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The Müntz-Legendre Tau method for Weakly singular Volterra integral equations

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Abstract. In this paper, a numerical method for improvement the result of Weakly singular integral equations by the tau method with müntz-Legendre polynomial base will be present. Althogh this method introduced with classic polynomials bases by S.Karimi Vanani and F.Soleymani, but with introduce and employing Müntz-Legendre base, not only will be have more accuracy in approximation but also we have exact solution in some cases.

Keywords: Spetral methods, weakly singular integral equations, Tau method, Müntz-Legendre polynomials.

1 Introduction

The spectral methods are a class of techniques used in applied mathematics and scientific computing.The methods solve certain ordinary differential equtions (ODEs), partial differential equations (PDEs), integral equations (IEs) and integro-differential equations (IDEs) that become incresingly popular in recent years numerically.

The tau method was initially formulated as a tool for the approximation of special functions of mathematical physics.It could be expressed in terms of simple differential equations presented in 1981 by Ortiz and Samara[5]. During recent yaers, considerable works have been done to develop it into a powerful and accurate tool for the numerical solution of complex differential and functional equations. Its main idea is to approximate or exactly the solution of a given problem by series sentence. Karimi Vanani and soleymani[9] presented tau method with arbitrary bases for solving weakly singular Volterra equations including Abel's equation and demonstrated if the exact solution is given in the form of a polynomial of degree. The proposed method yields the exact solution with terms of the Tau aproximation series solution.

1.1 Abel's integral equations

In this paper operational Tau method with Müntz-Legendre base is applied for the following form integral equation:

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\, dt, \tag{1}$$

where $f(x)$ is a predetermined data function, and $u(x)$ is the solution that will be determined. It is to be noted that Abel's integral eq. (1) is also called Volterra integral equation of the first kind. Second kind Abel's integral equation is:

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$$u(x) = f(x) + \lambda \int_0^x \frac{1}{\sqrt{x-t}} u(t)\, dt. \tag{2}$$

The organization of this paper is as follows. Section 2 is devoted to Basic formulation of the Müntz and Müntz-Legendre polynomials. Introduce some preliminaries about Tau method will be present in section 3. The error estimination of the method is also shown in section 4. In section 5, we present some example and their numerical result to demonstrate the high accuracy and efficiency of the proposed method. Finally present conclusion in section 6.

2 Basic formulation of the müntz and müntz-legendre polynomials

In this section, we recall Müntz polynomials and present their basic properties, which are needed in this study. All of the details presented in this section as well as further details can be found in[2, 8].

Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}$. be a complex sequence. We adopt the following definition for x^λ:

$$x^\lambda = e^{\lambda \log x}, \quad x \in (0,\infty),\ \lambda \in$$

and the value at $x = 0$ to be the limit of x^λ as $x \to 0$ from $(0,\infty)$ whenever the limits exists, and consider Müntz polynomials as linear combinations of the M´untz system $\left\{x^{\lambda_0}, x^{\lambda_1}, \ \ldots, x^{\lambda_n}\right\}.ByM_n(\Lambda)$ we denote the set of all such polynomials, ?By $M_n(\Gamma)$ i.e.,

$$M_n(\Lambda) := span \left\{x^{\lambda_0}, x^{\lambda_1}, \cdots, x^{\lambda_n}\right\}$$

where the linear span is over the real (or complex) numbers. The union of all $M_n(\Lambda)$ is denoted by $M(\Lambda)$, i.e., $M(\Lambda) = \bigcup\limits_{n=0}^{\infty} M_n(\Lambda)$.

Such generalized polynomials can be orthogonalized and applied to several approximation problems, including quadrature problems. The orthogonal M´uuntz systems were considered firstly by the Armenian mathematicians Badalyan[1] and Taslakyan[6]. Recently, they were investigated by McCarthy, Sayre and Shawyer[4] and more completely by Borwein, Erdelyi, and Zhang[2].

