MORE-FOR-LESS PARADOX IN A SOLID TRANSPORTATION PROBLEM

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Abstract. In this paper, we discuss more-for-less paradox in a solid transportation problem. Thereby, we demonstrate a theorem which gives a comfortable condition for the existence of paradox in this type of problem. Next we present an algorithm to find out all the paradoxical pairs as well as paradoxical range of flow and paradoxical pair for a specified flow if paradox exists. Also we illustrate a numerical example in support of the given algorithm.

Keywords: Solid Transportation Problem, Paradox in a Solid Transportation Problem, Paradoxical Range of Flow

1. Introduction


In some cases of the classical TP, an increase in the supplies and demands or in other words, increase in the flow results a decrease in the optimum transportation cost. This type of behavior which means paradoxical, is called transportation paradox. Basically, the papers of Charnes and Klingman[10] and Szwarc[17] are treated as the sources of transportation paradox for the researchers. In the paper of Charnes and Klingman, they name it “more-for-less” paradox and wrote “The paradox was first observed in the early days of linear programming history (by whom no one knows) and has been a part of the folklore known to some (e.g. A.Charnes and W.W.Cooper), but unknown to the great majority of workers in the field of linear programming”. Subsequently, in the paper of Appa[4], he mentioned that this paradox is known as “Doig Paradox” at the London School of Economics, named after Alison Doig. Gupta et al.[12] established a sufficient condition for a paradox in a linear fractional transportation problem with mixed constraints. Adlakha and Kowalski [3] derived a sufficient condition to identify the cases where the paradoxical situation exists. Deineko et al.[19] developed a necessary and sufficient condition for a cost matrix which is immersed against the transportation paradox. Basuet al.[7] provided an algorithm for obtaining paradoxical range of flow and paradoxical flow for a specified flow. Acharya et al. [1] developed an algorithm for obtaining paradoxical range of flow and paradoxical flow for a specified flow in a fixed charge transportation problem. They [2] also considered paradox in a fuzzy transportation problem with linear constraints.

In this paper, we present a method for solving solid transportation problem with linear constraints. Thereby, we state a sufficient condition for existence of paradox. Then we give an algorithm for obtaining all
paradoxical pairs, paradoxical range of flow and paradoxical pair for a specified flow in such type of problem. We also justify the theory by illustrating a numerical example.

2. Problem Formulation

In this paper, our goal is to obtain a transportation plan which satisfies all required demands and minimizes the overall transportation cost of the problem:

\[
P_1 : \text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} p_{ijk} x_{ijk}
\]

subject to the constraints,

\[
\sum_{j=1}^{n} \sum_{k=1}^{l} x_{ijk} = a_i \quad \forall \quad i \in I = (1, 2, 3, \ldots, m)
\]

\[
\sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk} = b_j \quad \forall \quad j \in J = (1, 2, 3, \ldots, n)
\]

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} = c_k \quad \forall \quad k \in K = (1, 2, 3, \ldots, l)
\]

and \(x_{ijk} \geq 0 \quad \forall \quad (i, j, k) \in I \times J \times K\)

where,

- \(x_{ijk}\) is the amount of k-th type of product transported from the i-th origin to the j-th destination,
- \(p_{ijk}\) is the cost involved in transporting per unit of the k-th type of product from the i-th origin to the j-th destination,
- \(a_i\) is the number of units available at the i-th origin,
- \(b_j\) is the number of units required at the j-th destination,
- \(c_k\) is the requirement of the number of units of the k-th product.

Hence this problem consists of \(m\) origins and \(n\) destinations along with \(l\) types of products. We assume that \(\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{k=1}^{l} c_k\) and \(a_i, b_j, c_k > 0\) for all \(i, j, k\) which is known as balanced STP. If the STP \(P_1\) is unbalanced then we convert it to a balanced STP by using dummy variables with zero cost.

