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A parallel descent-like prediction-correction method for structured variational inequalities with three blocks

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Abstract. This paper proposes a parallel descent-like prediction-correction method(PDPCM) for structured variational inequalities with three blocks. In the prediction step, the intermediate point $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$ is produced by solving three low-dimensional variational inequalities in parallel. In the correction step, a descent direction is first constructed using the iterate and the intermediate point, and then the new iterate is obtained along this descent direction. Global convergence of the proposed method is proved under mild assumptions, and we also establish its worse-case $O(1/t)$ convergence rate in the ergodic sense.

Keywords: variational inequality problems, prediction-correction method, global convergence, convergence rate.

1. Introduction

Let $S \subseteq R^n$ be a nonempty closed convex set and F be a continuous mapping from S into itself. The variational inequality problem, denoted by VI(S,F), is to find $u^* \in S$, such that

$$(u-u^*)^T F(u^*) \geq 0, \forall u \in S, \tag{1}$$

where `T' denotes the standard inner product. In this paper, we consider the VI(S,F) which has the following separable structure:

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, F(u) = \begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

$$S = \{(x,y,z) \mid x \in X, y \in Y, z \in Z, Ax+By+Cz=b\},$$

where $X \subseteq R^{n_1}, Y \subseteq R^{n_2}, Z \subseteq R^{n_3},$, are given nonempty closed convex sets; $A \in R^{m\times n_1}, B \in R^{m\times n_2}$, $C \in R^{m\times n_3}$ are given matrices with full rank; $b \in R^n$ is a given vector, and $f: R^{n_1} \to R^{n_1}, g: R^{n_2} \to R^{n_2}, h: R^{n_3} \to R^{n_3}$, are given monotone continuous mappings.

By attaching a Lagrangian multiplier $\lambda \in R^m$ to the linear constraint $Ax+By+Cz=b$, the above variational inequality problem can be converted into the following form (denoted by VI(W,Q): Find $w^* = (x^*, y^{*}, z^*, \lambda^*) \in W := X \times Y \times Z \times R^m$ such that

$$(w-w^*)^T Q(w^*) \geq 0, \forall x \in W, \tag{2}$$

Where

$$w = \begin{pmatrix} x \\ y \\ z \\ \lambda \end{pmatrix}, Q(w) = \begin{pmatrix} f(x) - A^T\lambda \\ g(y) - B^T\lambda \\ h(z) - C^T\lambda \\ Ax+By+Cz-b \end{pmatrix}$$

To solve VI (W,Q), the proximal alternating direction method(PADM)[2,5] finds a new iterate $(x^{k+1},y^{k+1},z^{k+1},\lambda^{k+1})\in W$ from a given iterate $(x^k,y^k,z^k,\lambda^k)\in W$ via the following procedure: Find $\tilde{x}^k\in X$ such that

$$(x'-\tilde{x}^k)^T\{f(\tilde{x}^k)-A^T[\lambda^k-\beta(A\tilde{x}^k+By^k+Cz^k-b)]+r(\tilde{x}^k-x^k)\}\geq 0,\forall x'\in X \tag{3}$$

and take $\tilde{x}^k$ as x^{k+1}. Find $\tilde{y}^k\in Y$ such that

$$(y'-\tilde{y}^k)^T\{g(\tilde{y}^k)-B^T[\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+Cz^k-b)]+s(\tilde{y}^k-y^k)\}\geq 0,\forall y'\in Y \tag{4}$$

and take $\tilde{y}^k$ as y^{k+1}. Find $\tilde{z}^k\in Z$ such that

$$(z'-\tilde{z}^k)^T\{h(\tilde{z}^k)-C^T[\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+C\tilde{z}^k-b)]+t(\tilde{z}^k-z^k)\}\geq 0,\forall z'\in Z \tag{5}$$

and take $\tilde{z}^k$ as z^{k+1}. Finally, update λ^k via

$$\lambda^{k+1}=\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+C\tilde{z}^k-b) \tag{6}$$

where $r\geq 0,s\geq 0,t\geq 0$ are given proximal parameters; $\beta>0$ is a given penalty parameter for the linear constraint $Ax+By+Cz=b$. Note that, when $r=s=t=0$ in (3)-(5), the classical alternating direction method is obtained. Obviously, the variables $(\tilde{x},\tilde{y},\tilde{z})$ of the involved low-dimensional variational inequalities (3)-(5) are crossed and thus the PADM is not eligible for parallel computing in the sense that the solutions of sub-variational inequalities (3)-(5) cannot be obtained simultaneously. For the purpose of parallel computing, the author of [3] proposed the following parallel form of the PADM, denoted by PPADM (in fact, in [3], $r=s=t=0$): Find $\tilde{x}^k\in X$ such that

$$(x'-\tilde{x}^k)^T\{f(\tilde{x}^k)-A^T[\lambda^k-\beta(A\tilde{x}^k+By^k+Cz^k-b)]+r(\tilde{x}^k-x^k)\}\geq 0,\forall x'\in X \tag{7}$$

