch of the membership and non-membership function is given below

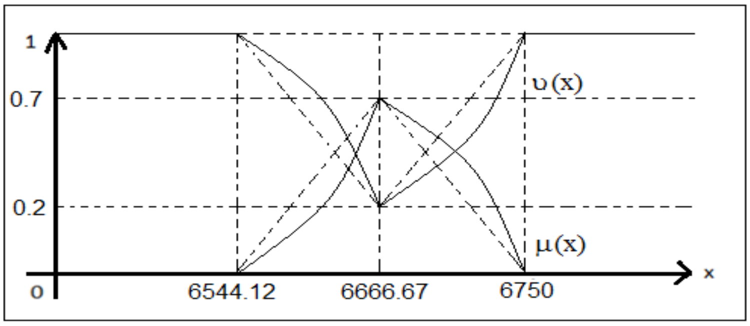


Fig-3.4:- Rough sketch of membership and non-membership function of $\tilde{P}^i$

3.4 Solution of Quadratic equation $\tilde{A}^i\tilde{X}^{i^2} = \tilde{B}^i$

Consider the equation

$$\tilde{A}^i\tilde{X}^{i^2} = \tilde{B}^i \tag{3.4.1}$$

where $\tilde{A}^i = (a_1, a_2, a_3; w_a, u_a)$, $\tilde{B}^i = (b_1, b_2, b_3; w_b, u_b)$ are positive GTIFNs.

We have

$$A^i_{\alpha,\beta} = \langle \left[a_1 + \frac{\alpha}{w_a}(a_2 - a_1), a_3 - \frac{\alpha}{w_a}(a_3 - a_2)\right], \left[a_1 + \frac{(1-\beta)}{(1-u_a)}(a_2 - a_1), a_3 - \frac{(1-\beta)}{(1-u_a)}(a_3 - a_2)\right] \rangle$$

$$B^i_{\alpha,\beta} = \langle \left[b_1 + \frac{\alpha}{w_b}(b_2 - b_1), b_3 - \frac{\alpha}{w_b}(b_3 - b_2)\right], \left[b_1 + \frac{(1-\beta)}{(1-u_b)}(b_2 - b_1), b_3 - \frac{(1-\beta)}{(1-u_b)}(b_3 - b_2)\right] \rangle$$

Let $w = \min(w_a, w_b)$ and $u = \max(u_a, u_b)$

Let $X^i_{\alpha,\beta} = \langle [X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)], [X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)] \rangle$ be the (α,β)-cut of the strong solution.

Here we only consider the positive solution.

∴ The equation (3.4.1) becomes

$$A^i_{\alpha,\beta}\, {X^i_{\alpha,\beta}}^2 = B^i_{\alpha,\beta} \tag{3.4.2}$$

$$\Rightarrow \langle \left[a_1 + \frac{\alpha}{w}(a_2 - a_1), a_3 - \frac{\alpha}{w}(a_3 - a_2)\right], \left[a_1 + \frac{(1-\beta)}{(1-u)}(a_2 - a_1), a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right] \rangle$$
$$\langle [{X_{\mu_L}}^2(\alpha), {X_{\mu_R}}^2(\alpha)], [{X_{\upsilon_L}}^2(\beta), {X_{\upsilon_R}}^2(\beta)] \rangle$$
$$= \langle \left[b_1 + \frac{\alpha}{w}(b_2 - b_1), b_3 - \frac{\alpha}{w}(b_3 - b_2)\right], \left[b_1 + \frac{(1-\beta)}{(1-u)}(b_2 - b_1), b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)\right] \rangle$$

$$\Rightarrow \left[a_1 + \frac{\alpha}{w}(a_2 - a_1), a_3 - \frac{\alpha}{w}(a_3 - a_2)\right][{X_{\mu_L}}^2(\alpha), {X_{\mu_R}}^2(\alpha)]$$
$$= \left[b_1 + \frac{\alpha}{w}(b_2 - b_1), b_3 - \frac{\alpha}{w}(b_3 - b_2)\right]$$

and,

$$\left[a_1 + \frac{(1-\beta)}{(1-u)}(a_2 - a_1), a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right][{X_{\upsilon_L}}^2(\beta), {X_{\upsilon_R}}^2(\beta)]$$
$$= \left[b_1 + \frac{(1-\beta)}{(1-u)}(b_2 - b_1), \quad b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)\right]$$
$$\Rightarrow \left\{a_1 + \frac{\alpha}{w}(a_2 - a_1)\right\}{X_{\mu_L}}^2(\alpha) = b_1 + \frac{\alpha}{w}(b_2 - b_1)$$

$$\left\{a_3 - \frac{\alpha}{w}(a_3 - a_2)\right\} X_{\mu_R}{}^2(\alpha) = b_3 - \frac{\alpha}{w}(b_3 - b_2)$$
$$\left\{a_1 + \frac{(1-\beta)}{(1-u)}(a_2 - a_1)\right\} X_{\upsilon_L}{}^2(\beta) = b_1 + \frac{(1-\beta)}{(1-u)}(b_2 - b_1)$$
$$\left\{a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right\} X_{\upsilon_R}{}^2(\beta) = b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)$$

or,

$$X_{\mu_L}(\alpha) = \sqrt{\frac{\left\{b_1+\frac{\alpha}{w}(b_2-b_1)\right\}}{\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}}}, X_{\mu_R}(\alpha) = \sqrt{\frac{\left\{b_3-\frac{\alpha}{w}(b_3-b_2)\right\}}{\left\{a_3-\frac{\alpha}{w}(a_3-a_2)\right\}}},$$

$$X_{\upsilon_L}(\beta) = \sqrt{\frac{\left\{b_1 + \frac{(1-\beta)}{(1-u)}(b_2 - b_1)\right\}}{\left\{a_1 + \frac{(1-\beta)}{(1-u)}(a_2 - a_1)\right\}}}, \quad X_{\upsilon_R}(\beta) = \sqrt{\frac{\left\{b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)\right\}}{\left\{a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right\}}}$$

Now

$$\frac{d}{d\alpha}\left[X_{\mu_L}(\alpha)\right] = \frac{1}{2w}\sqrt{\frac{\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}}{\left\{b_1+\frac{\alpha}{w}(b_2-b_1)\right\}}}\frac{a_1b_2-a_2b_1}{\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}^2} > 0 \Rightarrow a_1b_2 > a_2b_1 \quad ,$$

$$\frac{d}{d\alpha}\left[X_{\mu_R}(\alpha)\right] = \frac{1}{2w}\sqrt{\frac{\left\{a_3 - \frac{\alpha}{w}(a_3 - a_2)\right\}}{\left\{b_3 - \frac{\alpha}{w}(b_3 - b_2)\right\}}}\frac{a_3b_2 - a_2b_3}{\left\{a_3 - \frac{\alpha}{w}(a_3 - a_2)\right\}^2} < 0 \Rightarrow a_3b_2 < a_2b_3$$