Let \(X^a = \{x_{ijk}^a | (i, j, k) \in I \times J \times K\}\) be a basic feasible solution corresponding to the basis \(B\) of the problem \(P_1\), the corresponding value of the objective function \(Z^a = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} p_{ijk} x_{ijk}^a\) and the flow \(F^a = \sum_{i \in I} a_i = \sum_{j \in J} b_j = \sum_{k \in K} c_k\)

**Definition 2.1.** The pair \((Z^a, F^a)\) is called the cost-flow pair corresponding to the feasible solution \(X^a\).

**Condition of optimality:** The condition of optimality of the problem \(P_1\) is \(p_{ijk} = (u_i + v_j + w_k) - p_{ijk} \leq 0\) for all \((i, j, k) \in B\), where \(u_i, v_j, w_k\) are the dual variables corresponding to the basis \(B\) such that \(p_{ijk} = (u_i + v_j + w_k)[11]\).

Let \(Z^0 = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} p_{ijk} x_{ijk}^0\) and \(F^0 = \sum_{i \in I} a_i = \sum_{j \in J} b_j = \sum_{k \in K} c_k\) be the optimum cost and flow respectively corresponding to the optimum solution \(X^0 = \{x_{ijk}^0 | (i, j, k) \in I \times J \times K\}\).

**Definition 2.2** In a STP, if we can obtain flow \(F^1 > F^0\) with cost \(Z^1 < Z^0\) then we say paradox occurs.

**Definition 2.3** The pair \((Z^1, F^1)\) is called the paradoxical cost-flow pair if paradox exists.

**Definition 2.4** A cost-flow pair \((Z^2, F^2)\) is called an improved paradoxical cost-flow pair if \(Z^2 < Z^1\) and \(F^2 > F^1\).

**Definition 2.5** The paradoxical cost-flow pair \((Z^g, F^g)\) where \((Z^i, F^i), 1 < i < g\), such that \(Z^i < Z^{i-1}\) and \(F^i > F^{i-1}\) \(\forall i\) be all paradoxical cost-flow pairs and \(Z^* = Z^g\) and \(F^* = F^g\), is called the best paradoxical cost-flow pair.
Definition 2.5 The pair \([F^0, F^+]\) is defined as paradoxical range of flow.

Theorem 2.1. The sufficient condition for the existence of paradoxical solution of the problem \(P_1\) is that if \(\exists\) at least one cell \((r, s, t) \in \mathcal{E}B\) in the optimum table of \(P_1\) where \(a_r, b_s\) and \(c_t\) are replaced by \(a_r + q, b_s + q\) and \(c_t + q\) respectively \((q > 0)\) then \((u_r + v_s + w_t) < 0\).

Proof: Let \((Z^0, F^0)\) be the optimum cost-flow pair corresponding to the optimum solution \(X^0 = \{x^0_{ijk} | (i, j, k) \in I \times J \times K\}\) of the problem \(P_1\). The dual variables \(u_i, v_j\) and \(w_k\) satisfy the equation \(p_{ijk} = (u_i + v_j + w_k) \forall (i, j, k) \in B\).

Then
\[
Z^0 = \sum_i \sum_j \sum_k c_{ijk} x^0_{ijk} = \sum_i \sum_j \sum_k (u_i + v_j + w_k) x^0_{ijk} = \sum_i (\sum_j (\sum_k x^0_{ijk}) u_i + \sum_j (\sum_k x^0_{ijk}) v_j + \sum_k (\sum_i x^0_{ijk}) w_k) = \sum_i a_i u_i + \sum_j b_j v_j + \sum_k c_k w_k,
\]
and \(F^0 = \sum_i a_i = \sum_j b_j = \sum_k c_k\).

Now, let \(\exists\) at least one cell \((r, s, t) \in \mathcal{E}B\), where \(a_r, b_s, c_t\) are replaced by \(a_r + q, b_s + q\) and \(c_t + q\) respectively \((q > 0)\) in such a way that the optimum basis remains same, then the value of the objective function \(\hat{Z}\) is given by
\[
\hat{Z} = \sum_i a_i u_i + \sum_j b_j v_j + \sum_k c_k w_k + u_r (a_r + q) + v_s (b_s + q) + w_t (c_t + q) = [Z^0 + q(u_r + v_s + w_t)].
\]

The new flow \(\hat{F}\) is given by
\[
\hat{F} = \sum_i a_i + q = \sum_j b_j + q = \sum_k c_k + q. \text{ Therefore } \hat{F} - F^0 = q > 0.
\]

Hence for the existence of paradox we must have \(\hat{Z} - Z^0 < 0\). So the sufficient condition for the existence of paradox is that at least one cell \((r, s, t) \in \mathcal{E}B\) in the optimum table of \(P_1\) where \(a_r, b_s\) and \(c_t\) are replaced by \(a_r + q, b_s + q\) and \(c_t + q\) respectively \((q > 0)\) then \(q(u_r + v_s + w_t) < 0\), i.e. \((u_r + v_s + w_t) < 0\).

