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Modelling and Numerical Simulation on Five Species Syn Eco-System with Limited Resources

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Abstract. The present investigation is a mathematical modelling and numerical simulation on five species (S_1, S_2, S_3, S_4 and S_5) syn eco-system. In this system, S_1 is commensal of S_5, both S_2 and S_4 are hosts for S_3 and also S_5 is host for S_1. Further, S_4 and S_5 are neutral. Here all the five species have restricted resources. The model equations constitute a set of five first order non-linear simultaneous ordinary differential equations. Criteria for the asymptotic stability of all thirty two equilibrium states are established in six situations. Further, the global stability is discussed at the co-existing state by using suitable Liapunov's function and the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme by taking the fixed parameters.

Keywords: Asymptotically stable, Commensal, Equilibrium point, Host, Stable, Unstable

1 Introduction

Mathematical Modeling is one of the important interdisciplinary activity which involves a mathematical study of some aspects of diverse disciplines. Such disciplines are: Ecology, Biology, Bio-economics, Epidemiology, Genetics, Physiology, Immunology and Sociology. Ever since its inception, it has been extending its yeoman services for the development of science and technology in general and engineering in particular. It has become the backbone of modern scientific development. It has extended its sphere with manifold dimensions and every branch of mathematics has its own importance. Mathematical modeling has raised to the zenith in recent years and spread over all branches of life and drew the attention of every one.

This subject constantly endeavors to widen the areas to which, techniques of mathematics can be applied for gaining a better insight and help in deepening our understanding various phenomena that occur in nature. Real life-situations are quiet complex and we should have some insight into the situation before an attempt is made to formulate a new mathematical model. Employing proper mathematical techniques, the consequences of the model so formed can be deduced and the results compared with observations. The discrepancies between theoretical conclusions and the actual observations would suggest further improvements in the model from time to time. Several authors Ma [22], Moghadas [17], Murray [13] and Sze-Bi Hsu [21] were introduced the general concepts of Modeling in Biological Science. Srinivas [19] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Further, Kumar [18] studied some mathematical models of ecological commensalism. The present author Prasad [2–12] investigated continuous and discrete models on two, three and four species syn-ecosystems.

Ecology is the study of the interactions between organisms and their environment. The organisms include animals and plants, the environment includes the surroundings of animals. So ecology relates to the study of living beings (animals and plants) in relation to their habits and habitats. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in

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nature. Allied to the problem of population regulation is the problem of species distribution- commensalism, prey-predator, competition and so on. Significant researches in the area of theoretical ecology have been discussed by Gillman [15] and by Kot [16]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and Synecology, which are described in the treatises of Anna Sher [1], Arumugam [14] and Sharma [20].

2 Governing equations of the model

The model equations for the five species multi-environs is given by the following system of first order non-linear ordinary differential equations employing the following notation.

Notation Adopted

$N_i\,(t)$	:	The population strength of S_i, $i = 1, 2, 3, 4, 5$
t	:	Time instant
a_i	:	Natural growth rates of S_i, $i = 1, 2, 3, 4, 5$
a_{ii}	:	Self inhibition coefficients of S_i, $i = 1, 2, 3, 4, 5$
a_{15}	:	Interaction coefficient of S_1 due to S_5
a_{23}, a_{34}	:	Interaction coefficients of S_2 due to S_3 and S_3 due to S_4
$K_i = \frac{a_i}{a_{ii}}$	:	Carrying capacities of S_i, $i = 1, 2, 3, 4, 5$

Further the variables N_1, N_2, N_3, N_4, N_5 are non-negative and the model parameters a_1, a_2, a_3, a_4, a_5; $a_{11}, a_{22}, a_{33}, a_{44}, a_{55}$; a_{15}, a_{23}, a_{34} are assumed to be non-negative constants.

The fundamental equations for five species multi environs are given by the following system of non-linear ordinary differential equations.

(i) Equation for the first species (S_1):

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} {N_1}^2 + a_{15} N_1 N_5 \tag{1}$$

(ii) Equations for the species (S_i):

$$\frac{dN_i}{dt} = a_i N_i - a_{ii} {N_i}^2, \text{where} i = 2, 4, 5 \tag{2}$$

(iii) Equation for the third species (S_3):

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 + a_{23} N_2 N_3 + a_{34} N_3 N_4 \tag{3}$$

3 Equilibrium states

The equilibrium states of the system under investigation are given by

$$\frac{dN_i}{dt} = 0, \;\; i = 1, 2, 3, 4, 5 \tag{4}$$

There would be thirty two equilibrium states for this system.

(i) Fully washed out state

$$E_1 : \bar{N}_i = 0, i = 1, 2, 3, 4, 5$$

(ii) Semi/partially washed out states

(a) States in which four of the five species are washed out and fifth is not.

$$
\begin{aligned}
E_2 &: \bar{N}_1 = K_1, \bar{N}_i = 0, i = 2,3,4,5 \\
E_3 &: \bar{N}_2 = K_2, \bar{N}_i = 0, i = 1,3,4,5 \\
E_4 &: \bar{N}_3 = K_3, \bar{N}_i = 0, i = 1,2,4,5 \\
E_5 &: \bar{N}_4 = K_4, \bar{N}_i = 0, i = 1,2,3,5 \\
E_6 &: \bar{N}_5 = K_5, \bar{N}_i = 0, i = 1,2,3,4
\end{aligned}
$$

