ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 10, No. 2, 2015, *pp*. 098-105

Numerical Solution of Fractional Differential Equations by using Fractional Spline Model

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(*Received November 14, 2014, accepted March 01, 2015*

Abstract. In this paper, we consider a new suitable lacunary fractional interpolation with the idea of the spline function of polynomial form, and the method applied to solve linear fractional differential equations. The results obtained are in good agreement with the exact analytical solutions and the numerical results presented by several examples, results also show that the technique introduced here is robust and easy to apply.

Keywords: Fractional integral and derivative; Caputo Derivative; Taylor's expansion; Error bound; Spline functions

1. Introduction

Fractional differential equations are gaining considerable importance recently due to their wide range of applications in the fields of Physics, Engineering [3, 19], Chemistry, and/or Biochemistry [4], Control [5,6,7, 8], Medicine [9] and Biology [1]. Several techniques such as Adomian decomposition method (ADM) [11], Adams-Bashforth- Moulton method [14, 15], Fractional difference method [10], and Variational iteration method [12, 13] have been developed for solving non linear functional equations in general and solving fractional differential equations in particular.

In view of successful application of spline functions of polynomial form in system analysis [22], fractional differential equations ([24], [25]), and delay differential equations of fractional order [23], we hold that it should be applicable to solve fractional differential equations with the idea of the lacunary interpolation. For more details on lacunary interpolation we may refer to ([16]–[18]).

In this paper we shall apply fractional spline to find the approximate analytical solution of the fractional differential equation. Error bound and existence and uniqueness for the method will be performed.

2. Preliminaries

In this section, some definitions and Taylor's Theorem, used in our work, will be presented. There are many definitions for fractional derivatives, the most commonly used ones are the Riemann-Liouville and the Caputo derivatives, especially the Caputo derivative are involved in our work. Suppose that $\alpha > 0, x > a, \quad a, x \in \mathbb{R}$, then

Definition 1[21] *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *is defined by*

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N},$$

where Γ is the gamma function.

Definition 2[21] *The Riemann-Liouville fractional derivative of order* $\alpha > 0$ *is defined by*

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\alpha-1} f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N}.$$

Definition 3[2] *The Caputo fractional derivative of order* $\alpha > 0$ *is defined by*

$$D_*^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N}.$$

Definition 4[20] *Let* $\alpha \in \mathbb{R}^+$, $\Omega \subset \mathbb{R}$ *an interval such that* $a \in \Omega$, $a \leq x$, $x \in \Omega$. *Then the following set of functions are defined:*

$$_aI_\alpha = \{ f \in C(\Omega)\colon\ I^\alpha f(x)\ exists\ and\ is\ finite\ in\ \Omega \},$$
$$_aD_\alpha = \{ f \in C(\Omega)\colon\ D_*^\alpha f(x)\ exists\ and\ is\ finite\ in\ \Omega \}.$$

In view of these definitions we can conclude the following theorem:

Theorem 1 [20] *Let* $\alpha \in (0,1]$, $p \in \mathbb{N}$ *and* $f(x)$ *a continuous function in* $[a,b]$ *satisfying the following conditions:*

(1) $D_*^{m\alpha} f \in C([a,b])$ *and* $D_*^{m\alpha} f \in\ _aI_\alpha([a,b]), \forall\ m = 1,2,\ldots,p.$

(2) $D_*^{(p+1)\alpha} f(x)$ *is continuous on* $[a,b]$.

Then for each $x \in [a,b]$,

$$f(x) = \sum_{m=0}^{p} D_*^{m\alpha} f(a) \frac{(x-a)^{m\alpha}}{\Gamma(m\alpha+1)} + R_p(x,a),$$

with $R_p(x,a) = D_*^{(p+1)\alpha} f(\xi) \frac{(x-a)^{(p+1)\alpha}}{\Gamma((p+1)\alpha+1)}$, $a \leq \xi \leq x$.