Now we state two algorithms.

Algorithm 2.1 To obtain all the paradoxical pairs.

Step 1: Find the optimum cost-flow pair \((Z^0, F^0)\) for the optimum solution \(X^0\).
Step 2: \(i = 1\).
Step 3: Find all cells \((r, s, t) \in \mathcal{E}B\) such that \((u_r + v_s + w_t) < 0\) if it exists, otherwise go to step 8.
Step 4: \(F^i = F^{i-1} + 1\).
Step 5: Obtain \(X^i\) and \(Z^i\) corresponding to \(F^i\). Write \((Z^i, F^i)\).
Step 6: \(i = i + 1\).
Step 7: go to Step 3.
Step 8: Write the best paradoxical pair \((Z^*, F^*) = (Z^i, F^i)\) for the optimum solution \(X^* = X^i\).
Step 9: End.

Algorithm 2.2 To obtain the paradoxical pair for a specified flow \(\bar{F}\).

Step 1: Find the optimum cost-flow pair \((Z^0, F^0)\) for the optimum solution \(X^0\).
Step 2: \(i = 1\).
Step 3: Find all cells \((r, s, t) \in \mathcal{E}B\) such that \((u_r + v_s + w_t) < 0\).
Step 4: \(F^i = F^{i-1} + 1\).
Step 5: Obtain \(X^i\) and \(Z^i\) corresponding to \(F^i\). Write \((Z^i, F^i)\).
Step 6: If $\bar{F} = F^i$ go to step 9.
Step 7: $i = i + 1$.
Step 8: go to Step 3.
Step 9: We write the paradoxical solution for a specified flow $(\bar{Z}, \bar{F}) = (Z^i, F^i)$ corresponding to the optimum solution $\bar{X} = X^i$.
Step 10: End.

3. Numerical Example

We consider the following problem (Figure 1)

Applying algorithm 2.1, the optimum cost-flow pair $(Z^0, F^0) = (150, 49)$ corresponding to the optimum solution $X^0 = (x_{111} = 2, x_{112} = 7, x_{213} = 4, x_{222} = 18, x_{131} = 13, x_{133} = 5)$ (Figure 2).

We get the cells $(1, 2, 1), (1, 2, 3), (2, 2, 1)$ and $(2, 2, 3)$ with respective costs 148, 149, 147 and 148. The paradoxical cost-flow pair $(Z^1, F^1) = (147, 50)$ (Figure 3).
All the paradoxical pairs are (150,49), (147,50), (144,51), (141,52), (138,53), (135,54), (132,55), (129,56).

Hence the best paradoxical pair is $(Z^*, F^*) = (129, 56)$ corresponding to the optimum solution $X^* = (x_{111} = 9, x_{213} = 4, x_{222} = 25, x_{131} = 13, x_{133} = 5)$ (Figure 4).

4. Conclusion

In real life, we face many problems which belong to three dimensional (solid) rather than two dimensional classical transportation problem. Practically, paradox in a solid transportation problem may occur quite frequently. But till date, researchers do not give any attention in this area.

In this paper, we discuss a paradox (so called "more-for-less" paradox) in a solid transportation problem. Thereby, we develop a new efficient algorithm for solving paradox in a solid transportation problem if paradox exists. In this procedure, we not only obtain the best paradoxical pair but also all the paradoxical pairs as well as the paradoxical pair for a specified flow. Today, calculation is very simple and not time taking if one solves this type of problem using mathematical software. The managers in decisions such as...
increasing warehouse/plant capacity and/or advertising efforts to increase demand at some destinations and/or for some types of products may use this paradoxical analysis to increase his business under the same environment. Hence, in practically it is an important part of solid transportation problem.

5. References