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Optimal control measures to combat COVID-19 spread in Sri Lanka: a mathematical model considering the heterogeneity of cases

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Abstract. The COVID-19 pandemic caused by the novel corona virus (SARS-CoV-2) has been one of the major public health concerns across the globe, currently more than 20 million individuals have been infected, and the number of deaths has passed 750,000. The world wide burden of the disease has been massive, and the governments are in dilemma to protect the health system of the country while safeguarding the economy. There is no vaccine or antivirus drug found against this virus while multiple research groups are actively working on a suitable candidate. The only available mode of minimizing the disease burden has been to control its transmission among the population. Since the occurrence of first COVID-19 local case on 11 March 2020, the government of Sri Lanka introduced serious social distancing and public health interventions in its fullest capacity as a developing nation to effectively combat with the disease spread. This study focuses to develop a mathematical model to investigate the dynamic of this novel disease using an extended version of an SEIR compartmental structure considering the heterogeneity of cases such as asymptomatic, symptomatic with mild indications and the cases required intensive care treatment. All the measures and interventions are in progress with a significantly large social and economic costs, thus, optimal control techniques are used to identify the most appropriate strategies to minimize these costs. The results of the simulations prove that optimal control measures can be worked out as the epidemic curves are flattened while delaying the outbreak so that the health system might not be under pressure to treat and care the patients.

Keywords: Corona virus, public health, mathematical model, stability, optimal control

1 Introduction

The COVID-19 outbreak occurred in the city of Wuhan, Hubei province, China during late December 2019 from a cluster of pneumonia cases. The Chinese health authorities identified and informed World Health Organization (WHO) that the pneumonia condition was due to novel beta corona virus, the 2019 novel virus (2019-nCoV, recently renamed as SARS-CoV-2, the cause of corona virus disease COVID-19) [1]. It is claimed that the novel corona virus likely to have originated from a zoonetic type of transmission, occurred in a wet sea food market where wild animals are sold openly. Soon after few days, Chinese researchers found out that the corona virus effectively show the human-to human transmission, and this new virus was identified to be extremely contagious among people [2].

The novel corona virus transmitted to human through respiratory droplets of another. It had also been revealed later that these droplets can survive in variety of surfaces for multiple hours or even for days. Common symptoms of COVID-19 disease have been fever, cough and fatigue. There are some less common symptoms including sputum production, headache, hemophiliacs, and diarrhea [3]. According to WHO, COVID-19 has

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spread for more than 210 countries and independent territories while Italy, Spain, United States and Iran are the hardest hit apart from China where the disease is known to be emerged but now significantly controlled and stable. In numbers, currently more than 20 million people have been infected while there are 750 000 reported deaths worldwide [4].

Since this is a new virus, there is no vaccination found yet, however, there are number of scientific investigations are in progress including animal and human trials, across the globe to find a successful vaccine candidate to fight with the corona virus [5]. Researchers claim that, though they are able to find a suitable vaccine, it would take reasonable number of months to make them available to people. The only effective strategy to fight against COVID-19 is to control its transmission through social distancing measures and public health interventions. Contact tracing and isolation of cases is a common intervention for controlling infectious disease outbreaks which most of the countries have been following, however, it might need intensive public health efforts and community mobilization due to the requirement of figuring out all possible contacts. Current modeling outcomes suggest that at a minimum of 80% of symptomatic contacts must be traced, isolated and treat to maintain the efficacy of control and the stability of the disease spread [6].

In Sri Lanka, the first COVID-19 case was found on 26 January 2020, was a Chinese national and she recovered after few weeks. The first local patient was found on 11 March 2020 and the government of Sri Lanka took strong decisions to control the transmission of the disease over the community including shutting down all the places of public gatherings such as schools, universities and non essential services, imposing travel ban to high risk countries, introducing mandatory quarantine for all arrivals to the country, shutting down the air port and finally imposing island wide curfew [7]. The time line of COVID-19 related events and responses by the government is illustrated in Fig. 1. As of 15 August 2020, there are 2886 confirmed cases, 2666 recoveries and 11 deaths reported in the island while there are many suspected exposed cases are closely monitored [7]. Few of the high risk areas and villages have been locked down restricting any type of mobility. Even though the public health sector including the military forces are acting effectively, one of the major challenges to combat with the virus in Sri Lanka has been the significant rise in the asymptomatic infections who are not showing any COVID-19 symptoms but they are carries of the virus in the population [8].

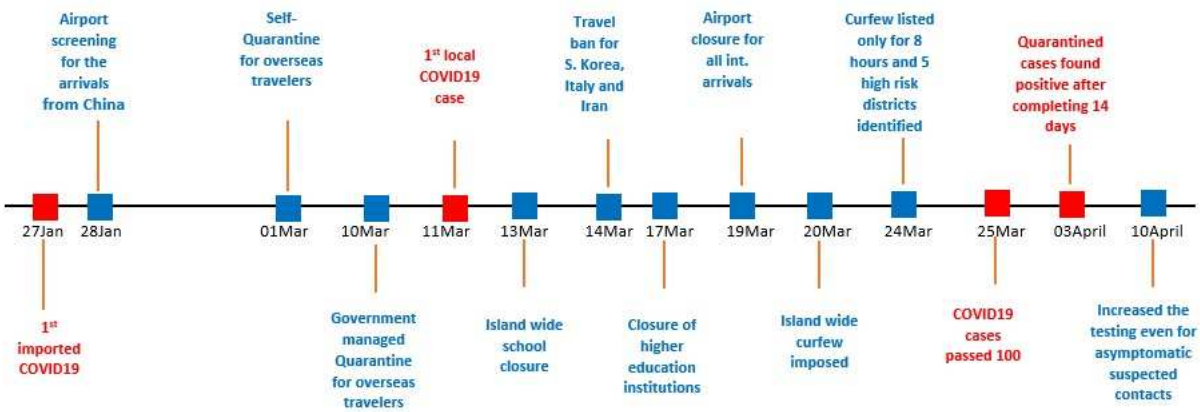

Fig. 1: COVID-19 events (red) and control measures (blue) in Sri Lanka from the first reported case.

Since COVID-19 is a new disease emerged in the world, lack of data are available related to the dynamic of the virus so that they fit to existing mathematical models to predict the outbreak. However, these models may be used to demonstrate the possible different scenarios of the disease transmission with respect to social distancing and public health intervention measures introduced by authorities [9].

In this study, we adopt the SEIR (Susceptible, Exposed, Infected, Recovered) compartmental approach to model the dynamic of COVID-19 in Sri Lanka. As it it critical to test, trace and isolate not only the symptomatic cases but also the asymptomatic, the infected population then is divided into asymptomatic (I_A) and the symptomatic with mild symptoms (I_M). In the context of Sri Lanka, all cases who are tested positive for the virus are isolated in designated hospitals and treated. The patients whose condition is not developed for the

severe level (I_H) are treated in isolated general wards however, the patients whose condition is worsen due to their demography and various other health related issues (I_C) are transferred to Intensive Care Units (ICUs) [10]. In order to deal with this COVID-19 outbreak, the government has decided to implement tough measures such as social distancing, personal protection, aggressive testing for the virus of all contacts and etc. In this context, the optimal control problem is considered to study the effect of said control measures to minimize the spread COVID-19 disease [11, 12].

This manuscript is organized as follows: In section 2, the deterministic mathematical model without control is discussed. Furthermore, basic analysis and the disease free equilibrium are presented by defining the basic reproduction number (R_0). In section 3, we present the optimal control problem with essential mathematical analysis. Numerical results and discussion are given in section 4 and finally the conclusion is presented in section 5.

2 Methods

2.1 Mathematical model with out control

First, we introduce the mathematical model of COVID-19 transmission with out any control measures. A more extended version of the SEIR (Susceptible-Exposed-Infected-Recovered) compartment model structure is used to formulate this dynamic [10, 13–15]. In Sri Lanka, the health authorities treated all the symptomatic COVID-19 cases in government hospitals, rather than advising them to be self-isolated. However, recent international travelers and close contacts of the identified COVID-19 patients are isolated in government managed quarantine centers in the different parts of the island [7]. If patients are identified from those groups then they are immediately taken to the hospitals and treated. However, it is also found that a reasonable number of individuals who are tested positive while they were asymptomatic [5, 7]. Based on this policy structure in Sri Lankan context, seven population compartments are considered for the model; Susceptible (S), Exposed (E), Infected with asymptomatic (I_A), Infected with mild symptoms (I_M), Isolated in designated hospitals (I_H), Patients with critical conditions treated in Intensive Care Units (I_C) and the patients who clinically determined as Recovered (R) [10]. Following the compartmental transition schematic diagram illustrated in Fig. 2, the seven dimensional differential system describing the COVID-19 transmission is given by

$$\begin{aligned}
\frac{dS}{dt} &= -(\beta_1 E + \beta_2 I_A + \beta_3 I_M)S - qS \\
\frac{dE}{dt} &= k + (\beta_1 E + \beta_2 I_A + \beta_3 I_M)S - \sigma E \\
\frac{dI_A}{dt} &= \phi\sigma E - \delta_1 I_A - \gamma_1 I_A \\
\frac{dI_M}{dt} &= (1-\phi)\sigma E - \delta_2 I_M \\
\frac{dI_H}{dt} &= \delta_1 I_A + \delta_2 I_M - \eta I_H - \gamma_2 I_H \\
\frac{dI_C}{dt} &= \eta I_H - \gamma_3 I_C - \mu I_C \\
\frac{dR}{dt} &= \gamma_1 I_A + \gamma_2 I_H + \gamma_3 I_C
\end{aligned} \tag{1}$$

where β_1, β_2 and β_3 represent the transmission rates from the exposed, infected and asymptomatic, and infected and symptomatic respectively while q is the rate of isolation of the susceptible individuals due to lock down, k is the rate of imported exposed cases, σ is the rate at which the exposed cases become infected, ϕ is the percentage of exposed individuals who become asymptomatic, δ_1 is the rate at which the asymptomatic cases are tested and hospitalized, δ_2 is the rate at which the symptomatic cases are tested and admitted to hospitals, η is the rate of patients condition becomes severe and require intensive care treatments, γ_1 is the recovery rate of asymptomatic cases who are not in hospitals, γ_2 is the recovery rate of mild symptomatic cases who are in general wards in hospitals, γ_3 is the recovery rate of critically sick patients and μ is the death rate of the disease.