Find $\tilde{y}^k\in Y$ such that

$$(y'-\tilde{y}^k)^T\{g(\tilde{y}^k)-B^T[\lambda^k-\beta(Ax^k+B\tilde{y}^k+Cz^k-b)]+s(\tilde{y}^k-y^k)\}\geq 0,\forall y'\in Y \tag{8}$$

Find $\tilde{z}^k\in Z$ such that

$$(z'-\tilde{z}^k)^T\{h(\tilde{z}^k)-C^T[\lambda^k-\beta(Ax^k+By^k+C\tilde{z}^k-b)]+t(\tilde{z}^k-z^k)\}\geq 0,\forall z'\in Z \tag{9}$$

Finally, update λ^k via

$$\lambda^{k+1}=\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+C\tilde{z}^k-b) \tag{10}$$

Obviously, the variables of the above sub-variational inequalities are not crossed. However, we cannot ensure convergence of the iterate sequence if we set $w^{k+1}=\tilde{w}^k$, therefore, in [3], the new iterate w^{k+1} is generated by a descent step. More specifically,

$$w^{k+1}=w^k-\alpha_k H^{-1}M(w^k-\tilde{w}^K), \tag{11}$$

where M is a predefined symmetric matrix; H is a given proper positive definite matrix, and α_k is a judiciously chosen step size.

The sub-variational inequalities (7)-(9) are easy to solve if the evaluations of $(A^TA+f/\beta)^{-1}(Av)$, $(B^TB+g/\beta)^{-1}(Bv)$, and $(C^TC+h/\beta)^{-1}(Cv)$, are simple, however, if A,B and C are not identity matrices, the above evaluations could be costly. In this paper, motivated by [6], we propose a modified version of the PPADM (denoted by PDPCM), and the new sub-variational inequalities in PDPCM are easy to solve under the assumption that the resolvent operators of f,g and h are easy to evaluate, where the resolvent operator of mapping T is defined as $(I+\lambda T)^{-1}(v)$. We prove the global convergence of the PDPCM, moreover, inspired by the strategy in [4,5], we establish its worst-case $O(1/t)$ convergence rate in ergodic sense.

The rest of the paper is organized as follows. We first give some basic concepts which are useful in the following analysis in Section 2. Then, in Section 3, we describe the parallel descent-like prediction-correction method(PDPCM) for structured variational inequalities, and the global convergence of the new method is proved. We establish the PDPCM's worst-case $O(1/t)$ convergence rate in ergodic sense in Section 4 and some conclusions are drawn in the last section.

2. Preliminaries

In this section, we give some standard assumptions and related definitions which will be used in the following discussion. We make the following standard assumptions throughout this paper:

Assumption

(A1) The solution set of problem VI(W,Q), denoted by W^*, is nonempty.

(A2) The underlying mappings f,g and h are monotone and continuous. A mapping $F: R^n \to R^n$ is said monotone on a closed convex set $\Omega \subseteq R^n$ if

$(u-v)^T(F(u)-F(v)) \geq 0, \forall u,v \in \Omega.$

(A3) The resolvent operators of f,g and h are easy to evaluate.

It is easy to prove that Q is also monotone when f,g and h are monotone, thus the solution set of VI (W,Q) is closed and convex under Assumption (A2).

Let H be a symmetric positive definite matrix, the H-norm of the vector u is denoted by $\|u\|_H = \sqrt{u^T Hu}$. For a matrix A, $\|A\|$ denotes its norm $\|A\| := \max\{\|Ax\| : \|x \leq 1\}\|$. Then, we provide a useful characterization on W^* as Theorem 2.1 in [4] and Theorem 2.3.5 in [1].

