Effect of environmental variation in a prey-predator fishery with harvesting in presence of toxicity*

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Abstract. The hazardous effects of toxic pollutants have become a major concern for marine aquaculture from both environmental and economical point of view. Normally chronic and sublethal changes take place very slowly and most of the organisms are quite capable of adapt themselves in these changing environment. However these effects may be catastrophic for many population if the fluctuation in marine environment is rapid and drastic due to extreme pollution. In the present article, complex dynamics of a prey-predator fishery with harvesting in presence of both toxicity and environmental variation is investigated in detail. It is assumed that both the species are infected by the toxic substances released in the ocean in the form of industrial waste or by some other toxin producing species. White noise terms are included into the growth equations of the populations to simulate the effect of fluctuating environmental condition. It is observed that system has positive global solution. It is further observed that system is persistent under certain parametric condition. Coefficients of toxicity are also perturbed by noise terms to represent the rapid increase of toxic level in ocean. It is observed that if the intensities of the noise terms are moderate or small then system may remain stable in mean square for long time. On the other hand if the intensities are extremely high, system may loose its stability and become unstable leading to the extinction of species.

Keywords: white noise; global solution; persistence; mean square stability

1 Introduction

The commercial aspects of harvesting of multispecies fisheries or marine aquaculture have been investigated by many researchers in recent years [4, 5, 13, 19]. However, the harmful effects of toxicants have become a major concern for these fisheries from both environmental and economical point of view. With the growing human needs the industries are producing a huge amount of toxicants. A major part of these toxic substances are released in marine water, affecting the species living therein. Rivers, lake and sea water are often contaminated with toxic chemicals including toxic metals, oil, and synthetic organic chemicals. Toxic contaminants lead to a severe reduction in the diversity of the organisms that live in affected regions. Especially habitat destruction in estuaries and contaminant disposal in the ocean have localized adverse effects on resource organisms. The adverse effects can spread, via the food chain, to fish, birds, and mammals that feed on contaminated sea life. Toxic chemicals may reduce reproductive rates in fish population. It can affect the process of protein synthesis, may cause digestive and assimilation problem and destabilize the metabolic processes in population as a whole. These pollutants may also interfere with gonad maturation and gamete production in the population. In the worst possible case these physiological disorders can lead to reproductive failure of the population.

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populations are also vulnerable to heavy metal exposure in aquatic environment. Exposure to heavy metals like zinc, cadmium and copper severely affect the metabolic processes of many fish populations. Also oxygen consumption rates (OCR) is significantly reduced in many organisms, such as Pacific white shrimp, Green-lipped mussel, and ridgetail white prawn by heavy metal exposure. There are a lot of literatures comprising the study of effects of toxic substances in aquatic organisms, for example one may see [26, 27].

There are many species in the ocean which produces toxin (toxin producing phytoplankton TPP) and toxin released by them may affect the growth of the other species significantly. For example phytoplankton species such as *Pseudo-nitzschia* sp, *Gambierdiscus toxicus*, *Alexandrium* sp, *Pfiesteria piscicida* are highly toxic in nature and they can significantly reduce the grazing pressure of zooplankton releasing toxic chemicals. TPP can act as a strong mediator of zooplankton feeding rate, and thus playing an important role in the species interaction. Investigation of ecotoxicological effects of toxicants released by the marine biological species can be found in [3, 10, 22, 24].

Various aspects of deterministic models have been analyzed by numerous researchers over the years, including stability, persistence, bifurcation, chaotic oscillation etc [2, 6, 11, 21, 23]. But it is important to note that population communities live in a environment that fluctuate rapidly throughout the year. Especially some variables in the sea environment are random in nature, such as availability of nutrients and minerals in water, level of nitrogen, oxygen and carbon dioxide concentration in certain ocean depth, conducive water temperature, speed of the current, food supply, sudden appearance of toxic substances, mating habits of species etc. To quantify the relationship between fluctuations and species concentration, the consideration of these variations suppose to deal with noisy quantities whose variance might at time be a sizable fraction of their mean levels. For example, the phenomena of birth and death processes are intrinsically stochastic processes which depend significantly on the average biomass level and may vary rapidly from time to time. Also there are many other factors that can change predator-prey dynamics significantly, such as effects of spatial structure of the habitat on the ecosystem. Since the interaction between populations are not uniformly distributed, they too introduce certain randomness. These processes altogether can be considered as a parameter that fluctuates irregularly and randomly in nature. To capture the effect of random variation of environment, model system under the influence of ‘white noise’ terms have been investigated by many researchers in recent years [1, 8, 9, 15–17, 20, 25].

