ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 10, No. 1, 2015, *pp*. 003-006

Bounded Extended Cesàro Operators From Q_K Spaces into Weighted Bloch Spaces

YaPing Cheng [1], TaiZhong Zhang [2+] and Yaling Zhang [2]

[1] Jiangsu Engineering Center of Network Monitoring, Nanjing University of Information Science and Technology, Nanjing, 210044, China

School of Computer and Software, Nanjing University of Information Science and Technology, Nanjing, 210044, China

[2] School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, China

(*Received August 03, 2014, accepted October 11, 2014*)

Abstract. Sufficient and necessary conditions for extended Cesàro operators from Q_K spaces into weighted Bloch spaces B_μ and logarithmic Bloch spaces $B_{\log}$ in the unit disc to be bounded are obtained.

Keywords: Cesàro operators, Q_K spaces, weighted Bloch spaces, logarithmic Bloch spaces

1. Introduction

Let D be the open unit disc of the complex plane C, $H(D)$ be the space of all analytic functions in D. A positive continuous decreasing function on the interval $[0,1)$ is called a normal function if there are constants a, b, δ such that $0<\delta<1$, $0<a<b<+\infty$, and $\frac{\mu(r)}{(1-r)^a}$ is decreasing and $\frac{\mu(r)}{(1-r)^b}$ is increasing on $[\delta,1)$,

Moreover, $\lim_{r\to 1^-}\frac{\mu(r)}{(1-r)^a}=0$ and $\lim_{r\to 1^-}\frac{\mu(r)}{(1-r)^b}=+\infty$. For $z\in D$, we can extend its definition, $\mu(z)=\mu(|z|)$.

The weighted Bloch spaces

$$B_\mu=\left\{f\in H(D)\middle|\|f\|_{B_\mu}=\sup_D\mu(z)|f'(z)|<+\infty\right\}$$

are Banach spaces under the norms $\|f\|_{B_\mu}=|f(0)|+\sup_D\mu(z)|f'(z)|$. Specially, when $\mu(z)=\left(1-|z|^2\right)^\alpha$, $0<\alpha<+\infty$, we get α-Bloch spaces B_α; when $\mu(z)=\left(1-|z|^2\right)\log\left(2/\left(1-|z|^2\right)\right)$, we get logarithmic Bloch space $B_{\log}$.

Let $Aut(D)$ be the holomorphic automorphism group on D under composite transformations of D. For $a\in D$, $\phi_a(z)=(z-a)/(1-\bar{a}z)\in Aut(D)$, green function $g(z,a)$ on D with pole $a\in D$ is given by

+ Corresponding author. Tel.: +86-25-58731160.
E-mail address: zhangspaces@163.com.

$g(z,a)=\log\frac{1}{|\phi_a(z)|}$, $dA=\frac{dxdy}{\pi}$ is the normalized Lebesgue area measure, the Banach spaces Q_K spaces consist of those $f\in H(D)$ such that

$$\|f\|_{Q_K}^2=\sup_D\int_D|f'(z)|^2K(g(z,a))dA<+\infty,$$

where $K:(0,+\infty)\to[0,+\infty)$ is right continuous nondecreasing function, $\varphi_K(s)=\sup_{0\le t\le 1}K(st)/K(t)$, $0<s<+\infty$, hence φ_K also is right continuous nondecreasing function. Suppose that φ_K always satisfies the following conditions: (for more dedails, please see [1] [2])

$$\int_0^1\varphi_K(s)\frac{ds}{s}<+\infty,\qquad \int_1^{+\infty}\varphi_K(s)\frac{ds}{s^2}<+\infty. \tag{1}$$

For a holomorphic function $f\in H(D)$ with Taylor expansion $f(z)=\sum_{n=0}^{+\infty}a_nz^n$, Cesàro operator C acting on f is defined by

$$C[f](z)=\sum_{n=0}^{+\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}a_k\right)z^n.$$

By computation, we see that

$$C[f](z)=\frac{1}{z}\int_0^z f(\zeta)\frac{1}{1-\zeta}d\zeta=\frac{1}{z}\int_0^z f(\zeta)\left(\log\frac{1}{1-\zeta}\right)'d\zeta.$$

On most holomorphic function spaces, $C[f]$ is bounded if and only if the integral operator $f\to\int_0^z f(\zeta)\left(\log\frac{1}{1-\zeta}\right)'d\zeta$ is bounded. From this point of view, it's natural to consider the extended Cesàro operator T_g with holomorphic symbol $g\in H(D)$:

$$(T_gf)(z)=\int_0^z f(\zeta)g'(\zeta)d\zeta.$$

Sufficient and necessary conditions for the Cesàro operator on Q_K space in the unit disc to be bounded were given in [3], boundedness of the Cesàro operator on α-Bloch spaces B_α was studied in [4][5], Sufficient and necessary conditions for the extended Cesàro operator T_g from Q_K space into α-Bloch spaces B_α to be bounded were obtained in [6]. However, in this paper, we generalize the results in [6], characterise boundedness of the extended Cesàro operator T_g from Q_K space into the weighted Bloch spaces B_μ in the unit disc, and discuss some relationships between bounded extended Cesàro operators.

2. Bounded extended Cesàro operators

Lemma 1[2] If K satisfies the condition (1), then we have $\log(1-z)\in Q_K$.