Theorem 1. The solution set of VI (W,Q) is convex and it can be characterized as

$$W^* = \bigcap_{w\in W} \{\tilde{w} \in W : (w-\tilde{w})^T Q(w) \geq 0\}. \tag{12}$$

Based on Theorem 1, $\tilde{w} \in W$ can be regarded as an ϵ-approximation solution of VI (W,Q) if it satisfies

$$\sup_{w\in D}\{(\tilde{w}-w)^T Q(w)\} \leq \varepsilon, \tag{13}$$

where $D \subseteq W$ is some compact set. As Definition in [11], we can take

$$D = B_W := \{w \in W \mid \|w-\tilde{w}\| \leq 1\}.$$

In our latter analysis, we shall establish the worse-case $O(1/t)$ convergence rate for the new algorithm to be presented in the sense that after t iterations, we can find a $\tilde{w} \in W$ such that

$$(\tilde{w}-w)^T Q(w) \leq \varepsilon, \forall w \in B_W(\tilde{w}),$$

with $\varepsilon = O(1/t).$

3. Algorithm and global convergence

In this section, we present a new parallel descent-like prediction-correction method(PDPCM) for VI (W,Q), and show its global convergence. To make the algorithm more succinct, we first define the some matrices.

$$M = \begin{pmatrix} rI_{n_1} & 0 & 0 & 0 \\ 0 & sI_{n_2} & 0 & 0 \\ 0 & 0 & tI_{n_3} & 0 \\ 0 & 0 & 0 & I_m/\beta \end{pmatrix}, P = \begin{pmatrix} rI_{n_1} & 0 & 0 & A^T \\ 0 & sI_{n_2} & 0 & B^T \\ 0 & 0 & tI_{n_3} & C^T \\ 0 & 0 & 0 & I_m/\beta \end{pmatrix} \tag{14}$$

Obvisouly, M and $P^T P$ are symmetric positive definite matrices whenever $r,s,t,\beta > 0$. In addition, let $n = n_1 + n_2 + n_3$, $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{z}^k)$ Now, we describe our algorithm detailed as follows:

The parallel descent-like prediction-correction method

Step 0.(Initialization step)} Let $\beta > 0, \gamma \in (0,2)$, and $H \in R^{(n+m)\times(n+m)}$ be symmetric and positive definite. Choose the parameters $r,s,t > 0$,such that

$$r \geq \sqrt{3}\beta\|A^T A\|, s \geq \sqrt{3}\beta\|B^T B\|, t \geq \sqrt{3}\beta\|C^T C\| \tag{15}$$

Given $\varepsilon > 0$, choose $w^0 = (x^0, y^0, z^0, \lambda^0) \in W$, and set k :=0.

Step 1.(Prediction step) Generate the predictor $\tilde{x}^k \in X$ such that

$$(x'-\tilde{x}^k)^T\{f(\tilde{x}^k)-A^T\lambda^k+r(\tilde{x}^k-x^k)\}\geq 0, \forall x'\in X \tag{16}$$

Find $\tilde{y}^k \in Y$ such that

$$(y'-\tilde{y}^k)^T\{g(\tilde{y}^k)-B^T\lambda^k+s(\tilde{y}^k-y^k)\}\geq 0, \forall y'\in Y \tag{17}$$

Find $\tilde{z}^k \in Z$ such that

$$(z'-\tilde{z}^k)^T\{h(\tilde{z}^k)-C^T\lambda^k+t(\tilde{z}^k-z^k)\}\geq 0, \forall z'\in Z \tag{18}$$

Finally, update λ^k via

$$\tilde{\lambda}^k=\lambda^k-\beta(A\tilde{x}^k+B\tilde{y}^k+C\tilde{z}^k-b) \tag{19}$$

Step 2.(Correction step) The new iterate is produced by:

$$w^{k+1}=w^k-\alpha_k H^{-1}P(w^k-\tilde{w}^k), \tag{20}$$

Where

$$\alpha_k=\gamma\alpha_k^*, \alpha_k^*=\frac{\varphi(w^k,\tilde{w}^k)}{\left\|H^{-1}P(w^k-\tilde{w}^k)\right\|_H^2},,$$

with

$$\varphi(w^k,\tilde{w}^k)=\left\|w^k-\tilde{w}^k\right\|_P^2. \tag{21}$$

Step 3. Convergence verification: If $\left\|w^k-\tilde{w}^k\right\|\leq\varepsilon$, then stop; otherwise, set $K:=k+1$ and goto Step 1.