Remark 1 *For simplicity we will use the operator* D *instead of* D_* *from now on.*

3. Description of the Method

Given the mesh points, $\Delta\colon 0 = x_0 < x_1 < \cdots < x_n = 1$ with $x_{k+1} - x_k = h,\ k = 0,1,\ldots,n-1$, and real numbers $\left\{f_k, D^{\frac{1}{2}} f_k, \left(D^{\frac{1}{2}}\right)^4 f_k\right\}_{k=0}^{n}$ associated with the knots. We are going to construct spline interpolant S_Δ for which $D^{m\cdot\frac{1}{2}} S_\Delta(x_i) = D^{m\cdot\frac{1}{2}} f_i$, $i = 0,1,\ldots,n$, and m=0,1,4. This construction is given in the following two cases:

Case 1:

In this case, we suppose that the conditions of Theorem 1 are satisfied with $p = 4$, and $\alpha = 0.5$. Then we can define the spline interpolant as follows:

$$S_\Delta = S_k(x) = y_k + D^{\frac{1}{2}} y_k \frac{2(x-x_k)^{\frac{1}{2}}}{\sqrt{\pi}} + a_k(x-x_k) + b_k \frac{4(x-x_k)^{\frac{3}{2}}}{3\sqrt{\pi}} + \left(D^{\frac{1}{2}}\right)^4 y_k \frac{(x-x_k)^2}{2}, \quad (1)$$

where $x_k \leq x \leq x_{k+1}$ and $k = 0,1,\ldots,n-1$.

4 Existence and Uniqueness

If we require that $S_\Delta(x)$ and $D^{\frac{1}{2}} S_\Delta(x)$ is continuous on $[0,1]$, then it is easy to prove that formula (1) *exists and is unique*. That is, clear from the continuity conditions of $S_\Delta(x)$ and $D^{\frac{1}{2}} S_\Delta(x)$ from which we get:

$$y_{k+1} = y_k + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} D^{\frac{1}{2}} y_k + h a_k + b_k \frac{4h^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{h^2}{2} \left(D^{\frac{1}{2}}\right)^4 y_k, \quad (2)$$

and

$$D^{\frac{1}{2}} y_{k+1} = D^{\frac{1}{2}} y_k + \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} a_k + h b_k + \frac{4h^{\frac{3}{2}}}{3\sqrt{\pi}} \left(D^{\frac{1}{2}}\right)^4 y_k. \quad (3)$$

Let

$$A_k = y_{k+1} - y_k - \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} D^{\frac{1}{2}} y_k - \frac{h^2}{2} \left(D^{\frac{1}{2}}\right)^4 y_k,$$

and let

$$B_k = D^{\frac{1}{2}} y_{k+1} - D^{\frac{1}{2}} y_k - \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} \left(D^{\frac{1}{2}}\right)^4 y_k.$$

Then the equations of (2) and (3), respectively, become

$$A_k = h a_k + \frac{4}{3\sqrt{\pi}} h^{\frac{3}{2}} b_k, \quad (4)$$

$$B_k = \frac{2}{\sqrt{\pi}} h^{\frac{1}{2}} a_k + h b_k. \quad (5)$$

Solving these two equations for a_k and b_k we obtain

$$a_k = \frac{\sqrt{\pi}\left(3\sqrt{\pi}A_k - 4B_k h^{\frac{1}{2}}\right)}{h(3\pi-8)}, \tag{6}$$

$$b_k = \frac{3\sqrt{\pi}\left(\sqrt{\pi}B_k - 2A_k h^{\frac{-1}{2}}\right)}{h(3\pi-8)}. \tag{7}$$

5 Error Bounds

Suppose that the conditions of Theorem 1 are satisfied with $p = 4$ and $D^{m\cdot\frac{1}{2}}S_\Delta(x_i) = D^{m\cdot\frac{1}{2}}f_i,\quad i = 0,1,\ldots,n-1$. We shall prove the following:

Theorem 2 *Let* $S_k(x)$ *be the fractional spline interpolant of the polynomial form (1) solving the lacunary case* $(0,\alpha,4\alpha)$ *. Then for all* $x \in [0,1]$ *the inequality*

$$|(D^\alpha)^m S_\Delta(x) - (D^\alpha)^m y(x)| \le c_{m\alpha} h^{2-m\alpha}\omega(h),$$

holds for all $m = 0,1,\ldots,4$ *and* $\alpha = 0.5$. *Where* $\omega(h)$ *is the modulus of continuity of* $\left(D^{\frac{1}{2}}\right)^4 y(x)$, *and*

$$c_0 = \frac{9\pi + 48\sqrt{\pi} + 4}{9\pi - 24},\quad c_{\frac{1}{2}} = \frac{14\sqrt{\pi}}{3\pi - 8},\quad c_1 = \frac{27\sqrt{\pi} + 68}{18\pi - 48},\quad c_{\frac{3}{2}} = \frac{7\sqrt{\pi}}{3\pi - 8} + \frac{2}{\sqrt{\pi}},\quad c_2 = 1.$$

To proof this theorem we shall need the following lemma.

Lemma 1 *The following estimates are valid for all* $k = 0(1)n-1$.

$$\left|a_k - \left(D^{\frac{1}{2}}\right)^2 y_k\right| \le \frac{9\pi+32}{18\pi-48} h\omega(h), \tag{8}$$

$$\left|b_k - \left(D^{\frac{1}{2}}\right)^3 y_k\right| \le \frac{7\sqrt{\pi}}{3\pi-8} h^{\frac{1}{2}}\omega(h), \tag{9}$$

for $k = 0,1,\ldots,n-1$.

Proof. From (6) we can find

$$\left|a_k - \left(D^{\frac{1}{2}}\right)^2 y_k\right| = \left|\frac{\sqrt{\pi}}{h(3\pi-8)}\left[3\sqrt{\pi}\left(y_{k+1} - y_k - \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}D^{\frac{1}{2}}y_k - h\left(D^{\frac{1}{2}}\right)^2 y_k - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y_k\right) - \right.\right.$$

4h12D12yk+1−D12yk−2πh12D12yk−43πh32D124yk.(10)

Taking:

$$y_{k+1} = y_k + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}D^{\frac{1}{2}}y_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^3 y_k + h\left(D^{\frac{1}{2}}\right)^2 y_k + \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y(\xi_k),$$

and

$$D^{\frac{1}{2}}y_{k+1} = D^{\frac{1}{2}}y_k + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}\left(D^{\frac{1}{2}}\right)^2 y_k + h\left(D^{\frac{1}{2}}\right)^3 y_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^4 y(\eta_k),$$

where $x_k < \xi_k, \eta_k < x_{k+1}$. Then (10) becomes

$$\begin{aligned}
&\left|a_k - \left(D^{\frac{1}{2}}\right)^2 y_k\right| \\
&\quad = \frac{\sqrt{\pi}}{h(3\pi-8)}\left|3\sqrt{\pi}\left(\frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y(\xi_k) - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y_k\right)\right. \\
&\quad\left. - 4h^{\frac{1}{2}}\left(\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^4 y(\eta_k - \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^4 y_k\right)\right| \le \frac{9\pi+32}{18\pi-48}h\omega(h).
\end{aligned}$$

This time from (7) we can find

$$\left|b_k - \left(D^{\frac{1}{2}}\right)^3 y_k\right|$$

$$= \left|\frac{3\sqrt{\pi}}{h(3\pi-8)}\left[\sqrt{\pi}\left(D^{\frac{1}{2}}y_{k+1} - D^{\frac{1}{2}}y_k - h\left(D^{\frac{1}{2}}\right)^3 y_k - \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^4 y_k\right) - 2h^{\frac{-1}{2}}\left(y_{k+1} - y_k - \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}D^{\frac{1}{2}}y_k - \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^3 y_k - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y_k\right)\right]\right|$$

$$= \frac{3\sqrt{\pi}}{h(3\pi-8)}\left|\sqrt{\pi}\left(\frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^4 y(\delta_k) - \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^4 y_k\right) - 2h^{\frac{-1}{2}}\left(\frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y(\gamma_k) - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y_k\right)\right| \le \frac{7\sqrt{\pi}}{3\pi-8}h^{\frac{1}{2}}\omega(h).$$