(b) Only three of the five species are washed out while the other two are not.

$$
\begin{aligned}
E_7 : \bar{N}_i = 0, \bar{N}_j = K_j, j = 1,2 \text{ and } \mathrm{i} = 3,4,5 \\
E_8 : \bar{N}_2 = K_2, \bar{N}_3 = K_3 + \frac{a_{23}K_2}{a_{33}}, \bar{N}_i = 0, i = 1,4,5 \\
E_9 : \bar{N}_3 = K_3 + \frac{a_{34}K_4}{a_{33}}, \bar{N}_4 = K_4, \bar{N}_i = 0, i = 1,2,5 \\
E_{10} : \bar{N}_i = 0, \bar{N}_j = K_j, i = 1,2,3 \text{ and } \mathrm{j} = 4,5 \\
E_{11} : \bar{N}_i = 0, \bar{N}_j = K_j, i = 2,4,5 \text{ and } \mathrm{j} = 1,3 \\
E_{12} : \bar{N}_i = 0, \bar{N}_j = K_j, i = 2,3,5 \text{ and } \mathrm{j} = 1,4 \\
E_{13} = K_1 + \frac{a_{15}K_5}{a_{11}}, \bar{N}_5 = K_5, \bar{N}_i = 0, i = 2,3,4 \\
E_{14} : \bar{N}_i = 0, \bar{N}_j = K_j, i = 1,3,5 \text{ and } \mathrm{j} = 2,4 \\
E_{15} : \bar{N}_i = 0, \bar{N}_j = K_j, i = 1,3,4 \text{ and } \mathrm{j} = 2,5 \\
E_{16} : \bar{N}_i = 0, \bar{N}_j = K_j, i = 1,2,4 \text{ and } \mathrm{j} = 3,5
\end{aligned}
$$

(c) Only two of the five species are washed out while the other three are not.

$$
\begin{aligned}
&E_{17} : \bar{N}_i = 0, \bar{N}_3 = K_3 + \frac{a_{23}K_2}{a_{33}}, \bar{N}_j = K_j, i = 1,2 \text{ and } \mathrm{j} = 4,5 \\
&E_{18} : \bar{N}_i = 0, \bar{N}_3 = K_3 + \frac{a_{23}K_2 + a_{34}K_4}{a_{33}}, \bar{N}_j = K_j, i = 1,5 \text{ and } \mathrm{j} = 2,4 \\
&E_{19} : \bar{N}_i = 0, \bar{N}_3 = K_3 + \frac{a_{34}K_4}{a_{33}}, \bar{N}_j = K_j, i = 1,2 \text{ and } \mathrm{j} = 4,5 \\
&E_{20} : \bar{N}_i = 0, \bar{N}_j = K_j, i = 1,3 \text{ and } \mathrm{j} = 2,4,5 \\
&E_{21} : \bar{N}_i = 0, \bar{N}_3 = K_3 + \frac{a_{23}K_2}{a_{33}}, \bar{N}_j = K_j, i = 1,4 \text{ and } \mathrm{j} = 2,5 \\
&E_{22} : \bar{N}_1 = K_1 + \frac{a_{15}K_5}{a_{11}}, \bar{N}_i = 0, \bar{N}_j = K_j, i = 2,3 \text{ and } \mathrm{j} = 4,5 \\
&E_{23} : \bar{N}_1 = K_1 + \frac{a_{15}K_5}{a_{11}}, \bar{N}_i = K_i, \bar{N}_j = 0, i = 2,5 \text{ and } \mathrm{j} = 3,4 \\
&E_{24} : \bar{N}_1 = K_1 + \frac{a_{15}K_5}{a_{11}}, \bar{N}_i = 0, \bar{N}_j = K_j, i = 2,4 \text{ and } \mathrm{j} = 3,5 \\
&E_{25} : \bar{N}_i = K_i, \bar{N}_3 = K_3 + \frac{a_{34}K_4}{a_{33}}, \bar{N}_j = 0, i = 1,4 \text{ and } \mathrm{j} = 2,5 \\
&E_{26} : \bar{N}_i = K_i, \bar{N}_j = 0, i = 1,2,4 \text{ and } \mathrm{j} = 3,5
\end{aligned}
$$

(d) Only one of the five species is washed out while the other four are not.

$$E_{27}:\bar{N}_1=0,\bar{N}_i=K_i,\bar{N}_3=K_3+\frac{a_{23}K_2+a_{34}K_4}{a_{33}},i=2,4,5$$

$$E_{28}:\bar{N}_1=K_1+\frac{a_{15}K_5}{a_{11}},\bar{N}_2=0,\bar{N}_3=K_3+\frac{a_{34}K_4}{a_{33}},\bar{N}_i=K_i,i=4,5$$

$$E_{29}:\bar{N}_1=K_1+\frac{a_{15}K_5}{a_{11}},\bar{N}_3=0,\bar{N}_i=K_i,i=2,4,5$$

$$E_{30}:\bar{N}_1=K_1+\frac{a_{15}K_5}{a_{11}},\bar{N}_i=K_i,\bar{N}_3=K_3+\frac{a_{23}K_2}{a_{33}},\bar{N}_4=0,i=2,5$$

$$E_{31}:\bar{N}_i=K_i,\bar{N}_3=K_3+\frac{a_{23}K_2+a_{34}K_4}{a_{33}},\bar{N}_5=0,i=1,2,4$$