The initial conditions for the model (1) is as $S(0) = S^0$, $E(0) = E^0$, $I_A(0) = I_A^0$, $I_M(0) = I_M^0$, $I_H(0) = I_H^0$, $I_C(0) = I_C^0$ and $R(0) = R^0$. We let the set of solutions denoted by Ω to the system of nonlinear differential equations in (1) as

$\Omega = \{(S, E, I_A, I_M, I_H, I_C, R) \in \mathbb{R}_+^7 : S + E + I_A + I_M + I_H + I_C + R \leq 1, S, E, I_A, I_M, I_H, I_C, R \geq 0\}$.

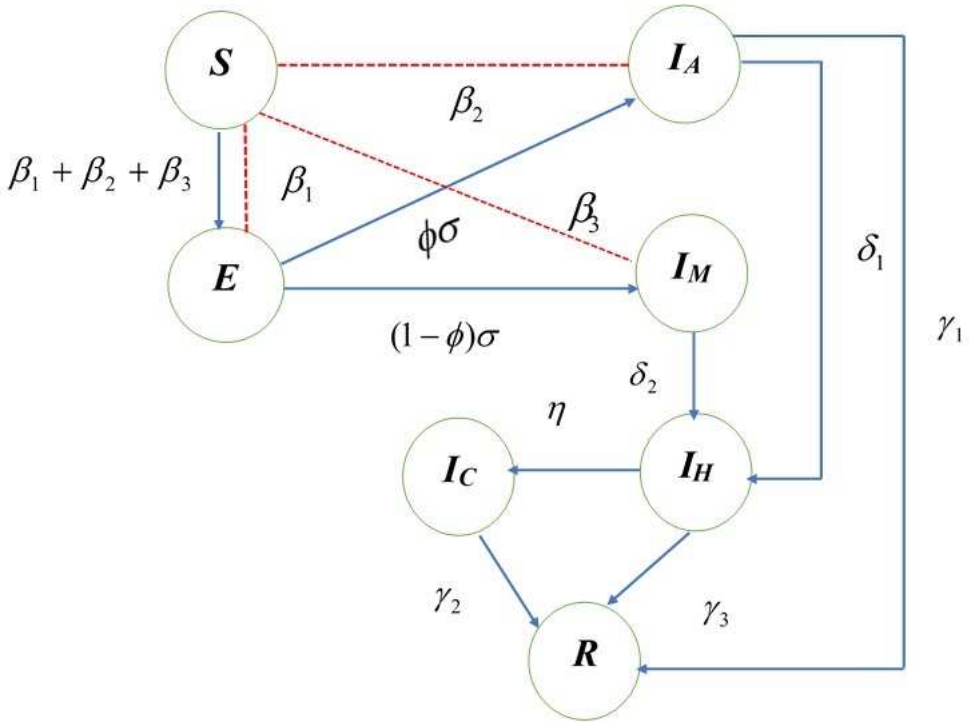

Fig. 2: Schematic diagram of COVID-19 transmission

2.2 Analysis of the model

2.2.1 Basic reproduction number

Basic reproduction number R_0 stands for the number of secondary infections those can be produced by a single infected patient on average [18]. It is very critical to distinguish new infections in the dynamic of the population to compute R_0. In general, we let $x = (x_1, \ldots, x_n)^T$, $x_i \geq 0$, be the number of individuals in each population class. For simplicity, we arrange the compartments in such a way that first m stand for the infected individuals. We also define the set

$\mathbf{X_0} = \{x \geq | x_i = 0, i = 1, \ldots, m\}$.

Let $\mathcal{F}_i(x)$ be the rate of arrival of new infections in compartment i, $\mathcal{V}_i^+(x)$ be the rate of transfer of individuals into compartment i in various other routes, and $\mathcal{V}_i^-(x)$ be the rate of transfer of individuals out of compartment i. The functions $\mathcal{F}_i(x)$, $\mathcal{V}_i^+(x)$ and $\mathcal{V}_i^-(x)$ are assumed to be continuous and at a minimum of twice differentiable on x. Now in general terms, the system of differential equations can be represented in the form

$$\dot{x}_i = f_i(x) \quad = \mathcal{F}_i(x) - \mathcal{V}_i(x), i = 1, \ldots, n, \tag{2}$$

where $\mathcal{V}_i(x) = \mathcal{V}_i^-(x) - \mathcal{V}_i^+(x)$ and the above functions must meet the assumptions A(1)-A(5) listed below.

A(1) Since each function represents a directed transfer of individuals in the population, they are all non-negative. That is, if $x \geq 0$, then $\mathcal{F}_i(x), \mathcal{V}_i^+(x), \mathcal{V}_i^-(x) \geq 0$ for $i = 1, \ldots, n$.

A(2) If a compartment is empty, then there can be no transfer of individuals out of the compartment by death, migration, infection, nor any other means. That is, if $x_i = 0$ then $\mathcal{V}_i^-(x) = 0$.

A(3) The incidence of infection for uninfected compartments is zero. That is, $\mathcal{F}_i(x) = 0$ if $i > m$.

A(4) If the population is free of disease then the population will remain free of disease. Thus, if $x \in \mathbf{X_0}$ then $\mathcal{F}_i(x) = 0$ and $\mathcal{V}_i^+(x) = 0$ for $i, \ldots, m$.

A(5) If the population is held closed to the Disease Free Equilibrium (DFE) then the population will get back to the DFE as ruled by the linearized system

$$\dot{x} = Df(x_0)(x - x_0) \tag{3}$$

where $Df(x_0) = \left[\frac{\partial f_i}{\partial x_i}\right]$ assessed at the DFE, x_0. This can be written as if $\mathcal{F}(x) = 0$ then all eigenvalues of $Df(x_0)$ have negative real parts.

Using th assumptions A(1)-A(5) enable us to partition the matrix $Df(x_0)$. This is given by the following lemma.

Lemma 1. *If x_0 is a DFE of the system (2) and $f_i(x)$ satisfies A(1)-A(5) then the derivatives $D\mathcal{F}(x_0)$ and $D\mathcal{V}(x_0)$ are partitioned as*

$$D\mathcal{F}(x_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, D\mathcal{V}(x_0) = \begin{pmatrix} V & 0 \\ J_3 & J_4 \end{pmatrix},$$

where F and V are the $m \times m$ matrices defined by $F = \left[\frac{\partial \mathcal{F}_i}{\partial x_j}(x_0)\right]$ and $V = \left[\frac{\partial \mathcal{V}_i}{\partial x_j}(x_0)\right]$ with $1 \leq i, j \leq m$. Further, F is non-negative, V is a non-singular M-matrix and all eigenvalues of J_4 have positive real part.

Proof. Let $x_0 \in \mathbf{X_0}$ be a DFE. By A(3) and A(4), $\frac{\partial \mathcal{F}_i}{\partial x_i}(x_0) = 0$ if either $i > m$ or $j > m$. Similarly, A(2) and A(4) gives that if $x \in \mathbf{X_0}$ then $\mathcal{V}_i(x) = 0$ for $i \leq m$. This provides $\frac{\partial \mathcal{V}_i}{\partial x_i}(x_0) = 0$ for $i \leq m$ and $j > m$. This shows the stated partition and zero blocks. The non-negativity of F follows from A(1) and A(4).
Let e_j be the Euclidean basis vectors. That is, e_j is the jth column of the $n \times n$ identity matrix. Then, for $i = 1, \ldots, m$

$$\left(\frac{\partial \mathcal{V}_i}{\partial x_i}\right)(x_0) = \lim_{h \to 0^+} \left(\frac{\mathcal{V}_i(x_0 + he_j) - \mathcal{V}_i(x_0)}{h}\right).$$

To show that V is a non-singular M-matrix, note that if x_0 is a DFE, then using A(2) and A(4), $\mathcal{V}_i(x_0) = 0$ for $i = 1, \ldots, m$ and if $i \neq j$, the the jth component of $x_0 + he_j = 0$ and $\mathcal{V}_i(x_0 + he_j) \leq 0$, by A(1) and A(2). Therefore, $\frac{\partial V_i}{\partial x_j} \leq 0$ for $i \leq m$and $j \neq i$ and V has the Z sign pattern [16]. Furthermore, by A(5), all eigenvalues of V have positive real parts. These two conditions provide that V is a non-singular M-matrix [16]. Finally, A(5) also implies that the eigenvalues of J_4 have positive real part. This completes the proof.