The n-th M´untz-Legendre polynomial of $M(\Lambda)$ is defined as follows:

$$L_n(x) = L_n(\Lambda, x) := \frac{1}{2\pi i} \int_D \prod_{k=0}^{n-1} \frac{t+\lambda_k+1}{t-\lambda_k} dt,$$

where the simple contour D surrounds all the zeros of the denominator in the above integrand. These elements satisfy the following orthogonality condition:

$$\int_\Omega L_n(x) L_m(x) dx = \frac{\delta_{n,m}}{2\lambda_n + 1}, (n \geq m),$$

where $\delta_{n,m}$ is the Kronecker delta.

In the case when the Müntz sequence satisfies the following conditions:

$$\lambda_n > -\frac{1}{2}(n \in N_0), \lambda_k \neq \lambda_j (k \neq j), \tag{3}$$

a straightforward application of the aforementioned Müntz theorem shows that the Müntz-Legendre polynomials are elements of the corresponding Müntz space, and thus we have the following representation:

$$L_n(\Lambda;x)=\sum_{k=0}^{n} c_{k,n}x^{\lambda_k}, c_{k,n}=\frac{\prod_{j=0}^{n-1}(\lambda_k+\lambda_j+1)}{\prod_{j=0 j\neq k}^{n}(\lambda_k-\lambda_j)}(n\in {}_0). \quad (4)$$

Throughout this study, we assume that (2.2) holds true, and thus we always have (4).

The Müntz-Legendre polynomials $L_n(\Lambda;x)$ satisfy the following recursive formula:

$$L_n(x)=L_{n-1}(x)-(\lambda_n+\lambda_{n-1}+1)x^{\lambda_n}\int_x^1 t^{-\lambda_n-1}L_{n-1}(t)dt,\ \ (x\in(0,1]). \quad (5)$$

Clearly, the recursive relation (5) computes the values of $L_n(x)$ accurately compared with the power form (4).

3 Preliminaries about tau method

In this section, we present some preliminaries and notation about applicable Tau method to solve Eq. (1) and Eq. (2). The main idea of the method is to seek a polynomial to approximate $u(x)$ in both mentioned equations. For this, we try to replace part by part of them with the matrix and vector of unknown coefficients that is sparse and easily invertible. ?Finally we obtain a system of algebraic equations in which its solution is very easy.

For any integrable functions $\psi(x)$ and $\phi(x)$ on $[a,b]$, we define the scaler product $\langle\ ,\rangle$ by

$$\langle\psi(x),\phi(x)\rangle_\omega=\int_a^b\psi(x)\phi(x)\omega(x)dx,$$

where $\|\psi\|_\omega^2=\langle\psi(x),\psi(x)\rangle_\omega$ and $\psi(x)$ is a weight function.

Lemma 1. *Suppose that $u(x)$ is a polynomial as $u(x)=\sum_{i=0}^{\infty}u_ix^i=\mathbf{uX_x}$, then we have:*

$$D^r(x)=\frac{d^r}{dx^r}u(x)=\mathbf{uM^rX_x}, r=0,1,2,\cdots,$$

and

$$x^su(x)=\mathbf{uN^sX_x}, s=0,1,2,\cdots,$$

and

$$\int_a^x u(t)dt=\mathbf{uPX_x}-\mathbf{uPX_a},$$

where $\mathbf{u}=[u_0,u_1,\cdots,u_n,\cdots]$, $\mathbf{X}_a=[1,a,a^2,\cdots]$, $a\in$ and M, N and P are infinite matrices with only nonzero elements that are:

$$\mathbf{M}_{i+1,i}=i+1,\mathbf{N}_{i,i+1}=1, ??\,\mathbf{P}_{i,i+1}=\frac{1}{i+1}, i=0,1,2,\cdots. \quad \text{None}$$

Proof. See [3].

Let us

$$u(x)=\sum_{i=0}^{\infty}uL_i(x)=\mathbf{uLX_x}, \tag{6}$$

be an orthogonal series expansion of the exact solution of Eq. (2), where $\mathbf{u}=\{u_i\}_{i=0}^{\infty}$ is a vector of unknown coefficients and $\mathbf{LX_x}$ is an orthogonal basis for polynomials in.