(iii) The co-existence state or Normal steady state.

$$E_{32}:\bar{N}_1=K_1+\frac{a_{15}K_5}{a_{11}},\bar{N}_3=K_3+\frac{a_{23}K_2+a_{34}K_4}{a_{33}},\bar{N}_i=K_i,i=2,4,5$$

4 Stability analysis of the equilibrium states

Let $N=(N_1,N_2,N_3,N_4,N_5)=\bar{N}+U$ where
$U=(u_1,u_2,u_3,u_4,u_5)$ is a perturbation over the equilibrium state $\bar{N}=\left(\bar{N}_1,\bar{N}_2,\bar{N}_3,\bar{N}_4,\bar{N}_5\right)$.
The Eqs. (1)-(3) are quasi-linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt}=AU \tag{5}$$

where

$$A=\begin{bmatrix} a_1-2a_{11}\bar{N}_1+a_{15}\bar{N}_5 & 0 & 0 & 0 & a_{15}\bar{N}_1 \\ 0 & a_2-2a_{22}\bar{N}_2 & 0 & 0 & 0 \\ 0 & a_{23}\bar{N}_3 & a_3-2a_{33}\bar{N}_3+a_{23}\bar{N}_2+a_{34}\bar{N}_4 & a_{34}\bar{N}_3 & 0 \\ 0 & 0 & 0 & a_4-2a_{44}\bar{N}_4 & 0 \\ 0 & 0 & 0 & 0 & a_5-2a_{55}\bar{N}_5 \end{bmatrix} \tag{6}$$

The characteristic equation for the system is

$$\det\left[A-\lambda I\right]=0 \tag{7}$$

The equilibrium state is stable, if both the roots of the Eq. (7) are negative in case they are real or have negative real parts in case they are complex.

Here six situations arise.

4.1 Situation 1: all the five roots are positive.

The corresponding characteristic equation of the Equilibrium point E_1 is

$$\Pi\left(\lambda-a_i\right)=0,\ i=1,2,3,4,5 \tag{8}$$

The roots a_1,a_2,a_3,a_4 and a_5 of which are all positive, hence the steady state E_1 is unstable and the Eq. (5) yield the solutions,

$$u_i=u_{i0}e^{a_it},\ i\ =\ 1,\ 2,\ 3,\ 4,5 \tag{9}$$

where $u_{10},u_{20},u_{30},u_{40},u_{50}$ are the initial values of u_1,u_2,u_3,u_4,u_5 respectively.

4.2 Situation 2: the four roots are positive.

In this situation four of the five roots of the characteristic equation of the Equilibrium states E_2, E_3, E_4, E_5, E_6 and E_{14} are positive, hence all these six steady states are unstable. The solution curves are,

$$u_1 = (u_{10} - A_{10})\, e^{-a_1 t} + A_{10} e^{a_5 t}, u_i = u_{i0} e^{a_i t},\ i\ =\ 2,\ 3,\ 4, 5;$$
$$u_i = u_{i0} e^{a_i t},\ u_2 = u_{20} e^{-a_2 t}, u_3 = u_{30} e^{(a_3 + a_{23} K_2) t},\ i\ =\ 1, 4, 5;$$
$$u_i = u_{i0} e^{a_i t}, u_3 = (u_{30} - A_{20} - A_{30})\, e^{-a_3 t} + A_{20} e^{a_2 t} + A_{30} e^{a_4 t},\ i\ =\ 1,\ 2,\ 4, 5;$$
$$u_i = u_{i0} e^{a_i t},\ u_4 = u_{40} e^{-a_4 t}, u_3 = u_{30} e^{(a_3 + a_{34} K_4) t},\ i\ =\ 1,\ 2, 5;$$
$$u_1 = u_{10} e^{(a_1 + a_{15} K_5) t}, u_i = u_{i0} e^{a_i t},\ u_5 = u_{50} e^{-a_5 t},\ i\ =\ 2, 3, 4;$$
$$u_i = u_{i0} e^{a_i t}, u_j = u_{j0} e^{-a_j t}, u_3 =\ u_{30} e^{(a_3 + a_{23} K_2 + a_{34} K_4) t},\ i\ =\ 1, 5 \text{ and j} = 2, 4.$$

Where

$$A_{10} = \frac{a_{15} K_1 u_{50}}{a_1 + a_5} > 0, A_{20} = \frac{a_{23} K_3 u_{20}}{a_2 + a_3} > 0 \text{ and } \mathrm{A}_{30} = \frac{\mathrm{a}_{34} \mathrm{K}_3 \mathrm{u}_{40}}{\mathrm{a}_3 + \mathrm{a}_4} > 0.$$