Lemma 2[7] If $f \in Q_K$, then for each $z \in D$, we have $|f(z)| \le \log\frac{1}{1-|z|}\|f\|_{Q_K}$.

Lemma 3[6] If K satisfies the condition (1), $0<\alpha<+\infty$, then $T_g : Q_K \to B_\alpha$ is a bounded linear operator if and only if $\sup_D \left(1-|z|^2\right)^\alpha |g'(z)|\log\frac{1}{1-|z|} < +\infty$.

Lemma 4 There exists $r_0 \in (0,1)$, such that for $|z| \ge r_0$, we have

$$\frac{\left(1-|z|^2\right)^b}{2^b} \le \mu(z) \le \left(1-|z|^2\right)^a.$$

Proof. The details can be omitted.

Lemma 5 There exists $r_0 \in (0,1)$, such that for $|z| \ge r_0$, arbitrary α, $0<\alpha<1$, we have

$$(\log 2)\left(1-|z|^2\right) \le \left(1-|z|^2\right)\log\frac{2}{1-|z|^2} \le \left(1-|z|^2\right)^\alpha.$$

Proof. The details can be omitted.

Theorem 1 If K satisfies the condition (1), then $T_g : Q_K \to B_\mu$ is a bounded linear operator if and only if $\sup_D \mu(z)|g'(z)|\log\frac{1}{1-|z|} < +\infty$.

Proof. Proof of necessity. If $T_g : Q_K \to B_\mu$ is bounded, taking $f(z) = \log\frac{1}{1-e^{-i\theta}z}$, by lemma 1, we get $f(z) \in Q_K$. By lemma 3, we obtain for arbitrary $z \in D$,

$$\left|\left(T_g f\right)'(z)\right|\mu(z) = |f(z)g'(z)|\mu(z) = \left|g'(z)\log\frac{1}{1-e^{-i\theta}z}\right|\mu(z) \le M\|f\|_{Q_K} < +\infty,$$

Here constant M is independent of f. Taking $z = re^{i\theta}$, $|g'(z)|\mu(z)\log\frac{1}{1-|z|} \le M\|f\|_{Q_K} < +\infty$, then we have $\sup_D \mu(z)|g'(z)|\log\frac{1}{1-|z|} < +\infty$.

Proof of sufficiency. Let $f(z) \in Q_K$, $\sup_D \mu(z)|g'(z)|\log\frac{1}{1-|z|} < +\infty$, by lemma 2, we get

$$\left|\left(T_g f\right)'(z)\right|\mu(z) = |f(z)g'(z)|\mu(z) \le |g'(z)|\mu(z)\|f\|_{Q_K} \log\frac{1}{1-|z|},$$

hence $T_g : Q_K \to B_\mu$ is bounded. We complete the proof.

As a consequence of Theorem 1, we obtain theorem 2 at once.

Theorem 2 If K satisfies the condition (1), then $T_g : Q_K \to B_{\log}$ is a bounded linear operator if and only if $\sup_D \left(1-|z|^2\right)|g'(z)|\log\frac{2}{1-|z|^2}\log\frac{1}{1-|z|} < +\infty$.

Theorem 3 If $T_g : Q_K \to B_\mu$ is bounded if and only if there exists $\alpha \in (0,+\infty)$ such that $T_g : Q_K \to B_\alpha$ is a bounded.

Proof. By theorem 1, lemma 3 and lemma 4, we complete the proof.

Theorem 4 (1) If $T_g : Q_K \to B_{\log}$ is bounded, then $T_g : Q_K \to B_1$ is a bounded, where $B_1 = B$ is the classical Bloch space. (2) If exists $\alpha \in (0,1)$ such that $T_g : Q_K \to B_\alpha$ is a bounded, then $T_g : Q_K \to B_{\log}$ is bounded.

Proof. By lemma 3, lemma 5 and theorem 2, we complete the proof.

Acknowledgements

This paper was supported in part by NSFC, Approved No. 41174165, a Natural Science Research Program of the Education Department of Jiangsu Province of P.R.China, Approved No. 07KJB110069 and a grant of NUIST, Approved No. 20080290. TaiZhong Zhang is the corresponding author.

3. References

[1] M. Essen, H. Wulan, *On analytic and meromorphic functions and spaces of Q_K -type,* Illinois J. Math. 46(2002), pp.1233-1258.

[2] M. Essen, H. Wulan and J. Xiao, *Several function-theoretic characterizations of Mobius invariant Q_K spaces,* J.Funct.Anal. 230(2006), pp.78-115.

[3] S. Li, H. Wulan, *Volterra type operators on Q_K space,* Taiwanese J. Math. 14(2010), pp.195-211.

[4] J. Xiao, *Cesàro operators on Hardy, BMOA and Bloch spaces,* Arch. Math. 68(1997), pp.392-318.

[5] S. Wang, Z. Hu, *Extended Cesàro operators on Bloch-type spaces,* Chinese Ann. Math. Ser. A 26 (2005), pp.613–624.

[6] M. Zhan, *Extended Cesàro operators between B^α spaces and Q_K spaces,* Math. Practice Theory 42(2012), pp.240-241.

[7] M. Kotilainen, *On composition operators in Q_K type spaces,* J. Funct. Anal. 23(2007), pp.103-112.