Thus we have proved the lemma.■

Proof of Theorem 2. In view of the above lemma we can see that, for $x_k \le x \le x_{k+1}$ and $k = 0,1,\ldots,n-1$,

$$|S_k(x) - y(x)| = \left|y_k + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}D^{\frac{1}{2}}y_k + ha_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^3 b_k + \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y_k - y_k - \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}D^{\frac{1}{2}}y_k - h\left(D^{\frac{1}{2}}\right)^2 y_k - \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^3 b_k - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y(\eta_k)\right|$$

$$\le h\left|a_k - \left(D^{\frac{1}{2}}\right)^2 y_k\right| + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left|b_k - \left(D^{\frac{1}{2}}\right)^3 y_k\right| + \frac{h^2}{2}\omega(h)$$

$$\le \frac{9\pi+48\sqrt{\pi}+4}{9\pi-24}h^2\,\omega(h). \quad (11)$$

Similarly, we can obtain

$$\left|D^{\frac{1}{2}}S_k(x) - D^{\frac{1}{2}}y(x)\right| \le \frac{14\sqrt{\pi}}{3\pi-8}h^{\frac{3}{2}}\,\omega(h), \quad (12)$$

$$\left|\left(D^{\frac{1}{2}}S_k(x)\right)^2 - \left(D^{\frac{1}{2}}y(x)\right)^2\right| \le \frac{27\sqrt{\pi}+68}{18\pi-48}\,h\,\omega(h), \quad (13)$$

$$\left|\left(D^{\frac{1}{2}}S_k(x)\right)^3 - \left(D^{\frac{1}{2}}y(x)\right)^3\right| \le \left(\frac{7\sqrt{\pi}+68}{3\pi-8} + \frac{2}{\sqrt{\pi}}\right)h^{\frac{1}{2}}\,\omega(h), \quad (14)$$

and finally,

$$\left|\left(D^{\frac{1}{2}}S_k(x)\right)^4 - \left(D^{\frac{1}{2}}y(x)\right)^4\right| \le \omega(h). \quad (15)$$

This completes the proof. ■

Case 2:

In this case, we suppose that the conditions of Theorem are fulfilled with $p = 5$, and $\alpha = 0.5$ then we can define the spline interpolant as follows:

$$S_\Delta = S_k(x) = y_k + D^{\frac{1}{2}}y_k\frac{2(x-x_k)^{\frac{1}{2}}}{\sqrt{\pi}} + a_k(x-x_k) + b_k\frac{4(x-x_k)^{\frac{3}{2}}}{3\sqrt{\pi}} + \left(D^{\frac{1}{2}}\right)^4 y_k\frac{(x-x_k)^2}{2} + c_k\frac{8(x-x_k)^{\frac{5}{2}}}{15\sqrt{\pi}}, \quad (16)$$

where$x_k \le x \le x_{k+1}$ and $k = 0,1,\ldots,n-1$.

Let

$$c_k = \frac{\sqrt{\pi}}{2}h^{-\frac{1}{2}}\left[\left(D^{\frac{1}{2}}\right)^4 y_{k+1} - \left(D^{\frac{1}{2}}\right)^4 y_k\right].$$

It can be easily shown that

$$\left|c_k - \left(D^{\frac{1}{2}}\right)^5 y_k \le \omega(h)\right|, \quad (17)$$

where$\omega(h)$ is the modulus of continuity of$\left(D^{\frac{1}{2}}\right)^5 y(x)$.