4.3 Situation 3: the three roots are positive.

Here three of the five roots of the characteristic equation of the Equilibrium states $E_7, E_8, E_9, E_{10}, E_{12}, E_{13}, E_{15}, E_{16}$ and E_{19} are positive, hence all these nine steady states are unstable and the solutions given by,

$$u_1 = (u_{10} - A_{10})\, e^{-a_1 t} + A_{10} e^{a_5 t}, u_2 = u_{20} e^{-a_2 t}, u_3 = u_{30} e^{(a_3 + a_{23} K_2) t}, u_i = u_{i0} e^{a_i t},\ i\ =\ 4, 5;$$
$$u_i = u_{i0} e^{a_i t}, u_2 = u_{20} e^{-a_2 t}, u_3 = (u_{30} - A_{40} - A_{50})\, e^{-(a_3 + a_{23} K_2) t} + A_{40} e^{-a_2 t} + A_{50} e^{a_4 t},\ i\ =\ 1,\ 4, 5;$$
$$u_i = u_{i0} e^{a_i t}, u_4 = u_{40} e^{-a_4 t}, u_3 = (u_{30} - B_{20} - B_{30})\, e^{-(a_3 + a_{34} K_4) t} + B_{20} e^{a_2 t} + B_{30} e^{-a_4 t},\ i\ =\ 1,\ 2, 5;$$
$$u_1 = u_{10} e^{(a_1 + a_{15} K_5) t}, u_2 = u_{20} e^{a_2 t},\ u_3 = u_{30} e^{(a_3 + a_{34} K_4) t}, u_i = u_{i0} e^{-a_i t},\ i\ =\ 4, 5;$$
$$u_1 = (u_{10} - A_{10})\, e^{-a_1 t} + A_{10} e^{a_5 t}, u_i = u_{i0} e^{a_i t},\ u_3 = u_{30} e^{(a_3 + a_{34} K_4) t}, u_4 = u_{40} e^{-a_4 t},\ i\ =\ 2, 5;$$
$$u_1 = H e^{-(a_1 + a_{15} K_5) t} + C_{10} e^{-a_5 t}, u_i = u_{i0} e^{a_i t},\ u_5 = u_{50} e^{-a_5 t},\ i\ =\ 2, 3, 4;$$
$$u_1 = u_{10} e^{(a_1 + a_{15} K_1) t}, u_i = u_{i0} e^{-a_i t},\ u_3 = u_{30} e^{(a_3 + a_{23} K_2) t}, u_4 = u_{40} e^{a_4 t},\ i\ =\ 2, 5;$$
$$u_1 = u_{10} e^{(a_1 + a_{15} K_5) t}, u_i = u_{i0} e^{a_i t}, u_3 = (u_{30} - B_{40} - B_{50})\, e^{-a_3 t} + B_{40} e^{a_2 t} + B_{50} e^{a_4 t}, u_5 = u_{50} e^{-a_5 t},\ i\ =\ 2, 4;$$
$$u_1 =\ u_{10} e^{(a_1 + a_{15} K_5) t}, u_2 =\ u_{20} e^{a_2 t}, u_3 = (u_{30} - B_{20} - B_{30})\, e^{-(a_3 + a_{34} K_4) t} +\ B_{20} e^{a_2 t} +\ B_{30} e^{-a_4 t}, u_i =\ u_{i0} e^{-a_i t},\ i\ =\ 4, 5.$$

Where

$$A_{40} = \frac{a_{23}(a_3 + a_{23} K_2) u_{20}}{a_{33}(a_3 - a_2 + a_{23} K_2)}, A_{50} = \frac{a_{34}(a_3 + a_{23} K_2) u_{40}}{a_{33}(a_3 + a_4 + a_{23} K_2)} > 0,$$
$$B_{20} = \frac{a_{23}(a_3 + a_{34} K_4) u_{20}}{a_{33}(a_2 + a_3 + a_{34} K_4)} > 0, B_{30} = \frac{a_{34}(a_3 + a_{34} K_4) u_{40}}{a_{33}(a_3 - a_4 + a_{34} K_4)} \text{ and } H = u_{10} - C_{10}.$$

4.4 Situation 4: the two roots are positive.

In this situation two of the five roots of the characteristic equation of the Equilibrium states $E_{11}, E_{17}, E_{18}, E_{20}, E_{21}, E_{22}, E_{23}, E_{24}, E_{25}$ and E_{26} are positive, hence all these ten steady states are unstable. The solution curves are given by,