Remark 1. It is easy to check that $\left\|w^k-\tilde{w}^k\right\|=0$ if and only if $x^k=\tilde{x}^k, y^k=\tilde{y}^k, z^k=\tilde{z}^k, \lambda^k=\tilde{\lambda}^k$. Then, from (16)-(19), we have that w^k is actually a solution of VI (W,Q) which means the iteration will be terminated. Thus, the stopping condition in Step 3 is reasonable.

Remark 2. Note that the inequalities (16)-(18) can generate the intermediate points $\tilde{x}^k, \tilde{y}^k, \tilde{z}^k$ simultaneously, and thus the PDPCM is eligible for parallel computation.

The following lemma is devoted to showing that $\varphi(w^k,\tilde{w}^k)$ is positive in the case $\left\|w^k-\tilde{w}^k\right\|_M^2\neq 0.$

Lemma 1. Let $\tilde{w}^k$ be generated by (16)-(19) from a given w^k and $\varphi(w^k,\tilde{w}^k)$ be defined in (21). Then we have

$$\varphi(w^k,\tilde{w}^k)\geq(1-\sqrt{3}\mu/2)\left\|w^k-\tilde{w}^k\right\|_M^2, \tag{22}$$

with

$$\mu=\sqrt{\max\{\frac{\sqrt{3}\beta\left\|A^TA\right\|}{r}.\frac{\sqrt{3}\beta\left\|B^TB\right\|}{s},\frac{\sqrt{3}\beta\left\|C^TC\right\|}{t}\}} \tag{23}$$

Proof. From the definitions of $\varphi(w^k,\tilde{w}^k)$ and matrices M, P, we have

$$\begin{aligned}&\varphi(w^k,\tilde{w}^k)\\&=(w^k-\tilde{w}^k)^T P(w^k-\tilde{w}^k)\\&=\left\|w^k-\tilde{w}^k\right\|_M^2+(\lambda^k-\tilde{\lambda}^k)^T A(x^k-\tilde{x}^k)+(\lambda^k-\tilde{\lambda}^k)^T B(y^k-\tilde{y}^k)+(\lambda^k-\tilde{\lambda}^k)^T C(z^k-\tilde{z}^k)\end{aligned} \tag{24}$$

By using the Cauchy-Schwartz inequality, we have

$$(\lambda^k-\tilde{\lambda}^k)^T A(x^k-\tilde{x}^k)\geq-\frac{1}{2}(\frac{\sqrt{3}\beta}{\mu}\left\|A(x^k-\tilde{x}^k)\right\|^2+\frac{\mu}{\sqrt{3}\beta}\left\|\lambda^k-\tilde{\lambda}^k\right\|^2.$$

$$(\lambda^k-\tilde{\lambda}^k)^T B(y^k-\tilde{y}^k)\geq-\frac{1}{2}(\frac{\sqrt{3}\beta}{\mu}\left\|B(y^k-\tilde{y}^k)\right\|^2+\frac{\mu}{\sqrt{3}\beta}\left\|\lambda^k-\tilde{\lambda}^k\right\|^2.$$

$$(\lambda^k-\tilde{\lambda}^k)^T C(z^k-\tilde{z}^k)\geq-\frac{1}{2}(\frac{\sqrt{3}\beta}{\mu}\left\|C(z^k-\tilde{z}^k)\right\|^2+\frac{\mu}{\sqrt{3}\beta}\left\|\lambda^k-\tilde{\lambda}^k\right\|^2.$$

Substituting the above three inequalities in (24), we have (22) by the definition of μ and the last equality follows from the definition M in (14). This completes the proof.

Remark 3. From (15) and (23), we have $\mu \in (0,1]$. Consequently, using (22), we get that $\varphi(w^k, \tilde{w}^k)$ is positive in the case $\left\| w^k - \tilde{w}^k \right\|_M^2 \neq 0.$

Remark 4. Since

$$\left\| H^{-1}P(w^k - \tilde{w}^k) \right\|_H^2 \leq \left\| P^T H^{-1} P \right\| \left\| w^k - \tilde{w}^k \right\|^2$$

and

$$\left\| w^k - \tilde{w}^k \right\|_M^2 \geq v \left\| w^k - \tilde{w}^k \right\|^2,$$

where $v = \min\{r, s, t, 1/\beta\}$. It follows from the definition of α_k^* that

$$\alpha_k^* \geq \frac{(2-\sqrt{3}\mu)v}{2\left\| P^T H^{-1} P \right\|} := \tau. \tag{25}$$