Now, if $S_\Delta(x) \in C[0,1]$ and $S_\Delta^{\frac{1}{2}}(x) \in C[0,1]$ then the *existence* and *uniqueness* of $S_\Delta(x)$ is easy to be proved, since here a_k and b_k are uniquely determined by

$$a_k = \frac{\sqrt{\pi}\left(3\sqrt{\pi}A_k - 4B_k h^{\frac{1}{2}}\right)}{h(3\pi-8)}, \tag{18}$$

$$b_k = \frac{3\sqrt{\pi}\left(\sqrt{\pi}B_k - 2A_k h^{\frac{-1}{2}}\right)}{h(3\pi-8)}, \tag{19}$$

where

$$A_k = y_{k+1} - y_k - \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}D^{\frac{1}{2}}y_k - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y_k - \frac{8h^{\frac{5}{2}}}{15\sqrt{\pi}}c_k,$$

$$B_k = D^{\frac{1}{2}}y_{k+1} - D^{\frac{1}{2}}y_k - \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^4 y_k - \frac{h^2}{2}c_k .$$

Then we conclude the following lemma:

Lemma 2 *The following estimates can be obtained for* $k = 0,1,\ldots,n-1$,

$$\left|a_k - \left(D^{\frac{1}{2}}\right)^2 y_k\right| \le \frac{18\sqrt{\pi}}{5(3\pi-8)}h^{\frac{3}{2}}\omega(h), \tag{20}$$

$$\left|b_k - \left(D^{\frac{1}{2}}\right)^3 y_k\right| \le \frac{3}{16}\frac{15\pi+32}{3\pi-8}h\omega(h), \tag{21}$$

for$k = 0,1,\ldots,n-1$.

Proof. From (18) we have

$$\left|a_k - \left(D^{\frac{1}{2}}\right)^2 y_k\right| = \left|\frac{\sqrt{\pi}}{h(3\pi-8)}\right|\left[3\sqrt{\pi}\left(y_{k+1} - y_k - \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}D^{\frac{1}{2}}y_k - h\left(D^{\frac{1}{2}}\right)^2 y_k - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y_k\right) - \right.$$

8h5215πck−4h12D12yk+1−D12yk−2πh12D12yk−43πh32D124yk−h22ck.

(22)

Taking:

$$y_{k+1} = y_k + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}D^{\frac{1}{2}}y_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^3 y_k + h\left(D^{\frac{1}{2}}\right)^2 y_k + \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^4 y_k - \frac{8h^{\frac{5}{2}}}{15\sqrt{\pi}}\left(D^{\frac{1}{2}}\right)^5 y(\xi_k),$$

and

$$D^{\frac{1}{2}}y_{k+1} = D^{\frac{1}{2}}y_k + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}\left(D^{\frac{1}{2}}\right)^2 y_k + h\left(D^{\frac{1}{2}}\right)^3 y_k + \frac{4}{3\sqrt{\pi}}h^{\frac{3}{2}}\left(D^{\frac{1}{2}}\right)^4 y_k - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^5 y(\eta_k).$$

where $x_k < \xi_k, \eta_k < x_{k+1}$. Then (22) becomes

$$\left|a_k - \left(D^{\frac{1}{2}}\right)^2 y_k\right|$$

$$= \frac{\sqrt{\pi}}{h(3\pi-8)}\left|\begin{array}{c} 3\sqrt{\pi}\left(\frac{8h^{\frac{5}{2}}}{15\sqrt{\pi}}\left(D^{\frac{1}{2}}\right)^5 y(\xi_k) - \frac{8h^{\frac{5}{2}}}{15\sqrt{\pi}}\left(D^{\frac{1}{2}}\right)^5 c_k\right) \\ -4h^{\frac{1}{2}}\left(\frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^5 y(\eta_k) - \frac{h^2}{2}\left(D^{\frac{1}{2}}\right)^5 c_k\right) \end{array}\right|$$

$$\le \frac{\sqrt{\pi}}{h(3\pi-8)}\left|\frac{8}{5}h^{\frac{5}{2}}\left(\left(D^{\frac{1}{2}}\right)^5 y(\xi_k) - \left(D^{\frac{1}{2}}\right)^5 c_k\right) - 2h^{\frac{5}{2}}\left(\left(D^{\frac{1}{2}}\right)^5 y(\eta_k) - \left(D^{\frac{1}{2}}\right)^5 c_k\right)\right|$$

$$\le \frac{\sqrt{\pi}}{h(3\pi-8)}\left[\frac{8}{5}h^{\frac{5}{2}}\omega(h)+2h^{\frac{5}{2}}\omega(h)\right], \qquad [using\ \ (17)]$$
$$=\frac{18\sqrt{\pi}}{5(3\pi-8)}h^{\frac{3}{2}}\omega(h).$$