Now we aim to compute the basic reproduction number for the system (1). The method of next generation matrix is used to derive R_0. For this purpose we now define the new vector of only infected variables $\underline{X} = (E, I_A, I_M)$ containing the classes which are responsible to transmit the virus in the population. It is assumed that the classes of I_H and I_C are fully isolated and it is unlikely that the virus is transmitted to the society anymore. Hence, we establish the following system of differential equations [16–18]:

$$\begin{aligned} \frac{dE}{dt} &= k + (\beta_1 E + \beta_2 I_A + \beta_3 I_M)S - \sigma E \\ \frac{dI_A}{dt} &= \phi\sigma E - \delta_1 I_A - \gamma_1 I_A \\ \frac{dI_M}{dt} &= (1 - \phi)\sigma E - \delta_2 I_M \end{aligned} \tag{4}$$

To apply the next generation matrix method, the necessary matrices F and V are obtained as follows [16, 17]:

$$F = \begin{pmatrix} \beta_1 S^0 & \beta_2 S^0 & \beta_2 S^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

and

$$V = \begin{pmatrix} \sigma & 0 & 0 \\ -\phi\sigma & (\delta_1 + \gamma_1) & 0 \\ -(1-\phi)\sigma & 0 & \delta_2 \end{pmatrix}. \tag{6}$$

Now the next generation matrix system is

$$FV^{-1} = \begin{pmatrix} \frac{S^0\beta_1}{\sigma} - \frac{S^0\beta_3(\phi-1)}{\delta_2} + \frac{S^0\phi\beta_2}{\delta_1+\gamma_1} & \frac{S^0\beta_2}{\delta_1+\gamma_1} & \frac{S^0\beta_3}{\delta_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

So, the basic reproduction number is the spectral radius ρ of the matrix FV^{-1}. Thus, we obtain

$$R_0 = S^0\Big[\frac{\beta_1}{\sigma} + \frac{(1-\phi)\beta_3}{\delta_2} + \frac{\phi\beta_2}{\delta_1+\gamma_1}\Big] \tag{8}$$

The expression for R_0 reveals very useful information about the dynamic of COVID-19 transmission such that the expected number of secondary infection is the addition of infections due to the exposed, asymptomatic, symptomatic cases respectively. As ϕ goes to 1, the secondary infections are not produced by the cases with mild symptoms as they have been tested and isolated early. Mathematically, it can be very easily shown that

$$\lim_{\phi\to 1} R_0 = S^0\Big[\frac{\beta_1}{\sigma} + \frac{\beta_2}{\delta_1+\gamma_1}\Big].$$

2.2.2 Stability analysis of the disease free equilibrium

Let us first obtain matrix M such that

$$M = F - V = \begin{pmatrix} S^0\beta_1 - \sigma & S^0\beta_2 & S^0\beta_3 \\ -\phi\sigma & (\delta_1+\gamma_1) & 0 \\ -(1-\phi)\sigma & 0 & \delta_2 \end{pmatrix}. \tag{9}$$

Now define $s(M) = \max\{Re(\alpha) : \alpha \text{ is an eigenvalue of } M\}$. Note that $s(M)$ is a simple eigenvalue of M with a positive eigenvector. In relation to R_0 we can establish the following equivalences: $R_0 > 1$ if and only if $s(M) > 0$ and $R_0 < 1$ if and only if $s(M) < 0$.

Let us now define the set of solution to system (4) by

$$\Omega_1 = \{(E, I_A, I_M) \in \mathbb{R}^3_+ : E + I_A + I_M \le 1, E, I_A, I_M \ge 0\}.$$

Theorem 1. *If $R_0 < 1$ then the DFE, $\mathcal{E}_0$ is locally asymptotically stable on Ω_1.*

Proof. To prove this we need to apply the assumptions A(1)-A(5) and A(1)-A(4) are easily verified. For A(5) we need to show that the matrix

$$J_{\mathcal{E}_0} = \begin{pmatrix} M & 0 \\ -J_3 & J_4 \end{pmatrix}.$$

have negative real parts with $J_3 = -F$,

$$J_4 = \begin{pmatrix} -\sigma & 0 & 0 \\ \phi\sigma & -(\delta_1+\gamma_1) & 0 \\ (1-\phi)\sigma & 0 & -\delta_2 \end{pmatrix}.$$

We then compute the eigenvalues of J_4 and yield,

$$s(J_4) = \max\{-\delta_2, -\sigma, -(\delta_1 + \gamma_1)\} < 0$$

Thus, if $R_0 < 1$ then the DFE, $\mathcal{E}_0$ is locally asymptotically stable.

3 Optimal control

It is very clear that the only possible strategy to combat the novel corona virus is to control its spread over the population as per the current development. Controlling can be achieved by reducing the transmission rates [19]. In our model in system (1), the spread of the virus is mainly due to three population compartments, exposed, infected with asymptomatic and infected with mild symptoms, and non of the three groups are isolated until the individuals are being clinically tested. The asymptomatic cases have been a very serious concern for the public health system across the globe including Sri Lanka. It has been estimated that around 20% of the cases may be asymptomatic hence they are undetected, however, with the potential of spreading the virus over the population. In this section, we introduce control measures to the system (1). The model is modified addressing the dynamic of transmission and necessary mathematical derivations, and analysis will be carried out.

3.1 Mathematical model with control

In the model with control, we introduce the combined factor $(1 - u_1)$ to reduce the transmission rates β_1, β_2 and β_3 respectively from exposed, infected with asymptomatic and infected with mild symptoms population classes. Thus, this u_1 measures the effort of personal protection such as wearing face marks, personal hygiene practices, social distancing methods and etc. The control variable u_2 measures the rate of identifying asymptomatic cases through contact tracing, testing and isolating them to treat in designated hospitals. The control variable u_3 measures the rate of tracing, testing and isolating of patients with mild symptoms. In this model, we assume that $u_2 I_A$ and $u_3 I_M$ are removed from I_A and I_M compartments and they are added to the compartment I_H. In addition, the critically sick patients who are currently at I_H compartment will be transferred to the class of patients in intensive care units with a rate of η. It is further assumed that asymptomatic cases who are undetected and could be recovered themselves with a rate of γ_1, patients who are in general wards with mild symptoms are recovered with a rate of γ_2 and the patients in ICUs are recovered with a rate of γ_3, and all are added to the recovery compartment. The modified version of the system (1) can now be established as in system (10).

$$\begin{aligned}
\frac{dS}{dt} &= -(1-u_1)(\beta_1 E + \beta_2 I_A + \beta_3 I_M)S - qS \\
\frac{dE}{dt} &= k + (\beta_1 E + \beta_2 I_A + \beta_3 I_M)S - \sigma E \\
\frac{dI_A}{dt} &= \phi\sigma E - u_2 I_A - \gamma_1 I_A \\
\frac{dI_M}{dt} &= (1-\phi)\sigma E - u_3 I_M \\
\frac{dI_H}{dt} &= u_2 I_A + u_3 I_M - \eta I_H - \gamma_2 I_H \\
\frac{dI_C}{dt} &= \eta I_H - \gamma_3 I_C - \mu I_C \\
\frac{dR}{dt} &= \gamma_1 I_A + \gamma_2 I_H + \gamma_3 I_C
\end{aligned} \tag{10}$$

3.2 Mathematical analysis of the model with control

It is clear that we have introduced three time invariant control variables $u(t) = (u_1, u_2, u_3) \in \mathcal{U}$ into system (1) and these variables are associated with the population compartments S, E, I_A, I_M and I_H. Further, the control variables are bounded and measurable such that

$$\mathcal{U} = \{(u_1, u_2, u_3) | u_k \text{is Lebsegue measurable on} [0,1], 0 \leq u_k(t) \leq 1, t \in [0,T], \\ k = 1,2,3\} \tag{11}$$

The objective functional for the control problem in (10) is now defined as

$$J(u_1, u_2, u_3) = \int_0^T \left[A_1 E + A_2 I_A + A_3 I_M + \frac{1}{2}\sum_{k=1}^{3} C_k u_k^2\right] dt \tag{12}$$

subject to (10).

It is aimed to minimize the cost functional in (12) which consists of populations exposed (E), asymptomatic infected (I_A) and mildly infected (I_M) as well as the socio-economic cost related to wearing masks, sanitizing methods, cost of social distancing measures, and etc given by $C_1 u_1^2$, public health cost on contact tracing, testing and isolation of asymptomatic cases given by $C_2 u_2^2$, and the same cost that is for cases with mild symptoms represented by $C_3 u_3^2$. The constants A_1, A_2, A_3, C_1, C_2 and C_3 are the weights and balancing parameters and they measure the associated relative cost of the interventions over the interval $[0, T]$. We find the optimal control measures $u^* = (u_1^*, u_2^*, u_3^*)$ such that

$$J(u_1^*, u_2^*, u_3^*) = \min_{\mathcal{U}} J(u_1, u_2, u_3) \tag{13}$$

Now we derive necessary conditions to find the solution for the optimal control problem using Pontryagins Maximum Principle [17, 19, 21, 22]. to show the existence of the control problem, we rewrite the system (10) as in the following form [17, 20].

$$d\mathcal{X} = B\mathcal{X} + F(\mathcal{X}) \tag{14}$$

where

$$\mathcal{X} = \begin{pmatrix} S(t) \\ E(t) \\ I_A(t) \\ I_M(t) \\ I_H(t) \\ I_C(t) \\ R(t) \end{pmatrix},$$

$$B = \begin{pmatrix} -q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(u_2+\gamma_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\eta+\gamma_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\mu+\gamma_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$F(\mathcal{X}) = \begin{pmatrix} -(1-u_1)(\beta_1 E + \beta_2 I_A + \beta_3 I_M)S \\ k + (\beta_1 E + \beta_2 I_A + \beta_3 I_M)S \\ \phi\sigma E \\ (1-\phi)\sigma E \\ u_2 I_A + u_3 I_M \\ \eta I_H \\ \gamma_1 I_A + \gamma_2 I_H + \gamma_3 I_C \end{pmatrix},$$

and $d\mathcal{X}$ is the derivative of $\mathcal{X}$ with respect to time.