Lemma 2. If $\tilde{w}^k$ is generated by (16)-(19) from a given w^k, then we have

$$\left\langle w' - \tilde{w}^k, Q(\tilde{w}^k) - P(w^k - \tilde{w}^k) \right\rangle \geq 0, \forall w' \in W. \tag{26}$$

Proof. Note that (15)-(18) can be rewritten into:

$$(x' - \tilde{x}^k)^T \{ f(\tilde{x}^k) - A^T \tilde{\lambda}^k - A^T(\lambda^k - \tilde{\lambda}^k) + r(\tilde{x}^k - x^k) \} \geq 0, \forall x' \in X$$

$$(y' - \tilde{y}^k)^T \{ g(\tilde{y}^k) - B^T \tilde{\lambda}^k - B^T(\lambda^k - \tilde{\lambda}^k) + s(\tilde{y}^k - y^k) \} \geq 0, \forall y' \in Y$$

$$(z' - \tilde{z}^k)^T \{ h(\tilde{z}^k) - C^T \tilde{\lambda}^k - C^T(\lambda^k - \tilde{\lambda}^k) + t(\tilde{z}^k - z^k) \} \geq 0, \forall z' \in Z$$

$$(\tilde{\lambda}^k - \lambda^k)^T [A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b - (\lambda^k - \tilde{\lambda}^k)/\beta] \geq 0.$$

Using the notations of Q and P, we can get (26) from the above four inequalities. The proof is complete.

Now, we prove the contraction of the PDPCM. First, we give an important lemma.

Lemma 3. Let the sequences $\{w^k\}$ and $\{\tilde{w}^k\}$ be generated by the PDPCM. Then, for any $w' \in W$, we have

$$\gamma\alpha_k^* (w' - \tilde{w}^k)^T Q(\tilde{w}^k) + \frac{1}{2}(\left\| w' - \tilde{w}^k \right\|_H^2 - \left\| w' - w^{k+1} \right\|_H^2) \geq \frac{1}{2}\gamma(2-\gamma)\alpha_k^*(1 - \frac{\sqrt{3}\mu}{2})\left\| w' - \tilde{w}^k \right\|_M^2. \tag{27}$$

Proof. First, using (26), we have

$$(w' - \tilde{w}^k)^T Q(\tilde{w}^k) \geq (w' - \tilde{w}^k)^T P(w^k - \tilde{w}^k)^T, \forall w \in W.$$

Then, from (20), we get

$$\gamma\alpha_k^* P(w^k - \tilde{w}^k) = H(w^k - w^{k+1}).$$

It follows that

$$\gamma\alpha_k^* (w' - w^k)^T P(w^k - \tilde{w}^k) = (w' - w^k)^T H(w^k - w^{k+1}).$$

Thus, it suffices to show that

$$(w' - \tilde{w}^k)^T H(w^k - w^{k+1}) + \frac{1}{2}(\left\| w' - \tilde{w}^k \right\|_H^2 - \left\| w' - w^{k+1} \right\|_H^2) \geq \frac{1}{2}\gamma(2-\gamma)\alpha_k^*(1 - \frac{\sqrt{3}\mu}{2})\left\| w' - \tilde{w}^k \right\|_M^2. \tag{28}$$

By setting $a = w', b = \tilde{w}^k, c = w^k$ and $d = w^{k+1}$ in the identity

$$(a-b)^T H(c-d) = \frac{1}{2}(\|a-d\|_H^2 - \|a-c\|_H^2) + \frac{1}{2}(\|c-b\|_H^2 - \|d-b\|_H^2),$$

we can derive that

$$(w' - \tilde{w}^k)^T H(w^k - w^{k+1}) = \frac{1}{2}(\left\| w' - w^{k+1} \right\|_H^2 - \left\| w' - w^k \right\|_H^2) + \frac{1}{2}(\left\| w^k - \tilde{w}^k \right\|_H^2 - \left\| w^{k+1} - \tilde{w}^k \right\|_H^2), \tag{29}$$