$u_1 = (u_{10} - B_{10})\, e^{(a_{15}K_5 - a_1)t} + B_{10}e^{-a_5 t}, u_i = u_{i0}e^{a_i t}, u_3 = De^{-a_3 t} + B_{40}e^{a_2 t} + B_{50}e^{a_4 t},$
$u_5 = u_{50}e^{-a_5 t},\ i = 2, 4;$
$u_1 = (u_{10} - A_{10})\, e^{-a_1 t} + A_{10}e^{a_5 t}, u_2 = u_{20}e^{-a_2 t},\ u_3 = Ee^{-(a_3 + a_{23}K_2)t} + C_{20}e^{-a_2 t} + C_{30}e^{a_4 t},$
$u_i = u_{i0}e^{a_i t},\ i = 4, 5;$
$u_i = u_{i0}e^{a_i t}, u_j = u_{j0}e^{-a_j t},\ u_3 = (u_{30} - C_{40} - C_{50})\, e^{-(a_3 + a_{23}K_2 + a_{34}K_4)t} + C_{40}e^{-a_2 t} + C_{50}e^{-a_4 t},$
$i = 1, 5, j = 2, 4;$
$u_1 = u_{10}e^{(a_1 + a_{15}K_5)t}, u_i = u_{i0}e^{-a_i t},\ u_3 = u_{30}e^{(a_3 + a_{23}K_2 + a_{34}K_4)t}, i = 2, 4, 5;$
$u_1 = u_{10}e^{(a_1 + a_{15}K_5)t}, u_i = u_{i0}e^{-a_i t}, u_3 = (u_{30} - A_{40} - A_{50})\, e^{-(a_3 + a_{23}K_2)t} + A_{40}e^{-a_2 t} + A_{50}e^{a_4 t},$
$u_4 = u_{40}e^{a_4 t},\ i = 2, 5;$
$u_1 = He^{-(a_1 + a_{15}K_5)t} + C_{10}e^{-a_5 t}, u_2 = u_{20}e^{a_2 t},\ u_3 = u_{30}e^{(a_3 + a_{23}K_4)t},\ u_i = u_{i0}e^{-a_i t},\ i = 4, 5;$
$u_1 = He^{-(a_1 + a_{15}K_5)t} + C_{10}e^{-a_5 t}, u_i = u_{i0}e^{-a_i t},\ u_3 = u_{30}e^{(a_3 + a_{23}K_2)t}, u_4 = u_{40}e^{a_4 t},\ i = 2, 5;$
$u_1 = He^{-(a_1 + a_{15}K_5)t} + C_{10}e^{-a_5 t}, u_3 = Fe^{-a_3 t} + A_{20}e^{a_2 t} + A_{30}e^{a_4 t}, u_i = u_{i0}e^{a_i t},\ u_5 = u_{50}e^{-a_5 t},\ i = 2, 4;$
$u_1 = (u_{10} - A_{10})\, e^{-a_1 t} + A_{10}e^{a_5 t}, u_i = u_{i0}e^{a_i t}, u_4 = u_{40}e^{-a_4 t}, u_3 = Ge^{-(a_3 + a_{34}K_4)t} + B_{20}e^{a_2 t} + B_{30}e^{-a_4 t},$
$i = 2, 5;$
$u_1 = (u_{10} - A_{10})\, e^{-a_1 t} + A_{10}e^{a_5 t}, u_i = u_{i0}e^{-a_i t},\ u_3 = u_{30}e^{-(a_3 + a_{23}K_2 + a_{34}K_4)t}, u_5 = u_{50}e^{a_5 t}, j = 2, 4.$

Where

$$B_{10} = \frac{a_{15}K_1 u_{50}}{a_{15}K_1 - a_1 - a_5}, B_{40} = \frac{a_{23}K_3 u_{20}}{a_2 + a_3} > 0, B_{50} = \frac{a_{34}K_3 u_{40}}{a_3 + a_4} > 0, C_{20} = \frac{a_{23}(a_3 + a_{23}K_2)u_{20}}{a_{33}(a_3 - a_2 + a_{23}K_2)},$$
$$C_{30} = \frac{a_{34}(a_3 + a_{23}K_2)u_{40}}{a_{33}(a_3 + a_4 + a_{23}K_2)} > 0, C_{40} = \frac{a_{23}(a_3 + a_{23}K_2 + a_{34}K_4)u_{20}}{a_{33}(a_3 - a_2 + a_{23}K_2 + a_{34}K_4)}, C_{50} = \frac{a_{34}(a_3 + a_{23}K_2 + a_{34}K_4)u_{40}}{a_{33}(a_3 - a_4 + a_{23}K_2 + a_{34}K_4)},$$
$D = u_{30} - B_{40} - B_{50}, E = u_{30} - C_{20} - C_{30}, F = u_{30} - A_{20} - A_{30}$ and $G = u_{30} - B_{20} - B_{30}$.

4.5 Situation 5: only one root is positive.

In this case one of the five roots of the characteristic equation of the Equilibrium states $E_{27}, E_{28}, E_{29}, E_{30}$ and E_{31} is positive, hence all these five steady states are unstable. The solution curves are given by,

$u_1 = u_{10}e^{(a_1 + a_{15}K_5)t}, u_i = u_{i0}e^{-a_i t},\ u_3 = Le^{-(a_3 + a_{23}K_2 + a_{34}K_4)t} + C_{40}e^{-a_2 t} + C_{50}e^{-a_4 t}, i = 2, 4, 5;$
$u_1 = He^{-(a_1 + a_{15}K_5)t} + C_{10}e^{-a_5 t}, u_2 = u_{20}e^{a_2 t}, u_3 = Ie^{-(a_3 + a_{34}K_4)t} + B_{20}e^{a_2 t} + B_{30}e^{-a_4 t},$
$u_i = u_{i0}e^{-a_i t},\ i = 4, 5;$
$u_1 = He^{-(a_1 + a_{15}K_5)t} + C_{10}e^{-a_5 t}, u_i = u_{i0}e^{-a_i t}, u_3 = u_{30}e^{-(a_3 + a_{23}K_2 + a_{34}K_4)t},\ i = 2, 4, 5;$
$u_1 = He^{-(a_1 + a_{15}K_5)t} + C_{10}e^{-a_5 t}, u_i = u_{i0}e^{-a_i t}, u_3 = Je^{-(a_3 + a_{23}K_2)t} + C_{20}e^{-a_2 t} + C_{30}e^{a_4 t},$
$u_4 = u_{40}e^{a_4 t},\ i = 2, 5;$
$u_1 = He^{-(a_1 + a_{15}K_5)t} + C_{10}e^{-a_5 t}, u_i = u_{i0}e^{-a_i t}, u_3 = Le^{-(a_3 + a_{23}K_2 + a_{34}K_4)t} + C_{40}e^{-a_2 t} + C_{50}e^{-a_4 t},$
$u_5 = u_{50}e^{a_5 t}, i = 2, 4.$