And

$$X_{\mu_L}(w) = X_{\mu_R}(w) = \sqrt{\frac{b_2}{a_2}}$$

again

$$\frac{d}{d\beta}\left[X_{\upsilon_L}(\beta)\right] = \frac{1}{2(1-u)}\sqrt{\frac{\left\{a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1)\right\}}{\left\{b_1+\frac{(1-\beta)}{(1-u)}(b_2-b_1)\right\}}}\frac{a_2b_1-a_1b_2}{\left\{a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1)\right\}^2} < 0 \Rightarrow a_1b_2 > a_2b_1 \quad ,$$

$$\frac{d}{d\beta}\left[X_{\upsilon_R}(\beta)\right] = \frac{1}{2(1-u)}\sqrt{\frac{\left\{a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right\}}{\left\{b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)\right\}}}\frac{a_2b_3 - a_3b_2}{\left\{a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right\}^2} > 0 \Rightarrow a_3b_2 < a_2b_3$$

And

$$X_{\upsilon_L}(u) = X_{\upsilon_R}(u) = \sqrt{\frac{b_2}{a_2}}$$

Hence in this case strong solution exists if $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \frac{b_3}{a_3}$

∴ The positive solution of the equation $\tilde{A}^i \tilde{X}^{i^2} = \tilde{B}^i$ is given by $\tilde{X}^i$ with its (α, β)-cut

$$X^i_{\alpha,\beta} = \langle\left[X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)\right], \left[X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)\right]\rangle$$

Where

$$\left[X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)\right] = \left\{x: \sqrt{\frac{\left\{b_1 + \frac{\alpha}{w}(b_2 - b_1)\right\}}{\left\{a_1 + \frac{\alpha}{w}(a_2 - a_1)\right\}}} \le x \le \sqrt{\frac{\left\{b_3 - \frac{\alpha}{w}(b_3 - b_2)\right\}}{\left\{a_3 - \frac{\alpha}{w}(a_3 - a_2)\right\}}}\right\}$$

$$X_{\mu_L}(0) = \sqrt{\frac{b_1}{a_1}}, X_{\mu_L}(w) = \sqrt{\frac{b_2}{a_2}}, X_{\mu_R}(w) = \sqrt{\frac{b_2}{a_2}}, X_{\mu_R}(0) = \sqrt{\frac{b_3}{a_3}}$$

And

$$[X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)] = \left\{x: \sqrt{\frac{\left\{b_1 + \frac{(1-\beta)}{(1-u)}(b_2 - b_1)\right\}}{\left\{a_1 + \frac{(1-\beta)}{(1-u)}(a_2 - a_1)\right\}}} \le x \le \sqrt{\frac{\left\{b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)\right\}}{\left\{a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right\}}}\right\}$$

$$X_{\upsilon_L}(1) = \sqrt{\frac{b_1}{a_1}}, X_{\upsilon_L}(u) = \sqrt{\frac{b_2}{a_2}}, X_{\upsilon_R}(u) = \sqrt{\frac{b_2}{a_2}}, X_{\upsilon_R}(1) = \sqrt{\frac{b_3}{a_3}}$$

$$\sqrt{\frac{\left\{b_1+\frac{\alpha}{w}(b_2-b_1)\right\}}{\left\{a_1+\frac{\alpha}{w}(a_2-a_1)\right\}}} \le x \Rightarrow w\frac{a_1x^2-b_1}{(b_2-b_1)-(a_2-a_1)x^2} \ge \alpha \text{ for } x \in \left[\sqrt{\frac{b_1}{a_1}}, \sqrt{\frac{b_2}{a_2}}\right]$$

$$\Rightarrow \mu^L_{\tilde{X}^i}(x) \ge \alpha$$

$$\frac{d}{dx}\left[\mu^L_{\tilde{X}^i}(x)\right] = \frac{2xw(a_1b_2-a_2b_1)}{\{(b_2-b_1)-x^2(a_2-a_1)\}^2} > 0 \quad [\because a_1b_2 > a_2b_1]$$

Again,

$$x \le \sqrt{\frac{\left\{b_3-\frac{\alpha}{w}(b_3-b_2)\right\}}{\left\{a_3-\frac{\alpha}{w}(a_3-a_2)\right\}}} \Rightarrow w\frac{b_3-a_3x^2}{(b_3-b_2)-(a_3-a_2)x^2} \ge \alpha \text{ for } x \in \left[\sqrt{\frac{b_2}{a_2}}, \sqrt{\frac{b_3}{a_3}}\right]$$

$$\Rightarrow \mu^R_{\tilde{X}^i}(x) \ge \alpha$$

$$\frac{d}{dx}\left[\mu^R_{\tilde{X}^i}(x)\right] = \frac{2xw(a_3b_2 - a_2b_3)}{\{(b_3-b_2)-(a_3-a_2)x^2\}^2} < 0 \quad [\because a_3b_2 < a_2b_3]$$

$$\mu^L_{\tilde{X}^i}\left(\sqrt{\frac{b_2}{a_2}}\right) = w = \mu^R_{\tilde{X}^i}\left(\sqrt{\frac{b_2}{a_2}}\right), \mu^L_{\tilde{X}^i}\left(\sqrt{\frac{b_1}{a_1}}\right) = 0, \mu^R_{\tilde{X}^i}\left(\sqrt{\frac{b_3}{a_3}}\right) = 0$$

$$\mu^L_{\tilde{X}^i}\left(\frac{\sqrt{\frac{b_1}{a_1}} + \sqrt{\frac{b_2}{a_2}}}{2}\right) = \frac{wa_1\left(\sqrt{a_1b_2} + 3\sqrt{a_2b_1}\right)}{a_1\left(\sqrt{a_1b_2} + 3\sqrt{a_2b_1}\right) + a_2\left(3\sqrt{a_1b_2} + \sqrt{a_2b_1}\right)} < \frac{w}{2}$$

and

$$\mu^R_{\tilde{X}^i}\left(\frac{\sqrt{\frac{b_3}{a_3}} + \sqrt{\frac{b_2}{a_2}}}{2}\right) = \frac{\omega a_3\left(\sqrt{a_3b_2} + 3\sqrt{a_2b_3}\right)}{a_2\left(3\sqrt{a_3b_2} + \sqrt{a_2b_3}\right) + a_3\left(\sqrt{a_3b_2} + 3\sqrt{a_2b_3}\right)} > \frac{w}{2}$$

$$\sqrt{\frac{\left\{b_1+\frac{(1-\beta)}{(1-u)}(b_2-b_1)\right\}}{\left\{a_1+\frac{(1-\beta)}{(1-u)}(a_2-a_1)\right\}}} \le x \Rightarrow 1-(1-u)\frac{a_1x^2-b_1}{(b_2-b_1)-(a_2-a_1)x^2} \le \beta \text{ for } x \in \left[\sqrt{\frac{b_1}{a_1}}, \sqrt{\frac{b_2}{a_2}}\right]$$

$$\Rightarrow \upsilon^L_{\tilde{X}^i}(x) \le \beta$$

$$\frac{d}{dx}\left[\upsilon^L_{\tilde{X}^i}(x)\right] = \frac{-2x(1-u)(a_1b_2-a_2b_1)}{\{(b_2-b_1)-x^2(a_2-a_1)\}^2} < 0 \quad [\because a_1b_2 > a_2b_1]$$