For the second term in the right side, from (20), we have the following inequality:

$$\left\| w^k - \tilde{w}^k \right\|_H^2 - \left\| w^{k+1} - \tilde{w}^k \right\|_H^2$$
$$= \left\| w^k - \tilde{w}^k \right\|_H^2 - \left\| w^k - \tilde{w}^k - \gamma\alpha_k^* H^{-1}P(w^k - \tilde{w}^k) \right\|_H^2$$

$$= 2\gamma\alpha_k^*(w^k - \tilde{w}^k)^T P(w^k - \tilde{w}^k) - (\gamma\alpha_k^*)^2 \|H^{-1}P(w^k - \tilde{w}^k)\|_H^2$$
$$= 2\gamma(\alpha_k^*)^2 (w^k - \tilde{w}^k)^T P(w^k - \tilde{w}^k) - \gamma^2\alpha_k^*\varphi(w^k, \tilde{w}^k)$$
$$= \gamma(2-\gamma)\alpha_k^*\varphi(w^k, \tilde{w}^k)$$
$$\ge \gamma(2-\gamma)\alpha_k^*(1 - \frac{\sqrt{3}\mu}{2})\|w^k - \tilde{w}^k\|_M^2,$$

where the last inequality follows from (22). Combining the above inequality and (29), we obtain (28). The assertion (27) follows immediately. The proof is completed.

With the above lemmas, now we can prove the contraction of the PDPCM.

Theorem 2. Let the sequences $\{w^k\}$ and $\{\tilde{w}^k\}$ be generated by the PDPCM. Then, for any $k \ge 0$, we have

$$\|w^{k+1} - w^*\|_M^2 \le \|w^k - w^*\|_M^2 - \tau\nu\gamma(2-\gamma)(1 - \frac{\sqrt{3}\mu}{2})\|w^k - \tilde{w}^k\|^2, \quad (30)$$

With $\nu = \min\{r, s, t, 1/\beta\}$..

Proof. Setting $w' = w^k$ in (27), we get

$$\|w^k - \tilde{w}^k\|_H^2 - \|w^k - w^{k+1}\|_H^2 \ge \gamma(2-\gamma)\alpha_k^*(1 - \frac{\sqrt{3}\mu}{2})\|w^k - \tilde{w}^k\|_M^2 + 2\gamma\alpha_k^*(w^k - \tilde{w}^k)^T Q(\tilde{w}^k).$$

Because $(\tilde{w}^k - w^*)^T Q(\tilde{w}^k) \ge (\tilde{w}^k - w^*)^T Q(w^*) \ge 0,$ it follows from the last inequality that

$$\|w^k - \tilde{w}^k\|_H^2 - \|w^k - w^{k+1}\|_H^{2\ k}$$
$$\ge \gamma(2-\gamma)\alpha_k^*(1 - \frac{\sqrt{3}\mu}{2})\|w^k - \tilde{w}^k\|_M^2$$
$$\ge \tau\nu\gamma(2-\gamma)(1 - \frac{\sqrt{3}\mu}{2})\|w^k - \tilde{w}^k\|_M^2$$

where the last inequality follows from (25) and the definition of ν. Thus, we obtain (30). The proof is completed.

Based on the above theorem, we are ready to prove the convergence of the proposed method.

Theorem 3. The sequence $\{w^k\}$ generated by the proposed method converges to a solution of VI(W, Q).

Proof. Since $\mu \in (0,1], \gamma \in (0,2)$, it follows from (30) that

$$\|w^{k+1} - w^*\|_M^2 \le \|w^k - w^*\|_M^2 \le \cdots \le \|w^0 - w^*\|_M^2 < +\infty.$$

This means that the sequence $\{w^k\}$ is bounded, and it has at least one cluster point. Let w^∞ be a cluster point of the sequence $\{w^k\}$ and the subsequence $\{w^{k_j}\}$ converges to w^∞. It follows from (30) again that

$$\lim_{k\to\infty}\|x^k - \tilde{x}^k\| = 0, \lim_{k\to\infty}\|y^k - \tilde{y}^k\| = 0, \lim_{k\to\infty}\|z^k - \tilde{z}^k\| = 0, \lim_{k\to\infty}\|\lambda^k - \tilde{\lambda}^k\| = 0, \quad (31)$$