To show the uniform Lipschitz continuity, we let

$$G(\mathcal{X}) = B\mathcal{X} + F(\mathcal{X}). \tag{15}$$

The function $F(\mathcal{X})$ in equation (15) satisfies
$|F(\mathcal{X}_1) - F(\mathcal{X}_2)| \leq Z_1|S_1 - S_2| + Z_2|E_1 - E_2| + Z_3|I_{A1} - I_{A2}|$
$+ Z_4|I_{M1} - I_{M2}| + Z_5|I_{H1} - I_{H2}| + Z_6|I_{C1} - I_{C2}| + Z_7|R_1 - R_2|$.
Now choose $Z > 0$ such that $Z = \max(Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7)$. Thus, we have
$|F(\mathcal{X}_1) - F(\mathcal{X}_2)| \leq Z(|S_1 - S_2| + |E_1 - E_2| + |I_{A1} - I_{A2}|$
$+ |I_{M1} - I_{M2}| + |I_{H1} - I_{H2}| + |I_{C1} - I_{C2}| + |R_1 - R_2|)$.
Further we have $|G(\mathcal{X}_1) - G(\mathcal{X}_2)| \leq Z|\mathcal{X}_1 - \mathcal{X}_2|$ with $Z = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + \|K\| < \infty$. Therefore, the function $G(\mathcal{X})$ is uniformly Lipschitz continuous. Hence we can state that the solution of the control system in (10) exists.

Theorem 2. *Given the objective functional $J(u_1, u_2, u_3)$ according to (12), where the control set $\mathcal{U}$ given by (11) is measurable subject to (10) with initial condition for the problem at $t = 0$, then there exists an optimal control*
$u^* = (u_1^*, u_2^*, u_3^*)$ *such that* $J(u_1^*, u_2^*, u_3^*) = \min\{J(u_1, u_2, u_3), (u_1, u_2, u_3) \in \mathcal{U}\}$

Proof. It is noted that the state variables and the control variables in the problem (10) are nonempty and the set $\mathcal{U}$ contains the control variables is closed and convex. The right hand side of system (10) is continuous, bounded above and can be written as a linear function of u with time invariant coefficients and are depending on state. There exist constants $l_1, l_2 > 0$ and $m > 1$ such that the intergrand $L(y, u, t)$ of the objective functional J is convex and it satisfies

$$L(y, u, t) \geq l_1(|u_1|^2 + |u_2|^2 + |u_3|^2)^{m/2} - l_2.$$

The state variables and the set of control $\mathcal{U}$ is clearly bounded and nonempty. The solutions are bounded, and convex. Thus, the system is bi-linear in control variables as the solutions are bounded. Now the following is verified so that

$$A_1E + A_2I_A + A_3I_M + \frac{1}{2}(C_1u_1^2 + C_2u_2^2 + C_3u_3^2) \geq l_1(|u_1|^2 + |u_2|^2 + |u_3|^2)^{m/2} - l_2$$

where $A_1, A_2, A_3, C_1, C_2, C_3, l_1, l_2 > 0$ and $m > 1$ [23, 24].

Now we discuss the method of obtaining the solution to the problem (10). For this, it is necessary to define the Lagrangian and Hamiltonian for the optimal control problem (10). Thus, the Lagrangian L is stated as

$$L(E, I_A, I_M, u_1, u_2, u_3) = A_1E + A_2I_A + A_3I_M + \frac{1}{2}(C_1u_1^2 + C_2u_2^2 + C_3u_3^2) \tag{16}$$

and for the Hamiltonian H we let $\underline{X} = (S, E, I_A, I_M, I_H, I_C, R)$, $\mathcal{U} = (u_1, u_2, u_3)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$, and we write

$$\begin{aligned} H(\underline{X}, \mathcal{U}, \lambda) = {} & L(E, I_A, I_M, u_1, u_2, u_3) \\ & + \lambda_1(-(1-u_1)(\beta_1E + \beta_2I_A + \beta_3I_M)S - qS) \\ & + \lambda_2(k + (\beta_1E + \beta_2I_A + \beta_3I_M)S - \sigma E) \\ & + \lambda_3(\phi\sigma E - u_2I_A - \gamma_1I_A) \\ & + \lambda_4((1-\phi)\sigma E - u_3I_M) \\ & + \lambda_5(u_2I_A + u_3I_M - \eta I_H - \gamma_2I_H) \\ & + \lambda_6(\eta I_H - \gamma_3I_C - \mu I_C) \\ & + \lambda_7(\gamma_1I_A + \gamma_2I_H + \gamma_3I_C) \end{aligned} \tag{17}$$

where $\lambda_j, j \in \{1, 2, 3, 4, 5, 6, 7\}$ are the adjoint variables. Next derivation is the necessary conditions for the Hamiltonian H given in (17).

Theorem 3. *Given an optimal control $u^* = (u_1^*, u_2^*, u_3^*)$ and a solution*
$\underline{X}^* = (S^*, E^*, I_A^*, I_m^*, I_H^*, I_C^*, R^*)$ *with respect to the system (10), there exist adjoint variables $\lambda_j, j \in \{1, 2, 3, 4, 5, 6, 7\}$ satisfying*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= (u_1-1)(\beta_1 E+\beta_2 I_A+\beta_3 I_M)(\lambda_2-\lambda_1)+q\lambda_1\\
\frac{d\lambda_2}{dt} &= -A_1+S\beta_1(u_1-1)(\lambda_2-\lambda_1)+\sigma(\lambda_2-\lambda_3\phi+\lambda_4(\phi-1))\\
\frac{d\lambda_3}{dt} &= -A_2+u_2(\lambda_3-\lambda_5)+(u_1-1)S\beta_2(\lambda_2-\lambda_1)+\gamma_1(\lambda_3-\lambda_7)\\
\frac{d\lambda_4}{dt} &= -A_3+S\beta_3(u_1-1)(\lambda_2-\lambda_1)+u_3(\lambda_4-\lambda_5)\\
\frac{d\lambda_5}{dt} &= \gamma_2(\lambda_5-\lambda_7)+\eta(\lambda_5-\lambda_6)\\
\frac{d\lambda_6}{dt} &= \gamma_3(\lambda_6-\lambda_7)+\mu\lambda_6\\
\frac{d\lambda_7}{dt} &= 0
\end{aligned} \tag{18}$$

with transversality conditions

$$\lambda_j(t_f)=0, j\in\{1,2,3,4,5,6,7\}. \tag{19}$$

In addition, the optimal control functions u_1^*, u_2^*, u_3^* are given by

$$\begin{aligned}
u_1^* &= \min\left\{1,\max\left\{0,\frac{S^*(\beta_1E^*+\beta_2I_A^*+\beta_3I_M^*)(\lambda_2-\lambda_1)}{C_1}\right\}\right\}\\
u_2^* &= \min\left\{1,\max\left\{0,\frac{I_A^*(\lambda_3-\lambda_3)}{C_2}\right\}\right\}\\
u_3^* &= \min\left\{1,\max\left\{0,\frac{I_M^*(\lambda_4-\lambda_5)}{C_3}\right\}\right\}
\end{aligned} \tag{20}$$

Proof. The control system (10) is obtained by taking the derivative

$$\frac{d\underline{X}}{dt}=\frac{\partial H(t,u_1^*,u_2^*,u_3^*,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7)}{\partial\lambda}$$

and the adjoint system (18) is obtained taking

$$\frac{d\lambda}{dt}=\frac{-\partial H(t,u_1^*,u_2^*,u_3^*,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7)}{\partial\underline{X}^*}$$

and the optimal control measures can be derived using

$$0=\frac{\partial H(t,u_1^*,u_2^*,u_3^*,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7)}{\partial\mathcal{U}}.$$

4 Numerical results and discussion

In this section, we obtain the numerical solutions for the problem with out control (1) and for the control problem (10). The Runge-Kutta algorithm of order four is implemented in MATLAB to solve the problem with out control and the numerical schemes presented in [25–27] are coupled with Runge-Kutta method of order four to carry out the simulation for the problem with control.

4.1 Algorithm for the optimal control problem

STEP 0: Guess an initial estimation to control parameters u and t_f.

STEP 1: Use initial conditions $S(0), E(0), I_A(0), I_M(0), I_H(0), I_C(0)$ and $R(0)$ and the stocked values by u and t_f.

Find the optimal states $S^*, E^*, I_A^*, I_M^*, I_H^*, I_C^*$ and R^* which iterate forward in the control problem (10)-(20).