Again,

$$x \le \sqrt{\frac{\left\{b_3-\frac{(1-\beta)}{(1-u)}(b_3-b_2)\right\}}{\left\{a_3-\frac{(1-\beta)}{(1-u)}(a_3-a_2)\right\}}} \Rightarrow 1-(1-u)\frac{b_3-a_3x^2}{(b_3-b_2)-(a_3-a_2)x^2} \le \beta \text{ for } x \in \left[\sqrt{\frac{b_2}{a_2}}, \sqrt{\frac{b_3}{a_3}}\right]$$

$$\Rightarrow \upsilon^R_{\tilde{X}^i}(x) \le \beta$$

$$\frac{d}{dx}\left[\upsilon^R_{\tilde{X}^i}(x)\right] = \frac{-2x(1-u)(a_3b_2 - a_2b_3)}{\{(b_3-b_2)-(a_3-a_2)x^2\}^2} > 0 \quad [\because a_3b_2 < a_2b_3]$$

$$\upsilon^L_{\tilde{X}^i}\left(\sqrt{\frac{b_2}{a_2}}\right) = u = \upsilon^R_{\tilde{X}^i}\left(\sqrt{\frac{b_2}{a_2}}\right), \upsilon^L_{\tilde{X}^i}\left(\sqrt{\frac{b_1}{a_1}}\right) = 1, \upsilon^R_{\tilde{X}^i}\left(\sqrt{\frac{b_3}{a_3}}\right) = 1$$

$$\upsilon^L_{\tilde{X}^i}\left(\frac{\sqrt{\frac{b_1}{a_1}}+\sqrt{\frac{b_2}{a_2}}}{2}\right) > \frac{1-u}{2} \text{ and } \upsilon^R_{\tilde{X}^i}\left(\frac{\sqrt{\frac{b_3}{a_3}}+\sqrt{\frac{b_2}{a_2}}}{2}\right) < \frac{1-u}{2}$$

The solution is a GTsIFN $\tilde{X}^i \approx \left(\sqrt{\frac{b_1}{a_1}}, \sqrt{\frac{b_2}{a_2}}, \sqrt{\frac{b_3}{a_3}}; w, u\right)$ with its membership function

$$\mu_{\tilde{X}^i}(x) = \begin{cases} 0 & x \le \sqrt{\frac{b_1}{a_1}} \\ w\frac{a_1x^2 - b_1}{(b_2-b_1)-(a_2-a_1)x^2}, & \sqrt{\frac{b_1}{a_1}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ w, & x = \sqrt{\frac{b_2}{a_2}} \\ w\frac{b_3 - a_3x^2}{(b_3-b_2)-(a_3-a_2)x^2}, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_3}{a_3}} \\ 0, & \sqrt{\frac{b_3}{a_3}} \le x \end{cases}$$

and non-membership function

$$\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1, & x \le \sqrt{\frac{b_1}{a_1}} \\ 1-(1-u)\frac{a_1x^2 - b_1}{(b_2-b_1)-(a_2-a_1)x^2}, & \sqrt{\frac{b_1}{a_1}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ u, & x = \sqrt{\frac{b_2}{a_2}} \\ 1-(1-u)\frac{b_3 - a_3x^2}{(b_3-b_2)-(a_3-a_2)x^2}, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_3}{a_3}} \\ 1, & \sqrt{\frac{b_3}{a_3}} \le x \end{cases}$$

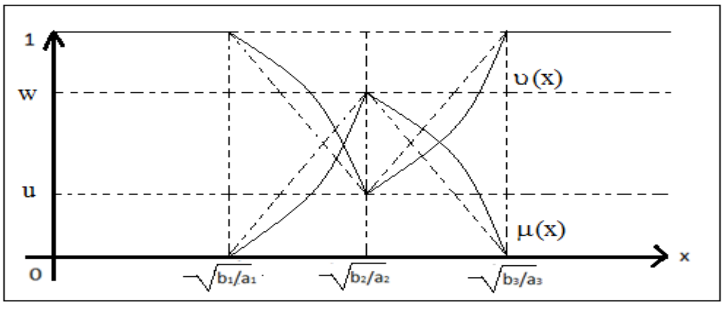
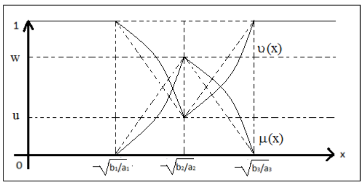
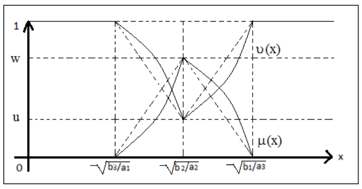
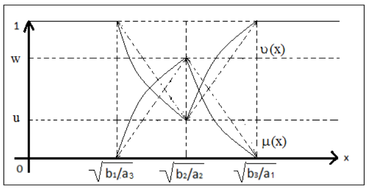
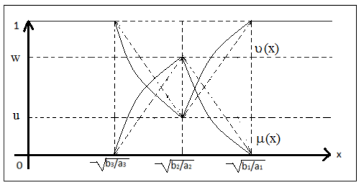


Fig-3.5:-Rough sketch of membership and non-membership function of $\tilde{X}^i$

We have solved the equation $\tilde{A}^i{\tilde{X}^i}^2 = \tilde{B}^i$ for all other cases and the results are given in table-3.2

Table-3.2:-Strong solution of $\tilde{A}^i {\tilde{X}^i}{}^2 = \tilde{B}^i$ for different $\tilde{A}^i, \tilde{B}^i$