Where

$$L = u_{30} - C_{40} - C_{50}, I = u_{30} - B_{20} - B_{30} \text{and} J = u_{30} - C_{20} - C_{30}.$$

4.6 Situation 6: all the five roots are negative.

The characteristic roots of the normal steady state are $-(a_1 + a_{15}K_5), -a_2, -(a_3 + a_{23}K_2 + a_{34}K_4), -a_4$ and $-a_5$. Since all the five roots are negative, hence the steady state is stable. The Eq. 5 yield the solutions,

$$u_1 = He^{-(a_1+a_{15}K_5)t} + C_{10}e^{-a_5 t},\ \ u_3 = Le^{-(a_3+a_{23}K_2+a_{34}K_4)t} + C_{40}e^{-a_2 t} + C_{50}e^{-a_4 t},$$
$$u_i = u_{i0}e^{-a_i t}, \quad i = 2, 4, 5.$$

5 Liapunov's function for global stability

In the section 4 we have discussed the local stability of the equilibrium state $E_{32}\left(\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4, \bar{N}_5\right)$. We now examine the global stability of dynamical system (1), (2) and (3) at this state by suitable Liapunov's function.

Theorem 1. *The equilibrium point* $E_{32}\left(\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4, \bar{N}_5\right)$ *is globally asymptotically stable.*

Proof: Let us consider the following Liapunov's function.

$$V\left(N_1, N_2, N_3, N_4, N_5\right) = N_1 - \bar{N}_1 - \bar{N}_1\ \ln\left(\frac{N_1}{\bar{N}_1}\right) + d_1\left[N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right)\right]$$
$$+ d_2\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right] + d_3\left[N_4 - \bar{N}_4 - \bar{N}_4 \ln\left(\frac{N_4}{\bar{N}_4}\right)\right] + d_4\left[N_5 - \bar{N}_5 - \bar{N}_5 \ln\left(\frac{N_5}{\bar{N}_5}\right)\right]$$

Where d_1, d_2, d_3 and d_4 are suitable constants to be determined as in the subsequent steps. The time derivative of V, along with solutions of Eqs. (1) - (3) can be written as,

$$\frac{dV}{dt} = \left(\frac{N_1 - \bar{N}_1}{N_1}\right)\frac{dN_1}{dt} + d_1\left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + d_2\left(\frac{N_3 - \bar{N}_3}{N_3}\right)\frac{dN_3}{dt} + d_3\left(\frac{N_4 - \bar{N}_4}{N_4}\right)\frac{dN_4}{dt}$$
$$+ d_4\left(\frac{N_5 - \bar{N}_5}{N_5}\right)\frac{dN_5}{dt}$$
$$= \left(N_1 - \bar{N}_1\right)\left(a_1 - a_{11}N_1 + a_{15}N_5\right) + d_1\left(N_2 - \bar{N}_2\right)\left(a_2 - a_{22}N_2\right)$$
$$+ d_2\left(N_3 - \bar{N}_3\right)\left(a_3 - a_{33}N_3 + a_{23}N_2 + a_{34}N_4\right)$$
$$+ d_3\left(N_4 - \bar{N}_4\right)\left(a_4 - a_{44}N_4\right) + d_4\left(N_5 - \bar{N}_5\right)\left(a_5 - a_{55}N_5\right)$$
$$= -\left[\sqrt{a_{11}}\left(N_1 - \bar{N}_1\right) - \sqrt{d_4 a_{55}}\left(N_5 - \bar{N}_5\right)\right]^2 - \left(2\sqrt{d_4 a_{11} a_{55}} - a_{15}\right)\left(N_1 - \bar{N}_1\right)\left(N_5 - \bar{N}_5\right)$$
$$- \left[\sqrt{d_1 a_{22}}\left(\bar{N}_2 - \bar{N}_2\right) - \sqrt{d_2 a_{33}}\left(N_3 - \bar{N}_3\right)\right]^2 - \left(2\sqrt{d_1 d_2 a_{22} a_{33}} - d_2 a_{23}\right)\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right)$$
$$- \left[\sqrt{d_2 a_{33}}\left(N_3 - \bar{N}_3\right) - \sqrt{d_3 a_{44}}\left(N_4 - \bar{N}_4\right)\right]^2 - \left(2\sqrt{d_2 d_3 a_{33} a_{44}} - d_2 a_{34}\right)\left(N_3 - \bar{N}_3\right)\left(N_4 - \bar{N}_4\right)$$
$$- d_2 a_{33}\left(N_3 - \bar{N}_3\right)$$

The constants d_1, d_2, d_3 and d_4 as so chosen that, the coefficients of $\left(N_1 - \bar{N}_1\right)\left(N_5 - \bar{N}_5\right)$, $\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right)$ and $\left(N_3 - \bar{N}_3\right)\left(N_4 - \bar{N}_4\right)$ in the above equation vanish.