Now we try to convert Eq. (2) to an algebric system using some operational matrices by the Tau method. It is sufficient to compute the

$$\int_0^x \frac{t^{\frac{m}{n}}}{(x-t)^{\alpha}}dt. \quad \text{None}$$

Lemma 2. *For* $(m\in{}^{+}\bigcup\{0\}, n\in{}^{+}, \alpha\in(0,1))$*, we have:*

$$\int_0^x \frac{t^{\frac{m}{n}}}{(x-t)^{\alpha}}dt=\frac{\Gamma\left(\frac{m}{n}+1\right)\Gamma\left(1-\alpha\right)}{\Gamma\left(\frac{m}{n}-\alpha+2\right)}\,x^{\frac{m}{n}-\alpha+1}. \tag{7}$$

Proof. At first we recall definition of Beta function and its with Gamma function as follows:

$$\beta\left(x,y\right)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \tag{8}$$

and relation between Beta and Gamma function is as follows:

$$\beta\left(x,y\right)=\frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x+y\right)}. \tag{9}$$

Now considering relations (8) and (9), we have:

$$\int_0^x \frac{t^{\frac{m}{n}}}{(x-t)^{\alpha}}dt=\int_0^x \frac{t^{\frac{m}{n}}}{x^{\alpha}(1-\frac{t}{x})^{\alpha}}dt.$$

Now by multiply numerator and denominator of fraction in $x^{\frac{m}{n}}$ conclude:

$$\int_0^x \frac{x^{\frac{m}{n}}\,t^{\frac{m}{n}}}{x^{\frac{m}{n}}\,x^{\alpha}(1-\frac{t}{x})^{\alpha}}dt=\int_0^x x^{\frac{m}{n}-\alpha}\times\frac{\left(\frac{t}{x}\right)^{\frac{m}{n}}}{(1-\frac{t}{x})^{\alpha}}dt.$$

With change of variables $(u=\frac{t}{x}\Rightarrow dt=xdu)$, we have:

$$\int_0^x x^{\frac{m}{n}-\alpha+1}\,u^{\frac{m}{n}}\,(1-u)^{-\alpha}du=x^{\frac{m}{n}-\alpha+1}\int_0^x u^{\frac{m}{n}}(1-u)^{-\alpha}du,$$

with regard to relation (8), we have:

$$x^{\frac{m}{n}-\alpha+1}\beta\left(\frac{m}{n}+1,1-\alpha\right).$$

Finally with attention to relation (9), we have:

$$\frac{\Gamma\left(\frac{m}{n}+1\right)\Gamma\left(1-\alpha\right)}{\Gamma\left(\frac{m}{n}-\alpha+2\right)}x^{\frac{m}{n}-\alpha+1},$$

and proof is complete. None

Now Tau method algorithm for resolution weakly singular volterra integral equations with M$\acute{}$untz-Legendre base will be presented. From relation (6) we have:

$$\int_0^x \frac{u(t)}{(x-t)^{\alpha}}dt=\int_0^x \frac{uLX_t}{(x-t)^{\alpha}}dt=uL\left[\int_0^x \frac{1}{(x-t)^{\alpha}}dt,\int_0^x \frac{t^{\frac{1}{n}}}{(x-t)^{\alpha}}dt,\dots\right.$$

$$\left.,\int_0^x \frac{t^{\frac{m}{n}}}{(x-t)^{\alpha}}dt,\ \dots\right]^T=uL\left[\frac{\Gamma(1-\alpha)\Gamma(1)}{\Gamma(-\alpha+2)}x^{-\alpha+1},\frac{\Gamma(1-\alpha)\Gamma(\frac{1}{n}+1)}{\Gamma(\frac{1}{n}-\alpha+2)}x^{\frac{1}{n}-\alpha+1},\dots\right.$$