STEP 2: Use the stocked values by u and the transversality conditions $\lambda_j(t_f)$ for $j = 1, 2, 3, 4, 5, 6, 7$ while searching the constant $\lambda_7(t_f)$ using the scant-method.
Find the adjoint variables $\lambda_j(t_f)$ for $j = 1, 2, 3, 4, 5, 6, 7$ which iterate backward in the control problem (10)-(20).

STEP 3: Update the control utilizing new state variables $S, E, I_A, I_M, I_H, I_C, R$ and $\lambda_j(t_f)$ for $j = 1, 2, 3, 4, 5, 6, 7$ in the characterization of optimal u^* given in (20).

STEP 4: Test the convergence. If the values of the sought variables in this iteration and the final iteration are sufficiently small, check out the recent values as solutions. If the values are not small, go back to STEP 1 [28–30].

4.2 Simulation of the covid 19 dynamic system with out control

Fig. 3 shows the simulation results of the problem with out control measures given in (1). It is found recently that there are a significant number of asymptomatic cases with in the populations who are also carriers of the virus. In the public health perspectives, it is very critical to clinically identify these cases through aggressive testing and isolate them if they are found to be positive for the virus. The outcome of this task depends on how many cases are asymptomatic as a proportion. Therefore, we aim to assess the sensitivity of this proportion in the parameter level. Thus, we let ϕ to be varying and consider the vector of values $\phi = (0.1, 0.25, 0.35, 0.4, 0.45, 0.5)$ for this simulation. The rest of the parameters are $\beta_1 = 0.5, \beta_2 = 0.6, \beta_3 = 0.45, \gamma_1 = 0.5, \gamma_2 = 0.2,$
$\gamma_3 = 0.05, \delta_1 = 0.15, \delta_2 = 0.25, \eta = 0.005, \mu = 0.04, \sigma = 1/5, \phi = 0.25, k = 0.00405$ and $q = 0.0004$. The initial conditions for the dimensionless form of the problem are $S(0) = 0.85, E(0) = 0, I_A(0) = 0, I_M(0) = 0, I_H(0) = 0, I_C(0) = 0$ and $R(0) = 0$ [10, 13]. No control measures u_1, u_2 and u_3 are inactive in this case.

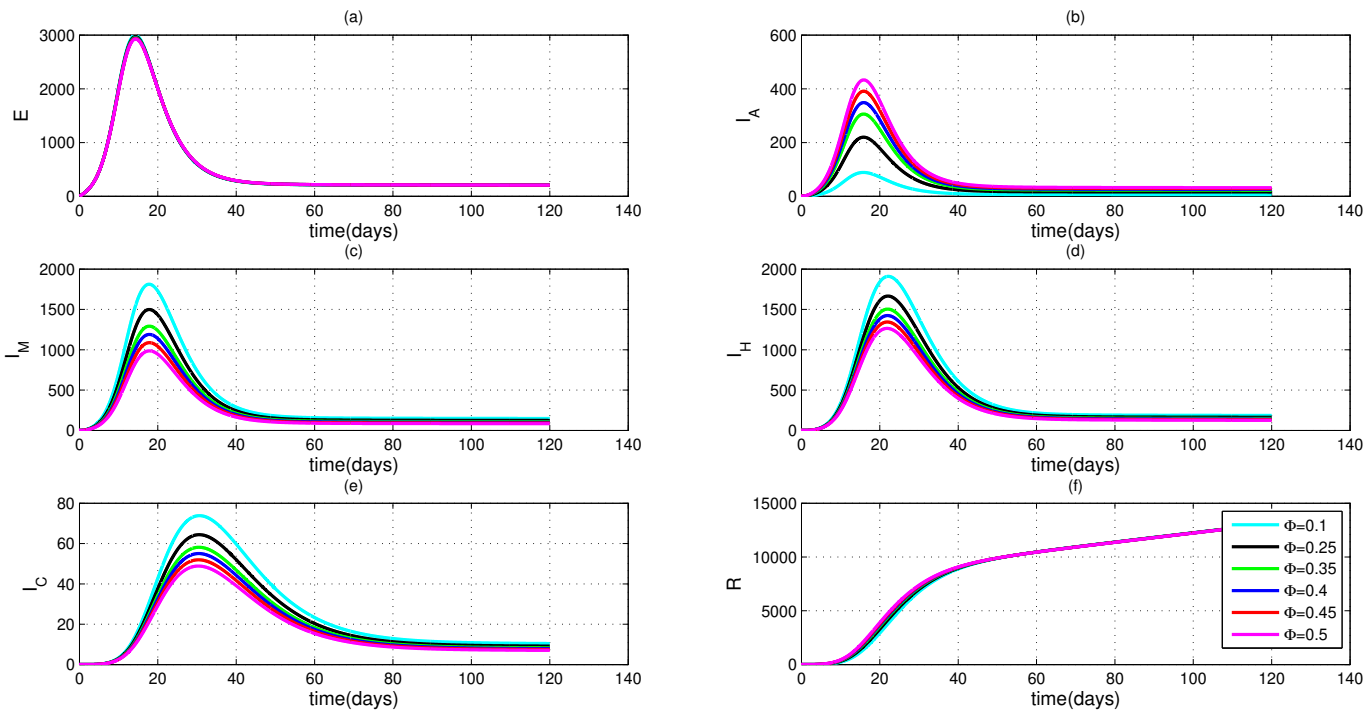

Fig. 3: The simulated solution curves for the Exposed (a), Asymptomatic (b), Symptomatic with mild (c), Isolated in hospitals (d), Treated in ICUs (e) and Recovered (f) as given in (1) with varying parameter $\phi = (0.1, 0.25, 0.35, 0.4, 0.45, 0.5)$.

It is very clearly seen from Fig. 3 that as ϕ increases, the number of asymptomatic cases also increase and this critical early diagnostic strategy has helped number of hospitalizations (I_H) and that of severely sick

patients (I_C) to reduce.

Solution trajectories of Exposed E population onto Asymptomatic I_A, Symptomatic with mild I_M, Isolated in hospitals I_H and Critically sick I_C are presented respectively in Fig. 4 (a)-(d).

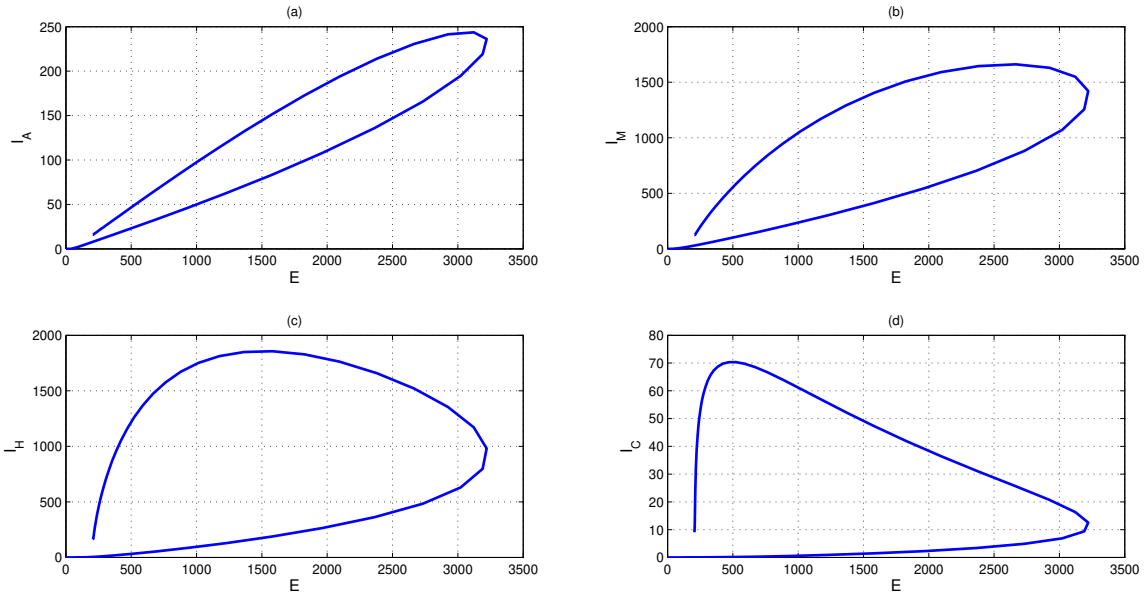

Fig. 4: Solution trajectories (E, I_A), (E, I_M), (E, I_H) and (E, I_C) for the problem 1 with fixed parameter $\phi = 0.25$.

4.3 Simulation of the control problem

In this section, we evaluate the efficacy of our three control measures, personal protection and social distancing, diagnostic and isolation of asymptomatic cases and diagnostic and isolation of mild symptomatic cases (that is u_1, u_2 and u_3 are all non-zero). First we simulate the problem in (10) considering non optimal control measures. We consider three combinations $(u_1 = 0.75, u_2 = 0.5, u_3 = 0.5)$, $(u_1 = 0.5, u_2 = 0.3, u_3 = 0.3)$ and $(u_1 = 0.25, u_2 = 0.2, u_3 = 0.2)$. The simulated results are given in Fig. 5. According to Fig. 5, it is clearly seen that when the control measures are increased the curves are flatten and the peak is occurred with a delay so that the public health system and hospitals can be prepared to handle the outbreak.

The cost functional given in (12) is used to compute the associate cost for the government if non optimal control measures are introduced. The cost incurred if $u_1 = 0.75, u_2 = 0.5, u_3 = 0.5$ is 4.9214×10^6, if $u_1 = 0.5, u_2 = 0.3,$
$u_3 = 0.3$ is 4.0192×10^6, and if $u_1 = 0.25, u_2 = 0.2, u_3 = 0.2$ is 3.6519×10^6.