Similarly for b_k, after using (17) and (19), we obtain

$$\left|b_k-\left(D^{\frac{1}{2}}\right)^3 y_k\right| \le \frac{3}{16}\frac{15\pi+32}{3\pi-8}h\omega(h).$$

Thus we have proved the lemma. ■

And then we can conclude the following theorem:

Theorem 3 *Let* $S_k(x)$ *be the fractional spline interpolant of the polynomial form (1) solving the lacunary case* $(0,\alpha,4\alpha)$ *for which the conditions of theorem are satisfied with* $p=5$. *Then for all* $x\in[0,1]$ *the inequality*

$$|(D^{\alpha})^m S_{\Delta}(x)-(D^{\alpha})^m y(x)| \le c_{m\alpha}h^{2.5-m\alpha}\omega(h),$$

holds for all $m=0,1,\dots,5$ *and* $\alpha=0.5$. *Where* $\omega(h)$ *is the modulus of continuity of* $(D^{1/2})^5y(x)$, *and*

$$c_0=\frac{75\pi+72\sqrt{\pi}+160}{20(3\pi-8)}+\frac{8}{15\sqrt{\pi}},\quad c_{\frac{1}{2}}=\frac{36}{5(3\pi-8)}+\frac{3}{16}\frac{15\pi+32}{3\pi-8}+\frac{1}{2},$$
$$c_1=\frac{18\sqrt{\pi}}{5(3\pi-8)}+\frac{3}{8\sqrt{\pi}}\frac{15\pi+32}{3\pi-8}+\frac{4}{3\sqrt{\pi}},\quad c_{\frac{3}{2}}=\frac{3}{18}\frac{15\pi+32}{3\pi-8}+1,\quad c_2=\frac{2}{\sqrt{\pi}},\quad c_{\frac{5}{2}}=1.$$

Proof. The proof is similar to that of Theorem 2, for this reason we omit it.

6 Numerical Results

In this section, the method is applied to two numerical examples. All calculations are implemented with MATLAB 12b. The numerical scheme leads to linear fractional differential equations and can apply for different step sizes *h*. To show the efficiency of the method, it is comparisedwith the exact solution we report absolute error and observed same convergence point with less iterations.

Example 1 *Consider the linear fractional differential equation*

$$D^2y(x)+2D^{\alpha}y(x)+y(x)=2x+\frac{4}{\Gamma(4-\alpha)}x^{3-\alpha}+\frac{1}{3}x^3, \qquad 0<\alpha<1, \qquad (23)$$

subject to

$$y(0)=y'(0)=0.$$

It is easily verified that the exact solution of this problem is

$$y(x)=\frac{1}{3}x^3.$$

The maximal absolute errors obtained for $\alpha=0.5$ *and for* $0\le x\le 1$ *in each case and these are shown in Table 1 and Table 2, to illustrate the accuracy of the spline method of polynomial form. Note that* $|e^{m\alpha}(x)|=|D^{m\alpha}S_k(x)-D^{m\alpha}y(x)|$, *for* $\alpha=0.5$, $m=0,1,\dots,4$ *for case 1, and* $m=0,1,\dots,5$ *for case 2.*

Table 1: Maximal absolute errors in case 1 for Example 1.

h	$\lvert e(x)\rvert$	$\lvert e^{a}(x)\rvert$	$\lvert e^{2a}(x)\rvert$	$\lvert e^{3a}(x)\rvert$	$\lvert e^{4a}(x)\rvert$
0.1	5.4910×10^{-2}	1.1015×10^{-1}	2.7105×10^{-1}	6.2211×10^{-1}	2×10^{-1}
0.01	5.4910×10^{-5}	3.4832×10^{-4}	2.7105×10^{-3}	1.9673×10^{-2}	2×10^{-2}
0.001	5.4910×10^{-8}	1.1015×10^{-6}	2.7105×10^{-5}	6.2211×10^{-4}	2×10^{-3}