Then we have $d_1 = \frac{a_{23}^2}{a_{22}} > 0, d_2 = 4a_{33} > 0, d_3 = \frac{a_{34}^2}{a_{44}} > 0$ and $d_4 = \frac{a_{15}^2}{4a_{11}a_{55}} > 0$

With this choice of the constants d_1, d_2, d_3 and d_4.

$$V = N_1 - \bar{N}_1 - \bar{N}_1\ \ln\left(\frac{N_1}{\bar{N}_1}\right) + \frac{a_{23}^2}{a_{22}}\left[N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right)\right] + 4a_{33}\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right]$$
$$+ \frac{a_{34}^2}{a_{44}}\left[N_4 - \bar{N}_4 - \bar{N}_4 \ln\left(\frac{N_4}{\bar{N}_4}\right)\right] + \frac{a_{15}^2}{4a_{11}a_{55}}\left[N_5 - \bar{N}_5 - \bar{N}_5 \ln\left(\frac{N_5}{\bar{N}_5}\right)\right]$$
$$\frac{dV}{dt} = -\left[\sqrt{a_{11}}\left(N_1 - \bar{N}_1\right) - \frac{a_{15}}{2\sqrt{a_{11}}}\left(N_5 - \bar{N}_5\right)\right]^2 - \left[a_{23}\left(N_2 - \bar{N}_2\right) - 2a_{33}\left(N_3 - \bar{N}_3\right)\right]^2$$
$$- \left[2a_{33}\left(N_3 - \bar{N}_3\right) - a_{34}\left(N_4 - \bar{N}_4\right)\right]^2 - 4a_{33}^2\left(N_3 - \bar{N}_3\right)^2$$

which is negative definite.

Hence, $E_{32}\left(\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4, \bar{N}_5\right)$ is globally asymptotically stable.

6 Numerical simulation of the growth rate equations

The numerical solutions of the growth rate basic Eqs. (1)-(3) computed employing the fourth order Runge-Kutta method. The results are illustrated from Fig. 1 to Fig. 6 and the observations are presented below.

Consider the fixed parameters, $a_1 = 4.76, a_2 = 4.36, a_3 = 7.16, a_4 = 6.64, a_5 = 5.56, a_{11} = 0.68, a_{15} = 5.44, a_{22} = 0.8, a_{23} = 29.12, a_{33} = 14.44, a_{34} = 12.2, a_{44} = 2.24, a_{55} = 31.64.$

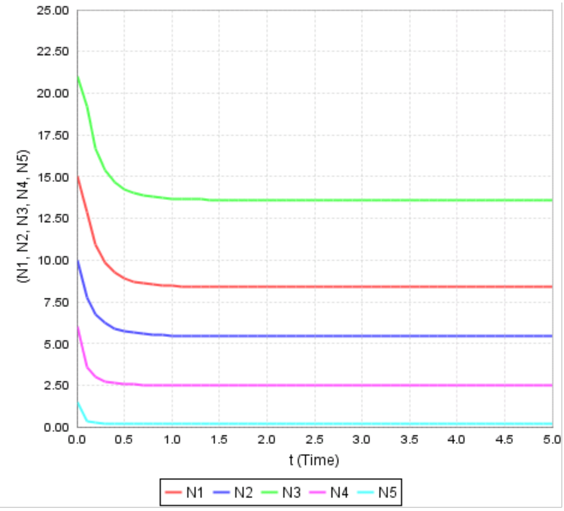

Fig. 1: Variation of time (t) against for N_{10}=15, N_{20}=10, N_{30}=21, N_{40}=6, N_{50}=1.5

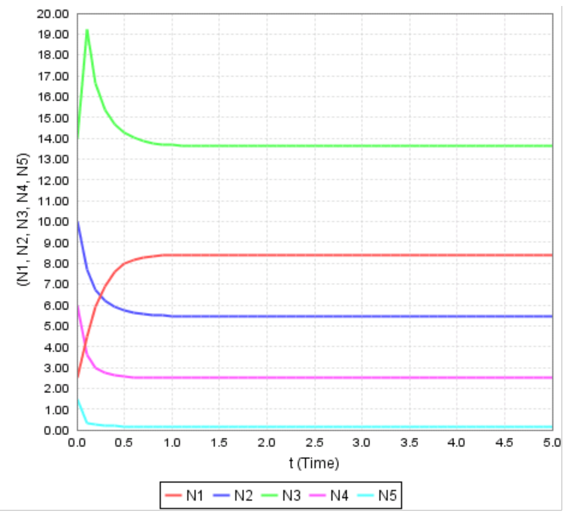

Fig. 2: Variation of time (t) against for N_{10}=25, N_{20}=10, N_{30}=14, N_{40}=6, N_{50}=1.5

6.1 Observations of the above graphs

Case 1: In this case the initial value of the fifth species is least. The second species (S_2) is dominate over S_4, S_5 and dominated by remaining other two species. Further all the five species decrease initially as shown in Fig. 1.

Case 2: In this case the initial values of S_5, S_4, S_2, S_3 and S_1 are ascending order. It is observed that initially the S_2 and S_4 are dominated by S1 up to time instant t* = 0.31 and t* = 0.11 after these dominate times we find reversal of the dominance. Further we notice that the third species have a steep rise initially. (Fig. 2).

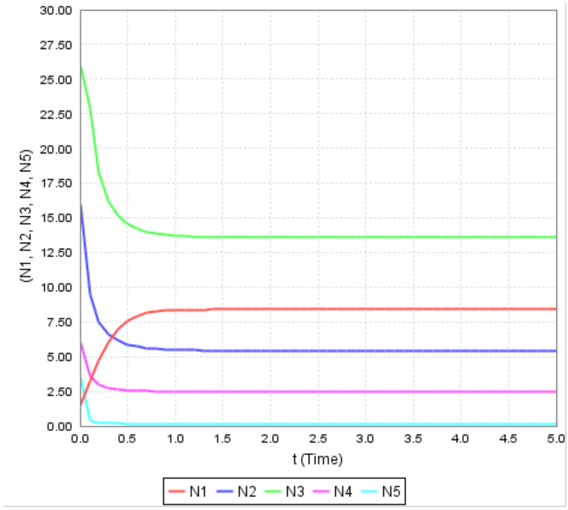

Fig. 3: Variation of time (t) against for N_{10}=1.5, N_{20}=16, N_{30}=26, N_{40}=6, N_{50}=3.5.