$$\left.,\frac{\Gamma(1-\alpha)\Gamma(\frac{m}{n}+1)}{\Gamma(\frac{m}{n}-\alpha+2)}x^{\frac{m}{n}-\alpha+1},\dots\right]^T=\mathbf{uL\Gamma\Pi},$$

that Γ is infinite diagonal matrix with element:

$$\mathbf{\Gamma_{m,m}}=\frac{\Gamma(1-\alpha)\Gamma(\frac{m}{\gamma}+1)}{\Gamma(\frac{m}{n}-\alpha+2)},m=0,1,\cdots,$$

and

$$\mathbf{\Pi}=\left[x^{-\alpha+1},x^{\frac{1}{n}-\alpha+1},\cdots,x^{\frac{m}{n}-\alpha+1},\cdots\right]^T,$$

and L is orthogonal M$\acute{}$untz-Legendre sentence that discussed in section 2.

Now with M$\acute{}$untz-Legendre polynomials approximate $x^{\frac{m}{n}-\alpha+1}$ that for the structure of these polynomials have more accuracy in approximation.

$$x^{\frac{m}{\gamma}-\alpha+1}=\sum_{i=0}^{\infty}a_{m,i}L_i(x)=\mathbf{a}_m\mathbf{L}\mathbf{X_x},$$

that $\mathbf{a}_m=[a_{m,0},a_{m,1},a_{m,2},\cdots]$. Then we have:

$$\mathbf{\Pi}=[\mathbf{a}_0\mathbf{L}X_x,\mathbf{a}_1\mathbf{L}X_x,\cdots,\mathbf{a}_m\mathbf{L}X_x,\cdots]^T=\mathbf{ALX_x},$$

that $\mathbf{A}=[\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_m,\cdots]^T$.

We obtain:

$$\int_0^x \frac{u(t)}{(x-t)^{\alpha}}dt = \mathbf{uL\Gamma\Pi} = \mathbf{uL\Gamma ALX_x} = \mathbf{uKLX_x}, \mathbf{K} = \mathbf{L\Gamma A}. \tag{10}$$

The function $f(x)$ can be also written as:

$$f(x) = \sum_{i=0}^{\infty} f_i L_i(x) = \mathbf{fLX_x}, \tag{11}$$

where $\mathbf{f} = [f_0, f_1, \cdots]$.
By substituting (10) and (11) in Eq. (2) we obtain:

$$\mathbf{uLX_x} = \mathbf{fLX_x} + \breve{\ }\mathbf{uKLX_x},$$

or equivently

$$\mathbf{uL} = \mathbf{fL} + \breve{\ }\mathbf{uKL}.$$

Let $\mathbf{\Delta} = id - \lambda\mathbf{K}$, then

$$\mathbf{u\Delta L} = \mathbf{fL}, \tag{12}$$

where id represent the identity operator.
Because of the orthogonality of $\{L_k\}_{k=0}^{\infty}$ and inner product property, we obtain:

$$\left\langle \sum_{i=0}^{\infty} \underset{-}{u}\mathbf{\Delta_i} L_i, L_k \right\rangle = \left\langle \sum_{i=0}^{\infty} f_i L_i, L_k \right\rangle \quad k = 0, 1, 2, \cdots, \tag{13}$$

where Δ_i is the i-th column of Δ. The orthogonality assumption of $\{L_k\}_{k=0}^{\infty}$ yields

$$\mathbf{u\Delta_k} = f_k, k = 0, 1, 2, \cdots, n. \tag{14}$$

The resolution of the system (14) yields the unknown vector $\mathbf{u} = \{u_k\}_{k=0}^{n}$. Note that in Abel's first kind integral equation calculations changed as below:

$$\int_0^x \frac{u(t)}{(x-t)^{\alpha}}dt = \mathbf{uKLX_x}, \mathbf{K} = \mathbf{L\Gamma A}. \tag{15}$$