The main goal of the optimal control problem presented in (10)-(20) is to minimize the number of exposed (E), asymptomatic infected cases (I_A) and mild symptomatic infected cases (I_M). In the public health point of view, it is aimed to reduce the number of patients who are in the community and able to transmit the virus, and make them isolated in designated hospitals. The simulation of the optimal control problem (10)-(20) is performed over three scenarios based on the relative importance of the three control measures. The parameters are $\beta_1 = 0.5, \beta_2 = 0.6, \beta_3 = 0.45, \gamma_1 = 0.5, \gamma_2 = 0.2, \gamma_3 = 0.05, \eta = 0.005, \mu = 0.04, \sigma = 1/5, \phi = 0.25$, $k = 0.00405$ and $q = 0.0004$. The initial conditions for the problem are $S(0) = 0.85, E(0) = 0, I_A(0) = 0, I_M(0) = 0, I_H(0) = 0, I_C(0) = 0$ and $R(0) = 0$.

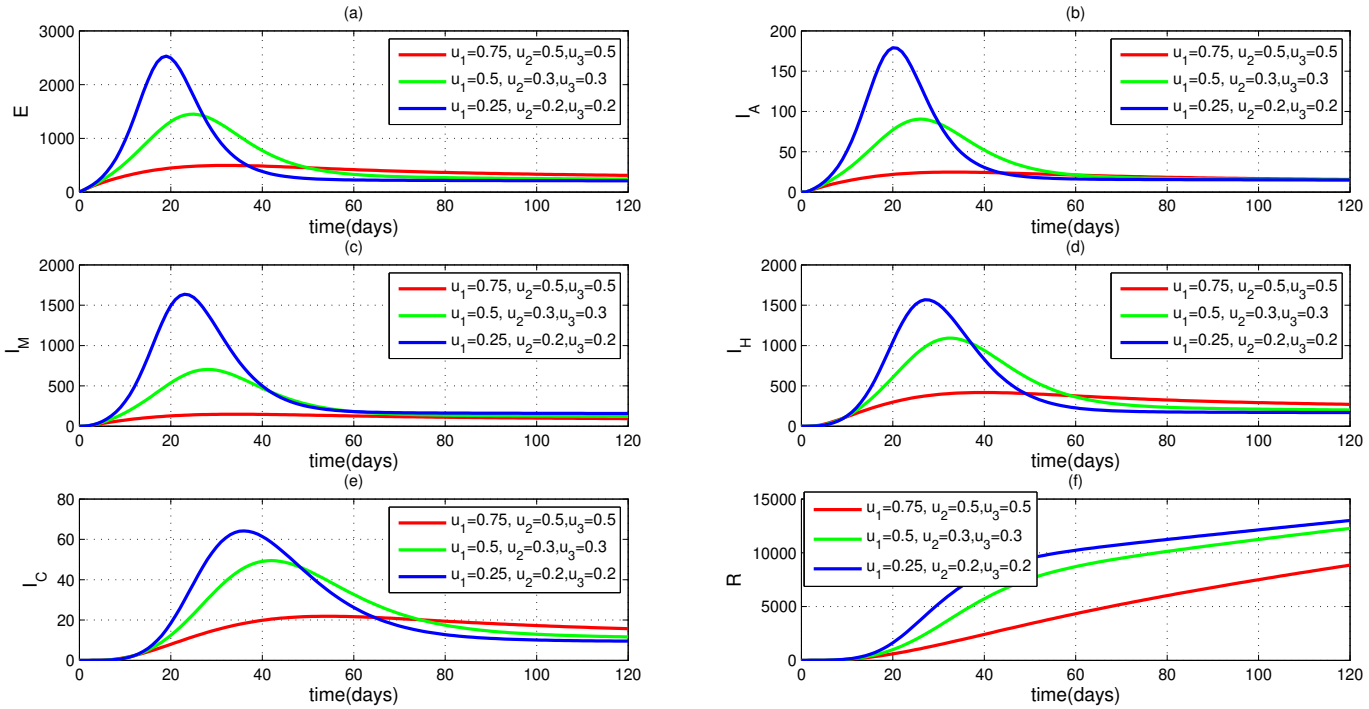

Fig. 5: The simulated solution curves for the Exposed (a), Asymptomatic (b), Symptomatic with mild (c), Isolated in hospitals (d), Treated in ICUs (e) and Recovered (f) as given in problem (10) considering combinations of non optimal control measures $(u_1 = 0.75, u_2 = 0.5, u_3 = 0.5)$, $(u_1 = 0.5, u_2 = 0.3, u_3 = 0.3)$ and $(u_1 = 0.25, u_2 = 0.2, u_3 = 0.2)$.

4.3.1 Scenario 1.

We assume the social distancing and personal protection measures are highly important while the costs on two diagnostic and isolation are equal. The simulated outcomes for each populations E, I_A, I_M, I_H, I_C and R are presented in Fig. 6 while the time invariant functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ are illustrated in Fig. 7.

It is seen from the Fig. 6 that the control interventions are effective since the number of cases for each E, I_A and I_M populations have reduced compared to they are for the problem with out control in (1). Further, it is seen that the peak of each curve has been reduced and it is delayed. Thus, the optimal control measures have helped to flatten the curve. The control functions in Fig. 7 suggest that tracing, testing and isolation of both asymptomatic and symptomatic infections are required for the entire period of time considered for the simulation.

4.3.2 Scenario 2.

We assume that tracing, testing and isolating asymptomatic cases are more critical. The simulated outcomes for each populations E, I_A, I_M, I_H, I_C and R are presented in Fig. 8 while the time invariant functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ are illustrated in Fig. 9.

It is also seen from the Fig. 6 that the control interventions are effective since the number of cases for each E, I_M, I_H, I_C populations have reduced compared to they are for the problem with out control. All three control interventions needed in their full capacity during the initial stage of the outbreak, according to Fig. 9.

4.3.3 Scenario 3.

We assume that social distancing with personal protection and tracing, testing and isolating mild asymptomatic cases are equally more critical. The simulated outcomes for each populations E, I_A, I_M, I_H, I_C and R are presented in Fig. 10 while the time invariant functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ are illustrated in Fig. 11.

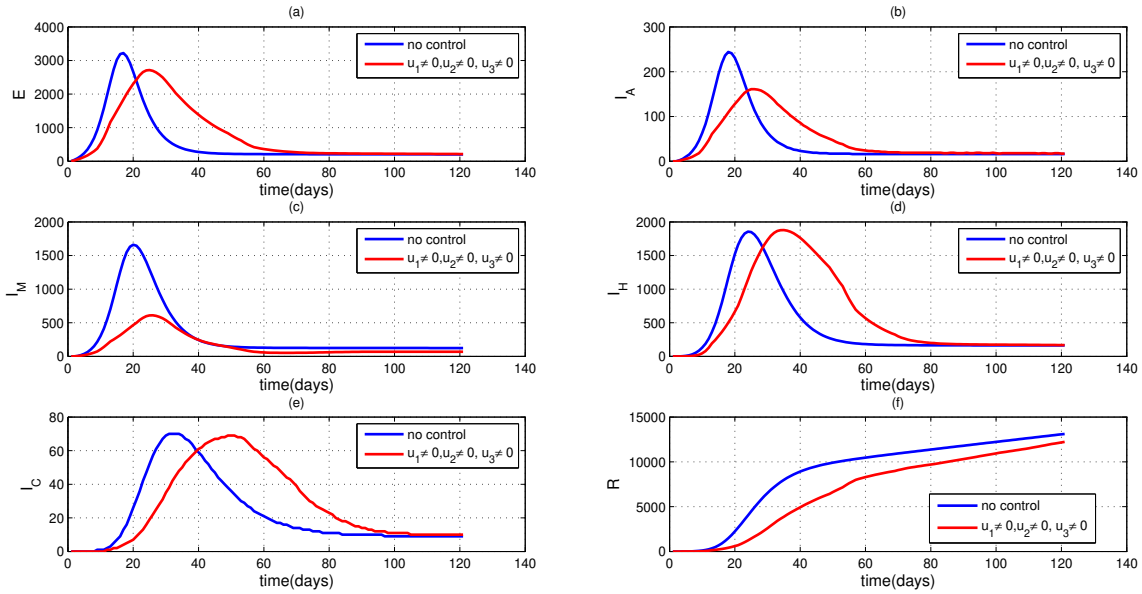

Fig. 6: The simulated solution curves for the Exposed (a), Asymptomatic (b), Symptomatic with mild (c), Isolated in hospitals (d), Treated in ICUs (e) and Recovered (f) for the optimal control problem given in (10)-(20) with $A_1 = 50$, $A_2 = 75$, $A_3 = 60$, $C_1 = 8$, $C_2 = C_3 = 2$. It is assumed that the relative cost for social distancing and personal protection is high.