Nature of $\tilde{A}^i, \tilde{B}^i$	**Condition for existence of strong solution**	**Type of solutions**	**Membership function and non-membership function of $\tilde{X}^i$**	**Rough sketch of membership and non-membership function of $\tilde{X}^i$**
$\tilde{A}^i > 0$, $\tilde{B}^i > 0$	$\frac{b_1}{a_1} < \frac{b_2}{a_2} < \frac{b_3}{a_3}$	$\pm\tilde{X}^i$ where $\tilde{X}^i \approx \left(\sqrt{\frac{b_1}{a_1}}, \sqrt{\frac{b_2}{a_2}}, \sqrt{\frac{b_3}{a_3}}; w, u\right)$	$\mu_{\tilde{X}^i}(x) = \begin{cases} 0 & x \le \sqrt{\frac{b_1}{a_1}} \\ w\frac{a_1x^2-b_1}{(b_2-b_1)-(a_2-a_1)x^2}, & \sqrt{\frac{b_1}{a_1}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ w, & x = \sqrt{\frac{b_2}{a_2}} \\ w\frac{b_3-a_3x^2}{(b_3-b_2)-(a_3-a_2)x^2}, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_3}{a_3}} \\ 0, & \sqrt{\frac{b_3}{a_3}} \le x \end{cases}$ $\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1, & x \le \sqrt{\frac{b_1}{a_1}} \\ 1-\frac{(1-u)(a_1x^2-b_1)}{(b_2-b_1)-(a_2-a_1)x^2}, & \sqrt{\frac{b_1}{a_1}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ u, & x = \sqrt{\frac{b_2}{a_2}} \\ 1-\frac{(1-u)(b_3-a_3x^2)}{(b_3-b_2)-(a_3-a_2)x^2}, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_3}{a_3}} \\ 1, & \sqrt{\frac{b_3}{a_3}} \le x \end{cases}$	
$\tilde{A}^i > 0$, $\tilde{B}^i < 0$	$\frac{b_3}{a_1} < \frac{b_2}{a_2} < \frac{b_1}{a_3}$	$\pm i\tilde{X}^i$ $\tilde{X}^i \approx \left(\sqrt{\frac{b_3}{a_1}}, \sqrt{\frac{b_2}{a_2}}, \sqrt{\frac{b_1}{a_3}}; w, u\right)$	$\mu_{\tilde{X}^i}(x) = \begin{cases} 0, & x \le \sqrt{\frac{b_3}{a_1}} \\ w\frac{b_3+a_1x^2}{(b_3-b_2)-(a_2-a_1)x^2}, & \sqrt{\frac{b_3}{a_1}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ w, & x = \sqrt{\frac{b_2}{a_2}} \\ w\frac{-a_3x^2-b_1}{(b_2-b_1)-(a_3-a_2)x^2}, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_1}{a_3}} \\ 0, & \sqrt{\frac{b_1}{a_3}} \le x \end{cases}$ $\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1, & x \le \sqrt{\frac{b_3}{a_1}} \\ 1-\frac{(1-u)(b_3+a_1x^2)}{(b_3-b_2)-(a_2-a_1)x^2}, & \sqrt{\frac{b_3}{a_1}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ u, & x = \sqrt{\frac{b_2}{a_2}} \\ 1-\frac{(1-u)(-a_3x^2-b_1)}{(b_2-b_1)-(a_3-a_2)x^2}, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_1}{a_3}} \\ 1, & \sqrt{\frac{b_1}{a_3}} \le x \end{cases}$	

$\tilde{A}^i < 0$, $\tilde{B}^i > 0$	$\frac{b_1}{a_3} < \frac{b_2}{a_2} < \frac{b_3}{a_1}$	$\pm i\tilde{X}^i$ where $\tilde{X}^i \approx \left(\sqrt{\frac{b_1}{a_3}}, \sqrt{\frac{b_2}{a_2}}, \sqrt{\frac{b_3}{a_1}}; w,u\right)$	$\mu_{\tilde{X}^i}(x) = \begin{cases} 0 \;, & x \le \sqrt{\frac{b_1}{a_3}} \\ w\frac{-a_3x^2-b_1}{(b_2-b_1)-(a_3-a_2)x^2} \;, & \sqrt{\frac{b_1}{a_3}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ w \;, & x = \sqrt{\frac{b_2}{a_2}} \\ w\frac{b_3+a_1x^2}{(b_3-b_2)-(a_2-a_1)x^2} \;, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_3}{a_1}} \\ 0 \;, & \sqrt{\frac{b_3}{a_1}} \le x \end{cases}$ $\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1 \;, & x \le \sqrt{\frac{b_1}{a_3}} \\ 1-\frac{(1-u)(-a_3x^2-b_1)}{(b_2-b_1)-(a_3-a_2)x^2} \;, & \sqrt{\frac{b_1}{a_3}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ u \;, & x = \sqrt{\frac{b_2}{a_2}} \\ 1-\frac{(1-u)(b_3+a_1x^2)}{(b_3-b_2)-(a_2-a_1)x^2} \;, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_3}{a_1}} \\ 1 \;, & \sqrt{\frac{b_3}{a_1}} \le x \end{cases}$	
$\tilde{A}^i < 0$, $\tilde{B}^i < 0$	$\frac{b_3}{a_3} < \frac{b_2}{a_2} < \frac{b_1}{a_1}$	$\pm\tilde{X}^i$ where $\tilde{X}^i \approx \left(\sqrt{\frac{b_3}{a_3}}, \sqrt{\frac{b_2}{a_2}}, \sqrt{\frac{b_1}{a_1}}; w,u\right)$	$\mu_{\tilde{X}^i}(x) = \begin{cases} 0 \;, & x \le \sqrt{\frac{b_3}{a_3}} \\ w\frac{a_3x^2-b_3}{(a_3-a_2)x^2-(b_3-b_2)} \;, & \sqrt{\frac{b_3}{a_3}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ w \;, & x = \sqrt{\frac{b_2}{a_2}} \\ w\frac{b_1-a_1x^2}{(a_2-a_1)x^2-(b_2-b_1)} \;, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_1}{a_1}} \\ 0 \;, & \sqrt{\frac{b_1}{a_1}} \le x \end{cases}$ $\upsilon_{\tilde{X}^i}(x) = \begin{cases} 0 \;, & x \le \sqrt{\frac{b_3}{a_3}} \\ 1-\frac{(1-u)(a_3x^2-b_3)}{(a_3-a_2)x^2-(b_3-b_2)} \;, & \sqrt{\frac{b_3}{a_3}} \le x \le \sqrt{\frac{b_2}{a_2}} \\ u \;, & x = \sqrt{\frac{b_2}{a_2}} \\ 1-\frac{(1-u)(b_1-a_1x^2)}{(a_2-a_1)x^2-(b_2-b_1)} \;, & \sqrt{\frac{b_2}{a_2}} \le x \le \sqrt{\frac{b_1}{a_1}} \\ 0 \;, & \sqrt{\frac{b_1}{a_1}} \le x \end{cases}$	

3.5 Solution of Quadratic equation $\tilde{X}^{i^2} + \tilde{A}^i = \tilde{B}^i$

Consider the equation

$$\tilde{X}^{i^2} + \tilde{A}^i = \tilde{B}^i \tag{3.5.1}$$

where $\tilde{A}^i = (a_1, a_2, a_3; w_a, u_a), \tilde{B}^i = (b_1, b_2, b_3; w_b, u_b)$ are positive GTIFNs and $\tilde{A}^i < \tilde{B}^i$.