The function $f(x)$ can be also written as follows:

$$f(x) = \sum_{i=0}^{\infty} f_i L_i(x) = \mathbf{fLX_x}, \tag{16}$$

that $\mathbf{f} = [f_0, f_1, \cdots]$.
By substituting (15) and (16) in Eq. (1) we have:

$$\mathbf{fLX_x} = \mathbf{uKLX_x},$$

or equivently

$$\mathbf{fL} = \mathbf{uKL}. \tag{17}$$

Because of the orthogonality of $\{L_k\}_{k=0}^{\infty}$ and inner product property, we obtain:

$$\left\langle \sum_{i=0}^{\infty} \underline{u}\mathbf{K_i}L_i, L_j \right\rangle = \left\langle \sum_{i=0}^{\infty} f_i L_i, L_j \right\rangle \quad , \quad k = 0, 1, 2, \cdots, \tag{18}$$

where K_i is the i-th column of K. The orthogonality assumption of $\{L_i\}_{i=0}^{\infty}$ yields?

$$\mathbf{u\Delta_i} = f_i, i = 0, 1, 2, \cdots, n. \tag{19}$$

With solve the system (19) yields the unknown vector $\mathbf{u} = \{u_k\}_{k=0}^{n}$.

4 Error estimination

In this section, state how calculate the error function for

$$u(x) = f(x) + \lambda \int_0^x \frac{u(t)}{(x-t)^{\alpha}} dt, \tag{20}$$

with Tau method. Let

$$e(x) = u(x) - u_n(x). \tag{21}$$

Then from (21), obtain:

$$u(x) = e(x) + u_n(x). \tag{22}$$

Now substituting (22) in Eq. (20) we have:

$$e(x) + u_n(x) = f(x) + \lambda \int_0^x \frac{e(t) + u_n(t)}{(x-t)^{\alpha}} dt. \tag{23}$$

Then

$$e(x) = f(x) - u_n(x) + \lambda \int_0^x \frac{u_n(t)}{(x-t)^{\alpha}} dt + \lambda \int_0^x \frac{e(t)}{(x-t)^{\alpha}} dt.$$

Since $f(x), u_n(x)$ are known, we can calculate $f(x) - u_n(x) + \lambda \int_0^x \frac{u_n(t)}{(x-t)^{\alpha}} dt$. Let $F(x) = f(x) - u_n(x) + \lambda \int_0^x \frac{u_n(t)}{(x-t)^{\alpha}} dt$. Now Eq. (23) converted to:

$$e(x) = F(x) + \lambda \int_0^x \frac{e(t)}{(x-t)^{\alpha}} dt, \tag{24}$$

that is another WSVIEs equation. By applying Tau method the unknown function $e(x)$ is found.

5 Numerical examples

In this section, five numerical examples and error functions are dicussioned to demonstrate the efficiency of the proposed method. We also compare our result with those obtained by the Tau method with classic bases that you can see them in [9]. All of the calculations were performed on a PC running Matlab software.

Example 1. Consider the following Abel's integral equation (First kind) ($\alpha = \frac{1}{2}$)[10],

$$\frac{2}{105}\sqrt{x}(48x^3 - 56x^2 + 105) = \int_0^x \frac{u(t)}{\sqrt{(x-t)}}dt, x \in [0,1].$$

The exact solution is $u(x) = x^3 - x^2 + 1$.

We have obtained the exact solution in seven terms $n = 7$, which shows the high accuracy and the efficiency of the method. This result can be easily verified that the method yields the desired accuracy only in a few terms. In fact, if the exact solution is given in the form of a polynomial of degree n, the proposed method yields the exact solution with $\left(\frac{n}{\alpha} + 1\right)$ terms of the Tau approximate series solution. None

Example 2. Consider the following special case of generalized Abel's integral equation (Second kind) ($\alpha = \frac{1}{2}$)[10],

$$u(x) = x^2 + \tfrac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \;\; x \in [0,1]\,.$$

The exact solution is $u(x) = x^2$.