Mukherjee [16] analyzed a prey-predator system in stochastic environment and proved that the deterministic model is robust with respect to stochastic perturbations. Ghosh et al. [8, 9] studied the effects of white noise terms in three species model systems and observed that stability and persistence of the system is significantly influenced by the intensities of the noise terms. Bahar and Mao [1] considered stochastic delayed Lotka-Volterra model and proved that the environmental noise will not only suppress a potential population explosion but also make the solutions to be stochastically ultimately bounded. Investigation of classical Lotka-Volterra type system in stochastic environment can be found in Mao et al. [17]. Issues like existence and uniqueness of the solution, persistence, stationary distribution etc are studied in detail to get an insight about asymptotic behaviour of the solution. Ton and Yagi [25] considered a stochastic Bedington-DeAngelis type predator-prey model and studied the boundedness of moments of population. They also provided some upper growth and exponential death rates of population in some cases. There are several other articles which deals with the analysis of model system in presence of white noise. But to the best of our knowledge investigation of marine fisheries system in presence of both toxicity and environmental noise has not been approached yet.

In the present paper we have considered a two species prey-predator system with harvesting in presence of both toxicity and environmental noise. The paper is organized as follows. Some properties of the deterministic system is given in section 2. Formulation of stochastic model is explained in section 3. Existence and uniqueness of the positive global solution of the system is proved in section 4. Stochastic persistence under certain parametric restriction is established in section 5. Mean square stability of the system is proved in section 6. Numerical simulation is carried out in section 7. A general discussion along with ecological interpretation about the whole analysis is given in concluding remarks in Section 8.
2 The deterministic model and some of its properties

The following two species deterministic model of harvesting of prey-predator fishery in presence of toxicity was analyzed by Das et al. [24].

\[
\begin{align*}
\frac{dx_1}{dt} &= r_1 x_1 (1 - \frac{x_1}{k}) - \alpha x_1 x_2 - c_1 E x_1 - \gamma_1 x_1^3, \\
\frac{dx_2}{dt} &= -r_2 x_2 + \beta x_1 x_2 - c_2 E x_2 - \gamma_2 x_2^2.
\end{align*}
\]  

(1)

Where \(x_1\) and \(x_2\) represent prey and predator population respectively, \(r_1\) is the intrinsic growth rate of prey population, \(k\) is its environmental carrying capacity and \(r_2\) is the death rate of predator population. The amount of prey consumed by a predator per unit time is given by \(\alpha x_1\), which may also be interpreted as the trophic function or the predator’s functional response to the prey population density. A fraction \(\frac{\beta}{r}\) (\(0 < \alpha < \beta \leq 1\)) of the energy consumed with this biomass goes into predator reproduction. Both the prey and the predator are subjected to a combined harvesting effort \(E\). \(c_1\) and \(c_2\) are the catchability coefficients of the two species. It is assumed that the prey is directly infected by some external toxic substances, while the predator feeding on this infected prey is indirectly affected by the toxic substance. The term \(\gamma_1 x_1^3\) represents the infection of the prey species by external toxic substance and \(\gamma_2 x_2^2\) represents the indirect infection of predator population. \(\gamma_1\) and \(\gamma_2\) (\(0 < \gamma_2 < \gamma_1 < 1\)) are the coefficients of toxicity to the prey and predator species respectively.

We now state some results of the deterministic model without proof, which have been derived in [24].

**Theorem 1.** \(S_2\) is either a locally stable node or a locally stable focus in the presence or absence of toxicity. \(S_2\) is globally asymptotically stable.

**Result** Local stability of the system is not directly dependent on the intensities of the toxicants provided \(x_1^* > 0\) and \(x_2^* > 0\). Which is possible if \(\alpha c_2 > c_1 \gamma_2\) and \((r_2 + c_2 E)\{\gamma_1 k(r_2 + c_2 E) + r_1 \beta\} < (r_1 - c_1 E)\beta^2 k\). As the amount of toxicity increases, the population densities of both the species will gradually decline and finally both the species will go to extinction.

3 Stochastic extension of the model

We wish to investigate the effect of environmental variations in the present system by adding noise terms into the system. In accordance with the method of formulation of stochastic model [9, 16, 17, 25] here we include white noise terms in the growth equations of the populations and obtained our desired stochastic model from the deterministic system (1). The stochastic model takes care of the fact that intrinsic growth rate of prey and mortality rate of predator are randomly fluctuating due to variability of environmental conditions. We get the following stochastic system subject to the positive initial condition \(x_1(0) > x_2(0) > 0\).