We have

$$A^i_{\alpha,\beta} = \langle\left[a_1 + \frac{\alpha}{w_a}(a_2 - a_1), a_3 - \frac{\alpha}{w_a}(a_3 - a_2)\right], \left[a_1 + \frac{(1-\beta)}{(1-u_a)}(a_2 - a_1), a_3 - \frac{(1-\beta)}{(1-u_a)}(a_3 - a_2)\right]\rangle$$

$$B^i_{\alpha,\beta} = \langle\left[b_1 + \frac{\alpha}{w_b}(b_2 - b_1), b_3 - \frac{\alpha}{w_b}(b_3 - b_2)\right], \left[b_1 + \frac{(1-\beta)}{(1-u_b)}(b_2 - b_1), b_3 - \frac{(1-\beta)}{(1-u_b)}(b_3 - b_2)\right]\rangle$$

Let $w = \min(w_a, w_b)$ and $u = \max(u_a, u_b)$

Let $X^i_{\alpha,\beta} = \langle\left[X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)\right], \left[X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)\right]\rangle$ be the (α, β)-cut of the strong solution.

Here we only consider the positive solution.

∴ The equation (3.5.1) becomes

$$X^{i}_{\alpha,\beta}{}^{2} + A^{i}_{\alpha,\beta} = B^{i}_{\alpha,\beta}$$

or,

$$\langle[X_{\mu_L}{}^2(\alpha), X_{\mu_R}{}^2(\alpha)], [X_{\upsilon_L}{}^2(\beta), X_{\upsilon_R}{}^2(\beta)]\rangle$$
$$+\langle\left[a_1 + \frac{\alpha}{w}(a_2 - a_1), a_3 - \frac{\alpha}{w}(a_3 - a_2)\right], \left[a_1 + \frac{(1-\beta)}{(1-u)}(a_2 - a_1), a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right]\rangle$$
$$= \langle\left[b_1 + \frac{\alpha}{w}(b_2 - b_1), b_3 - \frac{\alpha}{w}(b_3 - b_2)\right], \left[b_1 + \frac{(1-\beta)}{(1-u)}(b_2 - b_1), b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)\right]\rangle$$
$$\Rightarrow [X_{\mu_L}{}^2(\alpha), X_{\mu_R}{}^2(\alpha)] + \left[a_1 + \frac{\alpha}{w}(a_2 - a_1), a_3 - \frac{\alpha}{w}(a_3 - a_2)\right]$$
$$= \left[b_1 + \frac{\alpha}{w}(b_2 - b_1), b_3 - \frac{\alpha}{w}(b_3 - b_2)\right]$$

and,

$$[X_{\upsilon_L}{}^2(\beta), X_{\upsilon_R}{}^2(\beta)] + \left[a_1 + \frac{(1-\beta)}{(1-u)}(a_2 - a_1), a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2)\right]$$
$$= \left[b_1 + \frac{(1-\beta)}{(1-u)}(b_2 - b_1),\ b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)\right]$$
$$\Rightarrow X_{\mu_L}{}^2(\alpha) + a_1 + \frac{\alpha}{w}(a_2 - a_1) = b_1 + \frac{\alpha}{w}(b_2 - b_1)$$
$$X_{\mu_R}{}^2(\alpha) + a_3 - \frac{\alpha}{w}(a_3 - a_2) = b_3 - \frac{\alpha}{w}(b_3 - b_2)$$
$$X_{\upsilon_L}{}^2(\beta) + a_1 + \frac{(1-\beta)}{(1-u)}(a_2 - a_1) = b_1 + \frac{(1-\beta)}{(1-u)}(b_2 - b_1)$$
$$X_{\upsilon_R}{}^2(\beta) + a_3 - \frac{(1-\beta)}{(1-u)}(a_3 - a_2) = b_3 - \frac{(1-\beta)}{(1-u)}(b_3 - b_2)$$

or,

$$X_{\mu_L}(\alpha) = \sqrt{(b_1 - a_1) + \frac{\alpha}{w}(b_2 - b_1 - a_2 + a_1)},$$
$$X_{\mu_R}(\alpha) = \sqrt{(b_3 - a_3) - \frac{\alpha}{w}(b_3 - b_2 - a_3 + a_2)},$$
$$X_{\upsilon_L}(\beta) = \sqrt{(b_1 - a_1) + \frac{(1-\beta)}{(1-u)}(b_2 - b_1 - a_2 + a_1)},$$
$$X_{\upsilon_R}(\beta) = \sqrt{(b_3 - a_3) - \frac{(1-\beta)}{(1-u)}(b_3 - b_2 - a_3 + a_2)}$$

Now

$$\frac{d}{d\alpha}[X_{\mu_L}(\alpha)] = \frac{1}{2w}\frac{\{(b_2-b_1)-(a_2-a_1)\}}{\sqrt{(b_1-a_1)+\frac{\alpha}{w}(b_2-b_1-a_2+a_1)}} > 0 \text{ if } (b_2 - b_1) > (a_2 - a_1),$$
$$\frac{d}{d\alpha}[X_{\mu_R}(\alpha)] = \frac{1}{2w}\frac{\{(a_3-a_2)-(b_3-b_2)\}}{\sqrt{(b_3-a_3)-\frac{(1-\beta)}{(1-u)}(b_3-b_2-a_3+a_2)}} < 0 \text{ if } (b_3 - b_2) > (a_3 - a_2)$$

And

$$X_{\mu_L}(w) = X_{\mu_R}(w) = \sqrt{b_2 - a_2}$$

again

$$\frac{d}{d\beta}[X_{\upsilon_L}(\beta)] = \frac{1}{2(1-u)}\frac{-\{(b_2-b_1)-(a_2-a_1)\}}{\sqrt{(b_1-a_1)+\frac{\alpha}{w}(b_2-b_1-a_2+a_1)}} < 0 \text{ if } (b_2 - b_1) > (a_2 - a_1)$$
$$\frac{d}{d\beta}[X_{\upsilon_R}(\beta)] = \frac{1}{2(1-u)}\frac{-\{(a_3-a_2)-(b_3-b_2)\}}{\sqrt{(b_3-a_3)-\frac{(1-\beta)}{(1-u)}(b_3-b_2-a_3+a_2)}} < 0 \text{ if } (b_3 - b_2) > (a_3 - a_2)$$