Then (16)-(19) and (31) imply that

$$(x' - x^\infty)^T[f(x^\infty) - A^T\lambda^\infty] \ge 0, \forall x' \in X$$
$$(y' - y^\infty)^T[g(y^\infty) - B^T\lambda^\infty] \ge 0, \forall y' \in Y$$
$$(z' - z^\infty)^T[h(z^\infty) - C^T\lambda^\infty] \ge 0, \forall z' \in Z$$
$$Ax^\infty + By^\infty + Cz^\infty - b = 0.$$

which implies that $w^\infty \in W^*$, i.e., w^∞ is a solution of VI(W, Q). Now, we have to show that the sequence $\{w^k\}$ actually converges to w^∞. Suppose that $\hat{w}$ is another cluster point of $\{w^k\}$.Then, we have

$$\delta := \|w^\infty - \hat{w}\|_H > 0.\ \delta := \| w^\infty - \hat{w}\|_H > 0.$$

Because w^∞ is a cluster point of the sequence $\{w^k\}$, there is a $k_0 > 0$ such that

$$\left\|w^{k_0} - w^\infty\right\|_H \le \frac{\delta}{2}.$$

On the other hand, since $\{\left\|w^k - w^\infty\right\|_H\}$ is non-increased, we have $\left\|w^k - w^\infty\right\|_H \le \left\|w^{k_0} - w^\infty\right\|_H$ for all $k \ge k_0$, and it follows that

$$\left\|w^k - \hat{w}\right\|_H \ge \left\|w^\infty - \hat{w}\right\|_H - \left\|w^k - w^\infty\right\|_H \ge \frac{\delta}{2}, \forall k \ge k_0.$$

which is contradicts the fact that $\hat{w}$ is a cluster point of $\{w^k\}$. This contradiction indicates that the sequence $\{w^k\}$ converges its unique cluster point w^∞, which is a solution of VI(W, Q). This completes the proof.

5. Convergence rate of the PDPCM

Now, we are ready to present the $O(1/t)$ convergence rate for the PCPCM.

Theorem 4. For any integer $t > 0$, we have a $\tilde{w}_t \in W$ which satisfies

$$(\tilde{w} - w')^T Q(w') \le \frac{1}{2\gamma Y_t} \left\|w' - w^0\right\|_H^2, \forall w' \in W,$$

where

$$\tilde{w}_t = \frac{1}{Y_t} \sum_{k=0}^{t} \alpha_k \tilde{w}^k, Y_t = \sum_{k=0}^{t} \alpha_k$$

Proof. By using the monotonicity of Q, from (27), we get

$$\gamma \alpha_k^* (w' - \tilde{w}^k)^T Q(w') + \frac{1}{2}\left\|w' - w^k\right\|_H^2 \ge \frac{1}{2}\left\|w - w^{k+1}\right\|_H^2.$$

Summing the above inequality over $k = 0,1,\cdots,t$, we obtain

$$((\sum_{k=0}^{t} \gamma \alpha_k^*) w' - \sum_{k=0}^{t} \gamma \alpha_k^* \tilde{w}^k)^T Q(w') + \frac{1}{2}\left\|w' - w^0\right\|_H^2 \ge 0, \forall w' \in W.$$

Using the notation of Y_t and $\tilde{w}_t$ in the above inequality, we can derive

$$(\tilde{w} - w')^T Q(w') \le \frac{1}{2\gamma Y_t} \left\|w' - w^0\right\|_H^2, \forall w' \in W,$$

Indeed, $\tilde{w}_t \in W$ because it is a convex combination of $\tilde{w}_{0,} \tilde{w}_{1,} \cdots, \tilde{w}_t$.. This completes the proof.

It follows from (25) that $Y_t \ge \tau(t+1)$, and the method has $O(1/t)$ convergence rate. In fact, for a given compact set $D \subset W$, let $d = \sup\{\left\|w' - w^0\right\|_H \mid w' \in D\}$, where w^0 is the initial iterate. Then, the method reach

$$(\tilde{w}_t - w')^T Q(w') \le \varepsilon, \forall w' \in D,$$

in at most

$$t = \left\lceil \frac{d^2}{2\gamma\tau\varepsilon} \right\rceil$$

iteration. Thus a worst-case $O(1/t)$ convergence rate of PDPCM in ergodic sense is established.

6. Conclusions

In this paper, we presented a parallel descent-like prediction-correction method for structured monotone variational inequalities with three separate operators. At each iteration, the algorithm solve three strongly monotone sub-variational inequalities parallel to produce a predictor, and then make a simple correction to generate the new iterate. We also proved its global convergence and $O(1/t)$ convergence rate under mild conditions.

7. References

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