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 (r_1 - \frac{r_1 x_1}{k} - \alpha x_2 - c_1 E - \gamma_1 x_1^3 + \sigma_1 \xi_1(t)), \\
\frac{dx_2}{dt} &= x_2 (-r_2 + \beta x_1 - c_2 E - \gamma_2 x_2^2 + \sigma_2 \xi_2(t)),
\end{align*}
\]  

(2)

where \(\sigma_j\)'s \((j = 1, 2)\) are the intensities of environmental forcing, \(\xi_i(t)\)'s \((i = 1, 2)\) denote two independent white noise terms, characterized by

\[
\langle \xi_j(t) \rangle = 0 \quad (j = 1, 2) \quad \text{and} \quad \langle \xi_i(t_1) \xi_j(t_2) \rangle = \delta_{ij} \delta(t_1 - t_2),
\]  

(3)

\(\delta_{ij}\) and \(\delta(.)\) are the well known Kronecker delta and Dirac delta function respectively [16]. Now Following the approach of [16] we rewrite the eq. (2) in the form of following stochastic differential equation

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 (r_1 - \frac{r_1 x_1}{k} - \alpha x_2 - c_1 E - \gamma_1 x_1^3)dt + \sigma_1 x_1 dB_1(t), \\
\frac{dx_2}{dt} &= x_2 (-r_2 + \beta x_1 - c_2 E - \gamma_2 x_2^2)dt + \sigma_2 x_2 dB_2(t).
\end{align*}
\]  

(4)
Where $B_i(t)$’s ($r = 1, 2$) are two standard one-dimensional independent Wiener processes. The relations between the white noise terms and Wiener processes are given by $dB_r(t) = \xi_r(t)dt$, ($r = 1, 2$) \cite{12}. We note that any solution of eq. (4) with positive initial condition is an Itô process \cite{14}. It is assumed that $\sigma_1$ and $\sigma_2$ are positive. First we wish to establish the global existence of solutions. We only seek for non negative solution as other solutions are not biologically feasible. Hence we first intend to show that the solution of the system (4) is positive and global.

4 Existence and uniqueness of the positive global solution

If the functions present in the stochastic system satisfy linear growth condition and local Lipschitz condition, then the system of stochastic differential equations have a unique global solution for any given initial value. Existence of this solution imply ‘non-explosion’ \cite{7} in any finite time. In this section we first prove the existence of a unique positive local solution of the system (4).

Theorem 2. For $(x_1(0), x_2(0)) \in Int(R^2_+)$ the system (4) possesses unique positive local solution $(x_1(t), x_2(t))$ for $t \in [0, \tau_e)$ almost surely (a.s.), where $\tau_e$ is the explosion time. None

Proof. We first make following transformation of variables:

$$u = \log x_1 \quad \text{and} \quad v = \log x_2.$$  

Now applying Itô’s formula we obtain from eq. (4)

$$du = (r_1(1 - \frac{v}{K}) - \alpha e^v - c_1 E - \gamma_1 e^{2u} - \frac{1}{2} \sigma_1^2)dt + \sigma_1 dB_1(t),$$

$$dv = (-r_2 + \beta e^v - c_2 E - \gamma_2 e^{2v} - \frac{1}{2} \sigma_2^2)dt + \sigma_2 dB_2(t),$$

with initial condition $u(0) = \log x_1(0), v(0) = \log x_2(0)$. Now the functions associated with the drift part of the above system has linear growth and they satisfy local Lipschitz condition. Therefore there exists unique local solution $(u(t), v(t))$ defined in some interval $[0, \tau_e)$, ($\tau_e$ being any finite positive real number). Consequently $x_1(t) = e^{u(t)}, x_2(t) = e^{v(t)}$ is the unique positive local solution of (5) starting from an interior point of the first quadrant.

Our next aim is to prove that this local solution is actually the global solution of the given system. To this end we intend only to prove that $\tau_e = \infty$ almost surely (a.s),(see \cite{25}) which will eventually serve our purpose. Thus we have the following theorem.

Theorem 3. For any initial condition $(x_1(0), x_2(0)) \in Int(R^2_+)$ the system (3.3) possesses unique solution $(x_1(t), x_2(t))$ for $t \in [0, \infty)$ and the solution will remain in $Int(R^2_+)$ with probability one. Therefore $\forall t \geq 0$ $(x_1(t), x_2(t)) \in Int(R^2_+)$ a.s. None

Proof. First we choose a sufficiently large non negative integer $\hat{r}$ such that the closed region $B(\hat{r}) \in R^2_+$ contains $(x_1(0), x_2(0))$. In the usual manner for any $r' \geq \hat{r}$ the stopping time $\tau_{r'}$ is defined by,

$$\tau_{r'} = \inf\{t \in [0, \tau_e) : x_1 \notin (\frac{1}{r'}, r') \text{ or } x_2 \notin (\frac{1}{r'}, r')\},$$

where $\inf \emptyset = \infty$ ($\emptyset$ being the empty set). We can clearly see that $\tau_{r'}$ is increasing as $r' \to \infty$.

Let $\tau_{\infty} = \lim_{r' \to \infty} \tau_{r'}$, then $\tau_{\infty} \leq \tau_e$ a.s. We shall prove that $\tau_{\infty} = \infty$ a.s, which will eventually prove that $\tau_e = \infty$.