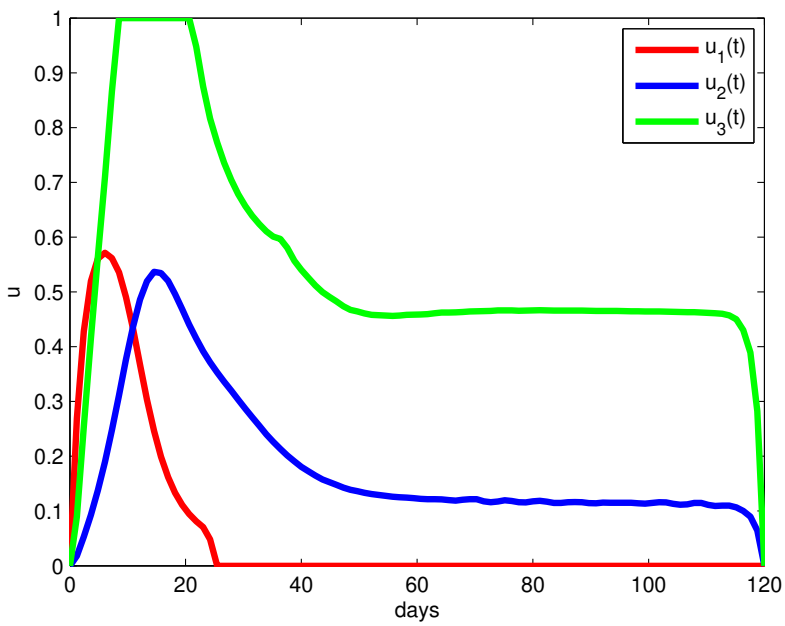

Fig. 7: The optimal control profiles $u_1(t)$, $u_2(t)$ and $u_3(t)$ with $A_1 = 50$, $A_2 = 75$, $A_3 = 60$, $C_1 = 8$, $C_2 = C_3 = 2$.

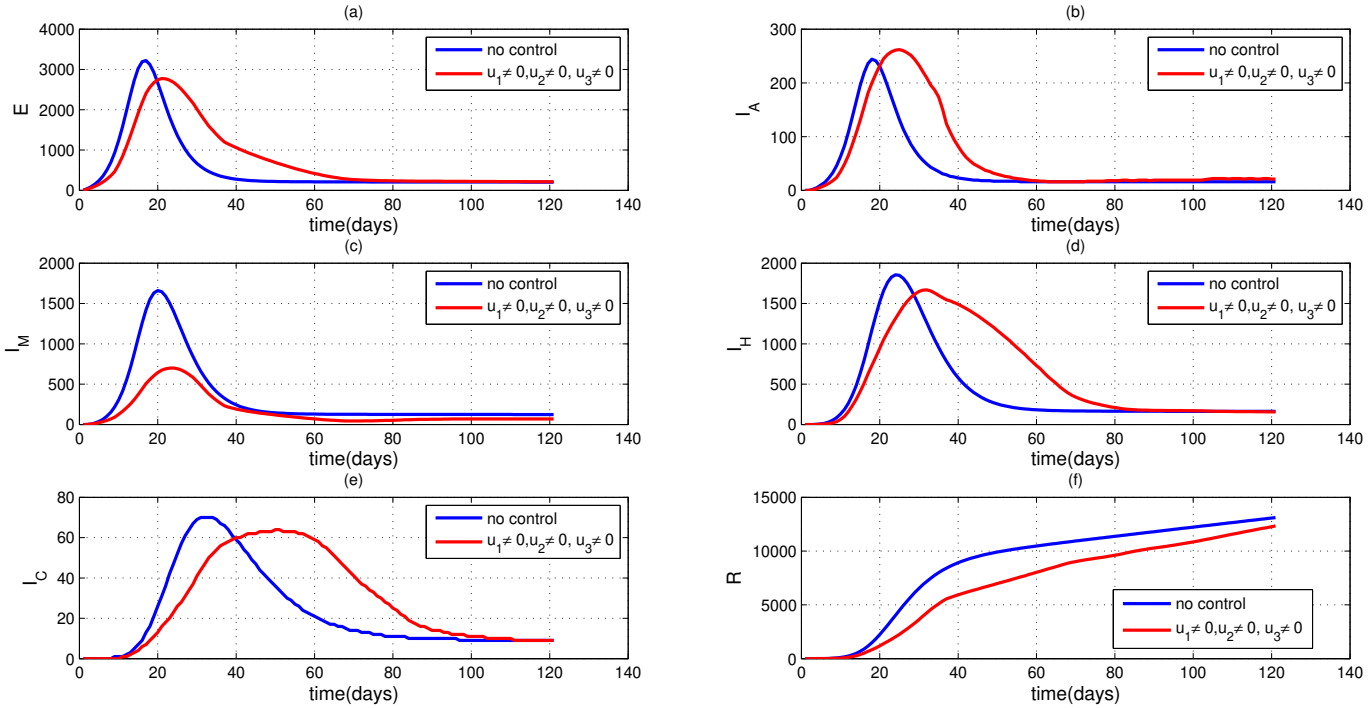

Fig. 8: The simulated solution curves for the Exposed (a), Asymptomatic (b), Symptomatic with mild (c), Isolated in hospitals (d), Treated in ICUs (e) and Recovered (f) for the optimal control problem given in (10)-(20) with $A_1 = 50$, $A_2 = 75$, $A_3 = 60$, $C_1 = 5$, $C_2 = 8$,and $C_3 = 2$. It is assumed that the relative cost for tracing and testing asymptomatic cases is high.

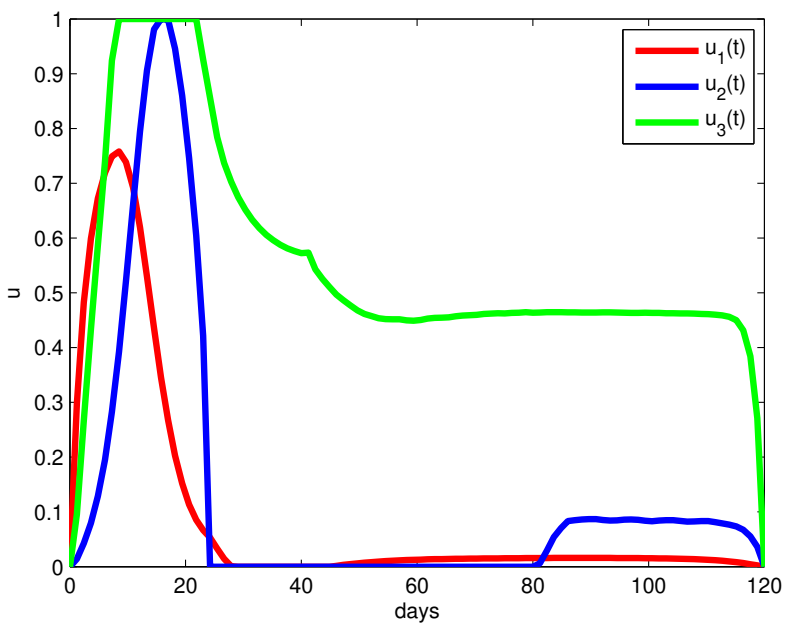

Fig. 9: The optimal control profiles $u_1(t)$, $u_2(t)$ and $u_3(t)$ with $A_1 = 50$, $A_2 = 75$, $A_3 = 60$, $C_1 = 5$, $C_2 = 8$, $C_3 = 2$.

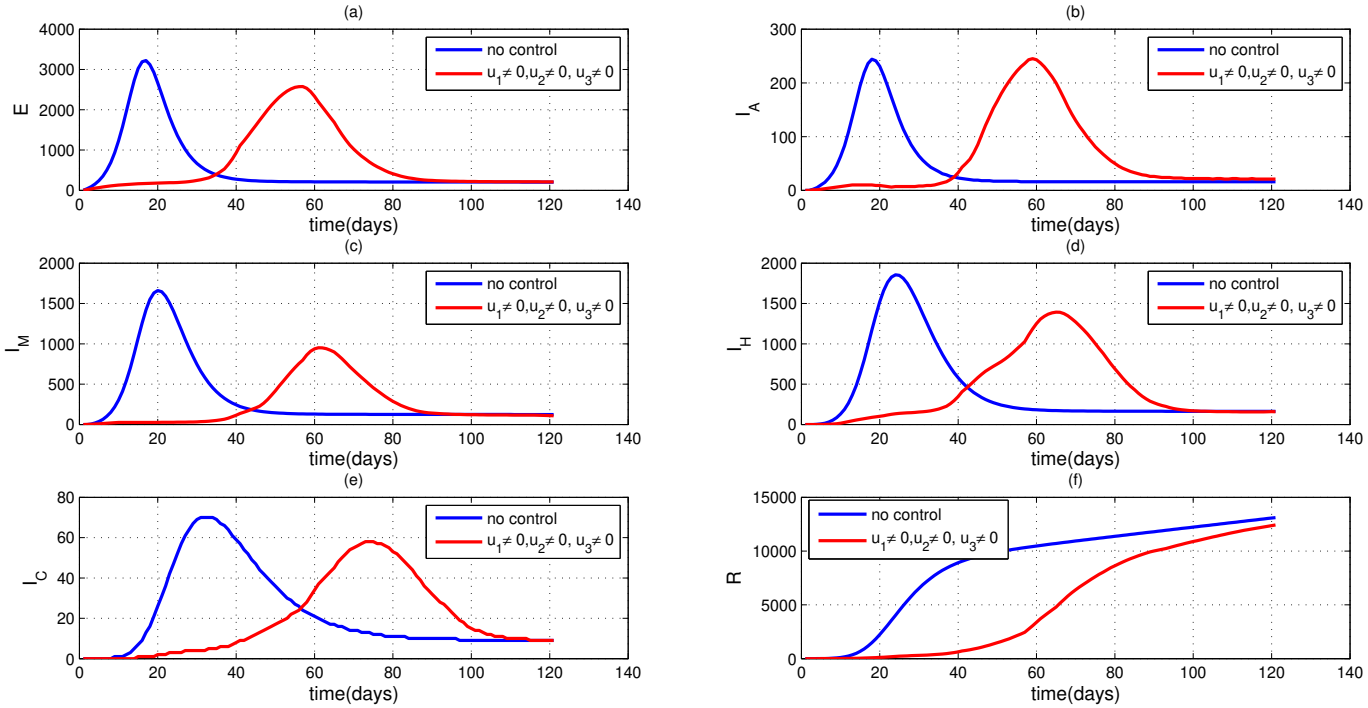

Fig. 10: The simulated solution curves for the Exposed (a), Asymptomatic (b), Symptomatic with mild (c), Isolated in hospitals (d), Treated in ICUs (e) and Recovered (f) for the optimal control problem given in (10)-(20) with $A_1 = 50$, $A_2 = 75$, $A_3 = 60$, $C_1 = 9$, $C_2 = 9$, and $C_3 = 3$. It is assumed that the relative cost for tracing and testing symptomatic cases is high while less importance is given for social distancing and personal protection.