And

$$X_{\upsilon_L}(u) = X_{\upsilon_R}(u) = \sqrt{b_2 - a_2}$$

Hence in this case strong solution exists if $\sqrt{b_1 - a_1} < \sqrt{b_2 - a_2} < \sqrt{b_3 - a_3}$

∴ The positive solution of the equation $\tilde{X}^{i^2} + \tilde{A}^i = \tilde{B}^i$ is given by $\tilde{X}^i$ with its (α,β)-cut
$X^i_{\alpha,\beta} = \langle[X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)], [X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)]\rangle$

Where

$$[X_{\mu_L}(\alpha), X_{\mu_R}(\alpha)] = \left\{x: \sqrt{(b_1 - a_1) + \frac{\alpha}{w}(b_2 - b_1 - a_2 + a_1)} \le x \le \sqrt{(b_3 - a_3) - \frac{\alpha}{w}(b_3 - b_2 - a_3 + a_2)}\right\}$$

and

$$[X_{\upsilon_L}(\beta), X_{\upsilon_R}(\beta)] = \left\{x: \sqrt{(b_1-a_1)+\frac{(1-\beta)}{(1-u)}(b_2-b_1-a_2+a_1)} \le x \le \sqrt{(b_3-a_3)-\frac{(1-\beta)}{(1-u)}(b_3-b_2-a_3+a_2)}\right\}$$

∴The solution $\tilde{X}^i$ is a GTsIFN $\left(\sqrt{b_1-a_1},\ \sqrt{b_2-a_2}, \sqrt{b_3-a_3}; w, u\right)$ whose membership function

$$\mu_{\tilde{X}^i}(x) = \begin{cases} 0\ , & x \le \sqrt{b_1-a_1} \\ w\dfrac{x^2-(b_1-a_1)}{(b_2-b_1)-(a_2-a_1)}\ , & \sqrt{b_1-a_1} \le x \le \sqrt{b_2-a_2} \\ w\ , & x = \sqrt{b_2-a_2} \\ w\ \dfrac{(b_3-a_3)-x^2}{(b_3-b_2)-(a_3-a_2)}\ , & \sqrt{b_2-a_2} \le x \le \sqrt{b_3-a_3} \\ 0\ , & \sqrt{b_3-a_3} \le x \end{cases}$$

and non-membership function

$$\upsilon_{\tilde{X}^i}(x) = \begin{cases} 1\ , & x \le \sqrt{b_1-a_1} \\ 1-(1-u)\dfrac{x^2-(b_1-a_1)}{(b_2-b_1)-(a_2-a_1)}\ , & \sqrt{b_1-a_1} \le x \le \sqrt{b_2-a_2} \\ u\ , & x = \sqrt{b_2-a_2} \\ 1-(1-u)\ \dfrac{(b_3-a_3)-x^2}{(b_3-b_2)-(a_3-a_2)}\ , & \sqrt{b_2-a_2} \le x \le \sqrt{b_3-a_3} \\ 1\ , & \sqrt{b_3-a_3} \le x \end{cases}$$

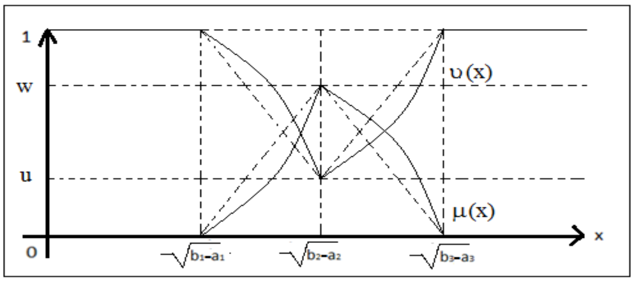


Fig-3.6:- Rough sketch of membership and non-membership function of $\tilde{X}^i$

3.6 Solution of Quadratic Equation $\tilde{A}^i\tilde{X}^{i^2} + \tilde{B}^i\tilde{X}^i = \tilde{C}^i$

Consider the equation

$$\tilde{A}^i\tilde{X}^{i^2} + \tilde{B}^i\tilde{X}^i = \tilde{C}^i \tag{3.6.1}$$

We can solve this equation in two steps:

In the first step we solve the equation $\tilde{B}^i\tilde{X}^i = \tilde{Y}^i$ and in the second step we solve the equation $\tilde{A}^i\tilde{X}^{i^2} + \tilde{Y}^i = \tilde{C}^i$

Here we have solved another problem following Buckley,Qu[7] using strong and weak solution concept.

Problem-3.2 (following [7]):- Suppose an investment firm wishes to set aside around $\tilde{A}$(0.8, 1.0, 1.2; 0.8,0.1) dollars to be invested at interest rate $\tilde{R}$ so that after one year they may withdraw approximately $\tilde{B}$ (0.15,0.20,0.25;0.7,0.2) dollars. And then after two years the amount that is left will accumulate to about $\tilde{C}$ (0.60, 0.90, 1.2; 0.8, 0.1) dollars. Find $\tilde{R}$.

Solution:- Let $\tilde{R}^i > 0$ be the unknown interest rate.

Then we have to solve

$$\left[\left(\tilde{A}^i - \tilde{B}^i\right) + \tilde{A}^i\tilde{R}^i\right] + \left[\left(\tilde{A}^i - \tilde{B}^i\right) + \tilde{A}^i\tilde{R}^i\right]\tilde{R}^i = \tilde{C}^i$$

or,

$$\tilde{A}^i\tilde{R}^{i^2} + \left(2\tilde{A}^i - \tilde{B}^i\right)\tilde{R}^i + \left(\tilde{A}^i - \tilde{B}^i\right) = \tilde{C}^i \tag{b}$$

Now

$$A^i_{\alpha,\beta} = \langle\left[0.8+0.2\tfrac{\alpha}{w}, 1.2-0.2\tfrac{\alpha}{w}\right], \left[0.8+0.2\tfrac{(1-\beta)}{(1-u)}, 1.2-0.2\tfrac{(1-\beta)}{(1-u)}\right]\rangle,$$

$$B^i_{\alpha,\beta} = \langle\left[0.15+0.05\tfrac{\alpha}{w}, 0.25-0.05\tfrac{\alpha}{w}\right], \left[0.15+0.05\tfrac{(1-\beta)}{(1-u)}, 0.25-0.05\tfrac{(1-\beta)}{(1-u)}\right]\rangle,$$

$$C^i_{\alpha,\beta} = \langle \left[0.6+0.3\frac{\alpha}{w}, 1.2-0.3\frac{\alpha}{w}\right], \left[0.6+0.3\frac{(1-\beta)}{(1-u)}, 1.2-0.3\frac{(1-\beta)}{(1-u)}\right] \rangle$$

where $w = \min(0.7, 0.8, 0.8) = 0.7$ and $u = \max(0.1, 0.2, 0.1) = 0.2$

Set $R^i_{\alpha,\beta} = \langle [R_{\mu_L}(\alpha), R_{\mu_R}(\alpha)], [R_{\upsilon_L}(\beta), R_{\upsilon_R}(\beta)] \rangle$

We 1st look at the strong solution.