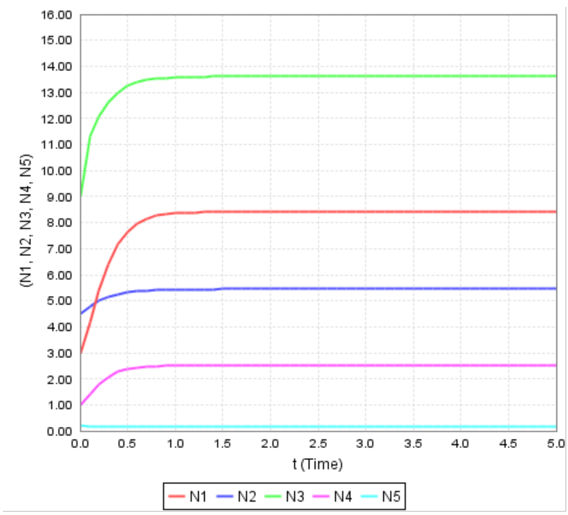

Fig. 4: Variation of time (t) against for N_{10}=3, N_{20}=4.5, N_{30}=9, N_{40}=1, N_{50}=0.2

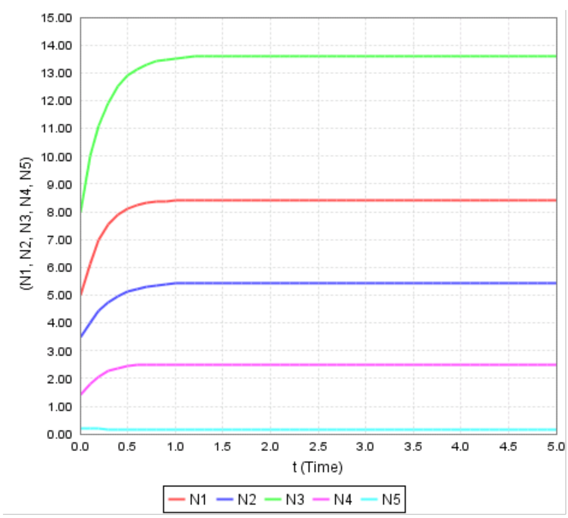

Fig. 5: Variation of time (t) against for N_{10}=5, N_{20}=3.5, N_{30}=8, N_{40}=1.4, N_{50}=0.2

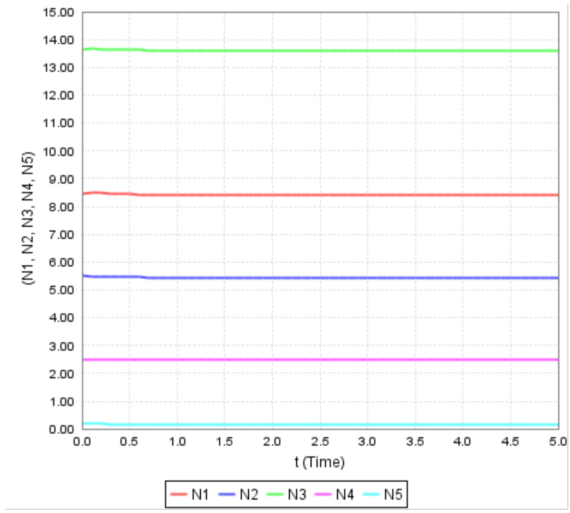

Fig. 6: Variation of time (t) against for N_{10}=8.5, N_{20}=5.5, N_{30}=13.7, N_{40}=2.5, N_{50}=0.2

Case 3: In this case the initial values of S_5, S_4, S_2, S_3 and S_1 are descending order. It is observed that initially the S_2, S_4 and S_5 are dominated by S1 up to time instant t* = 0.38, t* = 0.14 and t* = 0.08 after these dominate times we find reversal of the dominance. (Fig. 3).

Case 4: In this case the second species dominates over the first species up to the time instant t* = 0.16 after which the dominance is reversed. Further we notice that only the fifth species have no growth rate. (Fig. 4).

Case 5: In this case the initial values of S_5, S_4, S_2, S_1 and S_3 are ascending order. It is noticed that the fifth species is dominated by the fourth which itself dominated by the second, first and third. (Fig. 5).

Case 6: In this case the third species has the greatest initial value. Further we notice that all the five species have no growth rates. (Fig. 6).

7 Conclusion

The present paper deals with an investigation on the stability of five species syn eco-system in six situations depending on the roots of the characteristic equation of the equilibrium states. i.e, All the five roots are positive, the four, three and two roots are positive, only one root is positive and all the five roots are negative. It is observed that, in all thirty two equilibrium states, only $E_{32}\left(\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4, \bar{N}_5\right)$ is stable and rest of them unstable. Further, the global stability of normal state is discussed and the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order method by taking the fixed parameters and the time versus initial conditions

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