Table 2: Maximal absolute errors in case 2 for Example 1.

h	$\lvert e(x)\rvert$	$\lvert e^{a}(x)\rvert$	$\lvert e^{2a}(x)\rvert$
0.1	4.2117×10^{-2}	1.1394×10^{-1}	3.8320×10^{-1}
0.01	4.2117×10^{-5}	3.6031×10^{-4}	3.8320×10^{-3}

0.001	1.3318×10^{-7}	1.1394×10^{-6}	3.8320×10^{-5}
h	$\lvert e^{3a}(x) \rvert$	$\lvert e^{4a}(x) \rvert$	$\lvert e^{5a}(x) \rvert$
0.1	1.5609×10^{-1}	2.5464×10^{-1}	7.1364×10^{-1}
0.01	4.9362×10^{-3}	2.5464×10^{-2}	2.2567×10^{-1}
0.001	1.5609×10^{-4}	2.5464×10^{-3}	7.1364×10^{-2}

Example 2 *Consider the fractional differential equation*

$$D^{\alpha}y(x) = x^4 - \frac{1}{2}x^3 + \frac{24}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{3}{\Gamma(5-\alpha)}x^{4-\alpha} - y(x), \qquad 0 < \alpha < 1, \qquad (24)$$

with the initial condition $y(0) = 0$. *The exact solution is*

$$y(x) = x^4 - \frac{1}{2}x^3.$$

Similarly the maximal absolute errors obtained, for case 1, case 2 and for $\alpha = 0.5$, *are shown in Table 1and Table2, respectively, with* $|e^{m\alpha}(x)| = |D^{m\alpha}S_{\mathrm{k}}(x) - D^{m\alpha}y(x)|$, *for* $\alpha = 0.5$, $m = 0,1,\dots,4$ *for case 1, and* $m = 0,1,\dots,5$ *for case 2.*

Table 3: Maximal absolute error in case 1 for Example 2.

h	$\lvert e(x) \rvert$	$\lvert e^{a}(x) \rvert$	$\lvert e^{2a}(x) \rvert$	$\lvert e^{3a}(x) \rvert$	$\lvert e^{4a}(x) \rvert$
0.1	74.1286×10^{-2}	14.8702×10^{-2}	36.5918×10^{-2}	83.9857×10^{-1}	27×10^{-1}
0.01	74.1286×10^{-5}	47.0339×10^{-4}	36.5918×10^{-3}	26.5586×10^{-2}	27×10^{-2}
0.001	74.1286×10^{-8}	14.8702×10^{-6}	36.5918×10^{-5}	83.9857×10^{-4}	27×10^{-3}

Table 4: Maximal absolute error in case 2 for Example 2.

h	$\lvert e(x) \rvert$	$\lvert e^{a}(x) \rvert$	$\lvert e^{2a}(x) \rvert$
0.1	14.1960×10^{-2}	12.1447×10^{-1}	40.8441×10^{-1}
0.01	44.8919×10^{-5}	38.4049×10^{-4}	40.8441×10^{-3}
0.001	14.1960×10^{-7}	12.1447×10^{-6}	40.8441×10^{-5}
h	$\lvert e^{3a}(x) \rvert$	$\lvert e^{4a}(x) \rvert$	$\lvert e^{5a}(x) \rvert$
0.1	16.6379×10^{-1}	27.1421×10^{-1}	76.0656×10^{-1}
0.01	52.6138×10^{-3}	27.1421×10^{-2}	24.0540×10^{-1}
0.001	16.6379×10^{-4}	27.1421×10^{-3}	76.0656×10^{-2}

7 Conclusions

In this paper, we introduced a new kind of the fractional spline of polynomial form with the idea of lacunary interpolation. This method is very powerful and efficient in finding numerical solutions in fractional differential case as we have shown in the two examples. Maximal absolute errors are obtained for $\alpha = 0.5$ and different values of h, these illustrate the accuracy of the proposed method.

8. References

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