Then the equation (b) becomes

$$\langle \left[0.8+0.2\frac{\alpha}{w}, 1.2-0.2\frac{\alpha}{w}\right], \left[0.8+0.2\frac{(1-\beta)}{(1-u)}, 1.2-0.2\frac{(1-\beta)}{(1-u)}\right] \rangle \langle [R_{\mu_L}{}^2(\alpha), R_{\mu_R}{}^2(\alpha)], [R_{\upsilon_L}{}^2(\beta), R_{\upsilon_R}{}^2(\beta)] \rangle + \Big(2\langle \left[0.8+0.2\frac{\alpha}{w}, 1.2-0.2\frac{\alpha}{w}\right], \left[0.8+0.2\frac{(1-\beta)}{(1-u)}, 1.2-0.2\frac{(1-\beta)}{(1-u)}\right] \rangle - \langle \left[0.15+0.05\frac{\alpha}{w}, 0.25-0.05\frac{\alpha}{w}\right], \left[0.15+0.05\frac{(1-\beta)}{(1-u)}, 0.25 - 0.05\frac{(1-\beta)}{(1-u)}\right] \rangle \Big) \langle [R_{\mu_L}(\alpha), R_{\mu_R}(\alpha)], [R_{\upsilon_L}(\beta), R_{\upsilon_R}(\beta)] \rangle + \Big(\langle \left[0.8+0.2\frac{\alpha}{w}, 1.2-0.2\frac{\alpha}{w}\right], \left[0.8+0.2\frac{(1-\beta)}{(1-u)}, 1.2-0.2\frac{(1-\beta)}{(1-u)}\right] \rangle - \langle \left[0.15+0.05\frac{\alpha}{w}, 0.25-0.05\frac{\alpha}{w}\right], \left[0.15+0.05\frac{(1-\beta)}{(1-u)}, 0.25-0.05\frac{(1-\beta)}{(1-u)}\right] \rangle \Big) = \langle \left[0.6+0.3\frac{\alpha}{w}, 1.2-0.3\frac{\alpha}{w}\right], \left[0.6+0.3\frac{(1-\beta)}{(1-u)}, 1.2-0.3\frac{(1-\beta)}{(1-u)}\right] \rangle$$

or,

$$\left[0.8+0.2\frac{\alpha}{w}, 1.2-0.2\frac{\alpha}{w}\right][R_{\mu_L}{}^2(\alpha), R_{\mu_R}{}^2(\alpha)] + \left(2\left[0.8+0.2\frac{\alpha}{w}, 1.2-0.2\frac{\alpha}{w}\right] - \left[0.15+0.05\frac{\alpha}{w}, 0.25-0.05\frac{\alpha}{w}\right]\right)[R_{\mu_L}(\alpha), R_{\mu_R}(\alpha)] + \left(\left[0.8+0.2\frac{\alpha}{w}, 1.2-0.2\frac{\alpha}{w}\right] - \left[0.15+0.05\frac{\alpha}{w}, 0.25-0.05\frac{\alpha}{w}\right]\right) = \left[0.6+0.3\frac{\alpha}{w}, 1.2-0.3\frac{\alpha}{w}\right]$$

and

$$\left[0.8+0.2\frac{(1-\beta)}{(1-u)}, 1.2-0.2\frac{(1-\beta)}{(1-u)}\right][R_{\upsilon_L}{}^2(\beta), R_{\upsilon_R}{}^2(\beta)] + \left(2\left[0.8+0.2\frac{(1-\beta)}{(1-u)}, 1.2-0.2\frac{(1-\beta)}{(1-u)}\right] - \left[0.15+0.05\frac{(1-\beta)}{(1-u)}, 0.25-0.05\frac{(1-\beta)}{(1-u)}\right]\right) + \left(\left[0.8+0.2\frac{(1-\beta)}{(1-u)}, 1.2-0.2\frac{(1-\beta)}{(1-u)}\right] - \left[0.15+0.05\frac{(1-\beta)}{(1-u)}, 0.25-0.05\frac{(1-\beta)}{(1-u)}\right]\right) = \left[0.6+0.3\frac{(1-\beta)}{(1-u)}, 1.2-0.3\frac{(1-\beta)}{(1-u)}\right]$$

or,

$$\left(0.8+0.2\frac{\alpha}{w}\right)R_{\mu_L}{}^2(\alpha) + \left\{2\left(0.8+0.2\frac{\alpha}{w}\right) - \left(0.25-0.05\frac{\alpha}{w}\right)\right\}R_{\mu_L}(\alpha) + \left\{\left(0.8+0.2\frac{\alpha}{w}\right) - \left(0.25-0.05\frac{\alpha}{w}\right)\right\} = 0.6+0.3\frac{\alpha}{w}$$

$$\left(1.2-0.2\frac{\alpha}{w}\right)R_{\mu_R}{}^2(\alpha) + \left\{2\left(1.2-0.2\frac{\alpha}{w}\right) - \left(0.15+0.05\frac{\alpha}{w}\right)\right\}R_{\mu_R}(\alpha) + \left\{\left(1.2-0.2\frac{\alpha}{w}\right) - \left(0.15+0.05\frac{\alpha}{w}\right)\right\} = 1.2-0.3\frac{\alpha}{w}$$

$$\left(0.8+0.2\frac{(1-\beta)}{(1-u)}\right)R_{\upsilon_L}{}^2(\beta) + \left\{2\left(0.8+0.2\frac{(1-\beta)}{(1-u)}\right) - \left(0.25-0.05\frac{(1-\beta)}{(1-u)}\right)\right\}R_{\upsilon_L}(\beta) + \left\{\left(0.8+0.2\frac{(1-\beta)}{(1-u)}\right) - \left(0.25-0.05\frac{(1-\beta)}{(1-u)}\right)\right\} = 0.6+0.3\frac{(1-\beta)}{(1-u)}$$

and

$$\left(1.2-0.2\frac{(1-\beta)}{(1-u)}\right)R_{\upsilon_R}{}^2(\beta) + \left\{2\left(1.2-0.2\frac{(1-\beta)}{(1-u)}\right) - \left(0.15+0.05\frac{(1-\beta)}{(1-u)}\right)\right\}R_{\upsilon_R}(\beta) + \left\{\left(1.2-0.2\frac{(1-\beta)}{(1-u)}\right) - \left(0.15+0.05\frac{(1-\beta)}{(1-u)}\right)\right\} = 1.2-0.3\frac{(1-\beta)}{(1-u)}$$