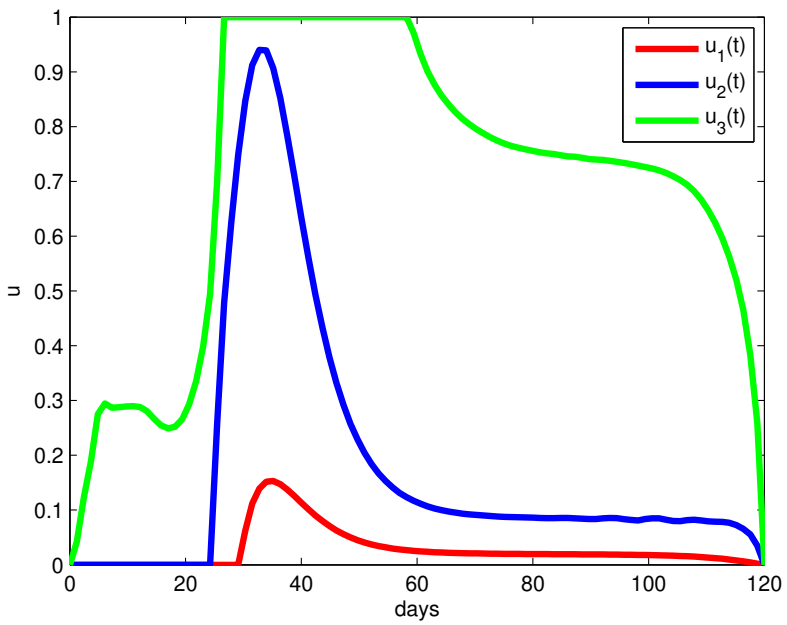

Fig. 11: The optimal control profiles $u_1(t)$, $u_2(t)$ and $u_3(t)$ with $A_1 = 50$, $A_2 = 75$, $A_3 = 60$, $C_1 = 9$, $C_2 = 9$, $C_3 = 3$.

According to Fig. 10, if the health system focuses equally more on social distancing and personal protection, tracing of asymptomatic cases then the peak of the exposed, asymptomatic, symptomatic, hospitalized, and ICU treated cases can be minimized on the other hand each peak can be delayed. Therefore, it can be stated, this control strategy is successful as the government needs to encourage more on social distancing and personal protection practices together with effective tracing, testing and isolation strategy for the patients who do not show symptoms.

The algorithm for the optimal control problem was iterated 100 times until the optimal solutions are found. The cost functional given in (12) is evaluated in each iteration and the behavior of execution is given in Fig. 12.

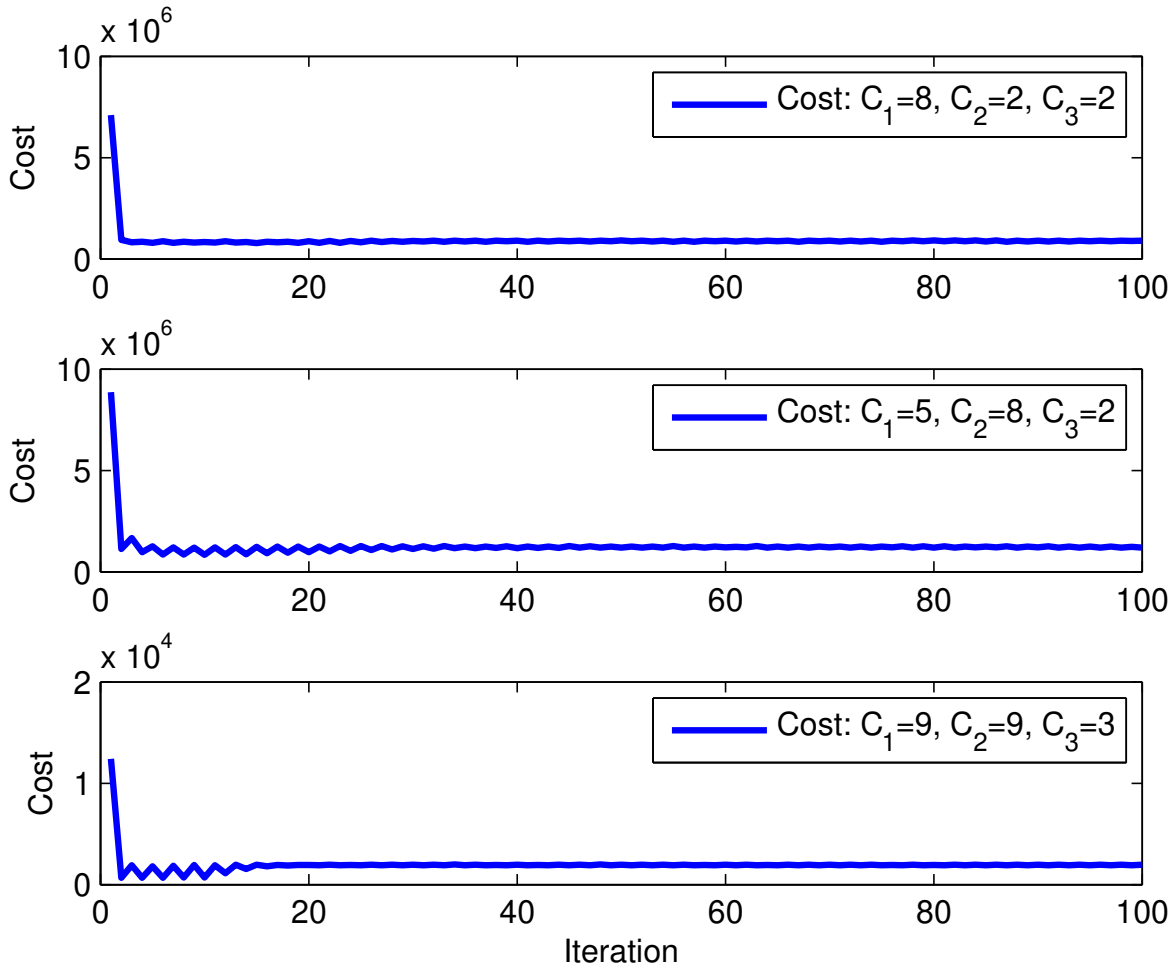

Fig. 12: The behavior of cost functional given in (12) with respect to iterations.

It can be clearly seen the convergence of cost to its optimal value 9.035×10^5 units for the scenario 1 while it is for scenario 2 obtained as 11.88×10^5 units, however, for the scenario 3, the cost is as small as 1.95×10^3.

5 Conclusion

Presently, COVID-19 disease caused by the novel corona virus has been a very serious public health concern across the globe. This outbreak is more than five months old since it was first claimed to be originated in the city of Wuhan, China in late December 2019. The development of suitable vaccine candidate is still in progress thus, strong social control measures and public health interventions are critically needed to effectively fight with the disease spread.

COVID-19 is a novel disease, therefore researchers are learning about the dynamic of this virus everyday. In an epidemiological state in this type, mathematical models are very useful to understand the dynamic of the disease and to evaluate the efficacy of different control measures such as social distancing, personal protection, and public health interventions such as contact tracing, isolation and treatments.

In this study, we develop an extended version of SIER conceptual model considering two main clinical, epidemiological and public health facts; firstly, the occurrence of asymptomatic and symptomatic infections of people and secondly, the individual demography such as age, life style and health condition found to have determined the patient's situation might turn into severe.

Since the government works hard utilizing most of its resources to control the spread over the population, an optimal control model is also constructed. Essential mathematical analysis is carried out for the models to check the stability of the equilibrium points, derive disease's R_0, investigate the existence of solutions to the optimal control problem, and etc while numerical simulations are performed in MATLAB. It is clearly seen from the simulations presented in Figures 7-11 that optimal control measures have reduced the exposed, asymptomatic, symptomatic cases significantly. The control scenario 3 provides a considerable effect on the epidemic curves, not only it minimizes the infections but also delaying the peak of the outbreak approximately by 40 days in contrast to the outcomes with out control. This enables the health system to be more equipped and prepared to combat with the epidemic. It should be noted that the simulations are carried out for a period as short as for 120 days.

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