We have obtained the exact solution in 5 terms ($n = 2$),only. In this problem the exact solution is a polynomial of degree 2. Therefore, it can be also obtained in five terms of the Tau approximate series solution with Ḿuntz-Legendre base. None

Example 3. ?Consider the following WSVIE ($\alpha = 0.2$)[10],

$$u\left(x\right) = \frac{\pi x}{5}\csc\left(\frac{\pi}{5}\right) + x^{\frac{1}{5}} - \int_0^x \frac{u\left(t\right)}{\left(x-t\right)^{0/2}}dt\,, \;\; x \in [0,1]\,. \quad \text{None}$$

The exact solution is $\sqrt[5]{x}$.

We have obtained the exact solution in 2 terms,only. In Table 1 we have maximal error of the Tau method with classic bases (Standard base, Chebyshev base and Legendre base) that show this fact that not only Müntz-Legendre base has exact solution in problem with polynomial as exact solution but also in problem that exact solution with structure of $\sum\limits_{k=0}^{N} x^{\lambda_k}$ such that $\lambda_k(k = 0, 1, \cdot, N)$ is multiple of α has exact solution.

Table 1: Maximal error of the Tau method with classic bases[9]

n	Standard base	Chebyshev base	Legendre base
8	5.78×10^{-5}	2.87×10^{-5}	2.50×10^{-5}
16	6.23×10^{-8}	5.40×10^{-8}	4.80×10^{-8}
24	5.12×10^{-11}	4.32×10^{-11}	4.08×10^{-11}
32	5.09×10^{-13}	3.01×10^{-13}	3.41×10^{-13}
40	4.81×10^{-14}	3.70×10^{-14}	3.25×10^{-14}

Example 4. Solve the following WSVIE ($\alpha = 0.5$)[7],

$$u(x) = \left(1 - \frac{\pi}{2}\right) x + \sqrt{x}\left(1 - \frac{4}{3}x\right) + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt. \quad \text{None}$$

The exact solution is $x + \sqrt{x}$.

Like before Tau method with employing Müntz-Legendre base has exact solution in this case. In Table 2 maximal error of this example with classis bases (Standard base, Chebyshev base and Legendre base) has been shown.

Table 2: Maximal error of the Tau method with classic bases[9]

n	Standard base	Chebyshev base	Legendre base
5	9.42×10^{-4}	6.25×10^{-4}	6.16×10^{-4}
8	7.30×10^{-6}	5.47×10^{-6}	4.09×10^{-6}
12	8.23×10^{-8}	5.56×10^{-8}	4.97×10^{-8}

Example 5. Consider the following WSVIE,

$$u(x) = e^x(1 + \sqrt{\pi}erf(\sqrt{x})) - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \;\; x \in [0, 1], \quad \text{None}$$

where $erf(x)$ is the error function and the exact solution is $u(x) = e^x$.

In Table 3 comparison between classic bases and Müntz-Legendre base show that in problem with approximate solution for these bases, Müntz-Legendre base has better result.

Table 3: Maximal error of the Tau method with classic bases[9] and Müntz-Legendre base

n	Standard base	Chebyshev base	Legendre base	Müntz-Legendre base
4	9.21×10^{-4}	4.76×10^{-4}	4.56×10^{-4}	5.78×10^{-4}
7	5.10×10^{-6}	3.88×10^{-6}	3.13×10^{-6}	4.22×10^{-6}
10	3.81×10^{-10}	2.09×10^{-10}	2.22×10^{-10}	3.15×10^{-10}

6 Conclusion

In this article, an extension of the Tau method was presented to solve weakly singular Volterra integral equations of first and second kind.As mentioned before,classic bases was employed for solving weakly singular Volterra integral equations,but in some cases we have approximate of exact solution.For this we employed Müntz-Legendre base as basis functions such that we can exactly answer in such cases. This is the main characteristic of the method. Some examples were solved to illustrate the validity and efficiency of the proposed technique.The obtained results show that the accuracy of the Tau approximate solution is independent of the selection of the basis functions.

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