$$\Rightarrow R_{\mu_L}(\alpha) = \frac{-\left(1.35+0.45\frac{\alpha}{w}\right)+\sqrt{\left(1.35+0.45\frac{\alpha}{w}\right)^2+4\left(0.80+0.20\frac{\alpha}{w}\right)\left(0.05+0.05\frac{\alpha}{w}\right)}}{2\left(0.80+0.20\frac{\alpha}{w}\right)}$$

$$R_{\mu_R}(\alpha)=\frac{-\left(2.25-0.45\frac{\alpha}{w}\right)+\sqrt{\left(2.25-0.45\frac{\alpha}{w}\right)^2+4\left(1.20-0.20\frac{\alpha}{w}\right)\left(0.15-0.05\frac{\alpha}{w}\right)}}{2\left(1.20-0.20\frac{\alpha}{w}\right)}$$

$$R_{\upsilon_L}(\beta)=\frac{-\left(1.35+0.45\frac{(1-\beta)}{(1-u)}\right)+\sqrt{\left(1.35+0.45\frac{(1-\beta)}{(1-u)}\right)^2+4\left(0.80+0.20\frac{(1-\beta)}{(1-u)}\right)\left(0.05+0.05\frac{(1-\beta)}{(1-u)}\right)}}{2\left(0.80+0.20\frac{(1-\beta)}{(1-u)}\right)}$$

and

$$R_{\upsilon_R}(\beta)=\frac{-\left(2.25-0.45\frac{(1-\beta)}{(1-u)}\right)+\sqrt{\left(2.25-0.45\frac{(1-\beta)}{(1-u)}\right)^2+4\left(1.20-0.20\frac{(1-\beta)}{(1-u)}\right)\left(0.15-0.05\frac{(1-\beta)}{(1-u)}\right)}}{2\left(1.20-0.20\frac{(1-\beta)}{(1-u)}\right)}$$

Now, $\frac{d}{d\alpha}\left[R_{\mu_L}(\alpha)\right]>0, \frac{d}{d\alpha}\left[R_{\mu_R}(\alpha)\right]<0, \frac{d}{d\beta}\left[R_{\upsilon_L}(\beta)\right]<0$ and $\frac{d}{d\beta}\left[R_{\upsilon_R}(\beta)\right]>0$
and $R_{\mu_L}(0)= 0.036258 = R_{\upsilon_L}(1),\ R_{\mu_R}(0)= 0.0644512 = R_{\upsilon_R}(1)$
$\therefore R_{\mu_L}(0.7) = R_{\mu_R}(0.7) = 0.0539392 = R_{\upsilon_L}(0.2) = R_{\upsilon_R}(0.2)$
Hence the strong solution exists here and the (α,β)- cut of the strong solution is

$$R^i_{\alpha,\beta}=\langle\left[R_{\mu_L}(\alpha),R_{\mu_R}(\alpha)\right],\left[R_{\upsilon_L}(\beta),R_{\upsilon_R}(\beta)\right]\rangle$$

where

$$\left[R_{\mu_L}(\alpha),R_{\mu_R}(\alpha)\right]=\left\{x:\frac{-\left(1.35+0.45\frac{\alpha}{w}\right)+\sqrt{\left(1.35+0.45\frac{\alpha}{w}\right)^2+4\left(0.80+0.20\frac{\alpha}{w}\right)\left(0.05+0.05\frac{\alpha}{w}\right)}}{2\left(0.80+0.20\frac{\alpha}{w}\right)}\le x\le\frac{-\left(2.25-0.45\frac{\alpha}{w}\right)+\sqrt{\left(2.25-0.45\frac{\alpha}{w}\right)^2+4\left(1.20-0.20\frac{\alpha}{w}\right)\left(0.15-0.05\frac{\alpha}{w}\right)}}{2\left(1.20-0.20\frac{\alpha}{w}\right)}\right\}$$

and

$$\left[R_{\upsilon_L}(\beta),R_{\upsilon_R}(\beta)\right]=\left\{x:\frac{-\left(1.35+0.45\frac{(1-\beta)}{(1-u)}\right)+\sqrt{\left(1.35+0.45\frac{(1-\beta)}{(1-u)}\right)^2+4\left(0.80+0.20\frac{(1-\beta)}{(1-u)}\right)\left(0.05+0.05\frac{(1-\beta)}{(1-u)}\right)}}{2\left(0.80+0.20\frac{(1-\beta)}{(1-u)}\right)}\le x\le\frac{-\left(2.25-0.45\frac{(1-\beta)}{(1-u)}\right)+\sqrt{\left(2.25-0.45\frac{(1-\beta)}{(1-u)}\right)^2+4\left(1.20-0.20\frac{(1-\beta)}{(1-u)}\right)\left(0.15-0.05\frac{(1-\beta)}{(1-u)}\right)}}{2\left(1.20-0.20\frac{(1-\beta)}{(1-u)}\right)}\right\}$$

$\therefore$ The solution $\tilde{R}^i$ is a GTsIFN $(0.036258,\ 0.0539392,\ 0.0644512; 0.7, 0.2)$ and the rough sketch of the membership and non-membership function of $\tilde{R}^i$ is given below :

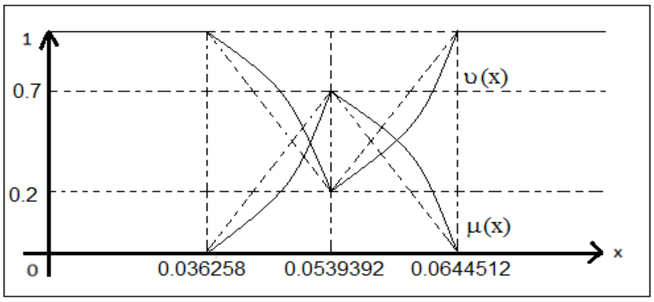


Fig-3.7: Rough sketch of membership and non-membership function of $\tilde{R}^i$

4. Conclusion

In this paper, we have solved Intuitionistic Fuzzy Linear and Quadratic Equations using the concept of strong and weak solution taking coefficients as GTIFNs. Further we have discussed two problems following Buckley in Intuitionistic fuzzy environment. For future work, these Intuitionistic fuzzy equations can be solved by taking the coefficients as Intuitionistic L-R Fuzzy Numbers. In Engineering, Physical and

Mathematical Sciences there are so many problems involving linear and quadratic equations which can be solved in Intuitionistic fuzzy environment.

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