If possible let $\tau_{\infty} \not= \infty$. Then we can find two constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_{\infty} \leq T\} > \epsilon.$$  

In this case there exists an integer $r_1 \geq \hat{r}$ such that $\forall r \geq r_1$
\[ P\{\tau_{\epsilon} \leq T\} \geq \epsilon. \] (7)

Let us define a \( C^2 \) function \( V: \text{Int}(R^2_+) \to \text{Int}(R_+) \) by,
\[ V(x_1, x_2) = (x_1 + 1 - \log x_1) + (x_2 + 1 - \log x_2). \]

The function \( V(x_1, x_2) \) is positive definite \( \forall (x_1, x_2) \in \text{Int}(R^2_+) \).

Using Itô’s formula and taking differential of \( V(x_1, x_2) \) along the trajectories of equation (4) we get
\[
dV(x_1, x_2) = [(x_1 - 1)(r_1 - \frac{\gamma_1 k^2}{k} - \alpha x_2 - c_1 E - \gamma_1 x_1^2) + (x_2 - 1)(-r_2 + \beta x_1 - c_2 E - \gamma_2 x_2^2) + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}] dt + \sigma_1 (x_1 - 1) dB_1(t) + \sigma_2 (x_2 - 1) dB_2(t),
\]
\[
dV(x_1, x_2) \leq \{r_1 x_1(1 + \frac{1}{k}) + \alpha x_2 + \beta x_1 x_2 + (c_1 + c_2) E + \gamma_1 x_1^2 + \gamma_2 x_2^2 + r_2 + \frac{\sigma_2^2}{2}\} dt + \sigma_1 (x_1 - 1) dB_1(t) + \sigma_2 (x_2 - 1) dB_2(t). \] (8)

Now proceeding exactly in a similar manner as \([25]\) we can actually prove that \( \tau_\infty = \infty \) a.s. Which effectively prove that the above local solution is indeed the global solution of the present system. This completes the proof.

5 Stochastic persistence

Though there are several concepts of stochastic persistence (Liu et al. \([15]\)), here we consider the notion of stochastic persistence in mean. Stochastic persistence means, if we chose any arbitrary positive initial condition, that is, starting from any arbitrarily chosen interior point of the first quadrant, the solution trajectories of the stochastic model will always remain within the interior of the first quadrant and remain bounded at all future time.

Definition 1. Let \( \langle x(t) \rangle = \frac{1}{t} \int_0^t x(s) ds \) and \( \langle x(t) \rangle_\ast = \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t x(s) ds \). The population \( x(t) \) is said to be strongly persistent in the mean if \( \langle x(t) \rangle_\ast > 0 \).

Before proving the main theorem we shall first state the following lemma \([15]\). We shall follow the approach used in \([19]\) to derive the persistence conditions.

Lemma 1. Suppose \( x(t) \in C([0, t] \times R_+, R^2_+) \), where \( R^2_+ = \{a : a > 0, a \in R\} \)

(i) if there are positive constants \( \mu, T \) and \( \lambda \geq 0 \) such that
\[
\ln x(t) \leq \lambda t - \mu \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t)
\]
for \( t \geq T \), where \( \beta_i \)’s are constants, \( 1 \leq i \leq n \), then \( \langle x(t) \rangle_\ast \leq \frac{\mu}{\lambda} \), a.s.

(ii) if there are positive constants \( \mu, T \) and \( \lambda \geq 0 \) such that
\[
\ln x(t) \geq \lambda t - \mu \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t)
\]
for \( t \geq T \), where \( \beta_i \)’s are constants, \( 1 \leq i \leq n \), then \( \langle x(t) \rangle_\ast \geq \frac{\mu}{\lambda} \), a.s.

Definition 2. \( \langle x(t) \rangle_\ast \) is defined by \( \langle x(t) \rangle_\ast = \lim \sup_{t \to \infty} \frac{1}{t} \int_0^t x(s) ds \)

Now we state the strong stochastic persistence result for the stochastic system 4 in the following theorem.

Theorem 4. Let \( m_x \) be a positive constant and \( (r_1 - c_1 E - \gamma_1 k^2) > \frac{\sigma_1^2}{2}, (\beta m_x - r_2 - c_2 E) > \frac{\sigma_2^2}{2} \), then solutions of stochastic model system (4) starting from any interior point of first quadrant are strongly persistent in mean.
Proof. Let us define \( v(x_1(t)) = \ln(x_1(t)) \) for \( x_1(t) \in (0, \infty) \). Then by Itô’s formula we get from the first eq. (4)

\[
d(ln x_1) = (r_1 (1 – \frac{x_1}{k}) – \alpha x_2 – c_1 E – \gamma_1 x_1^2 – \frac{\sigma_1^2}{2})dt + \sigma_1 dB_1(t) \quad \text{or,}
\]

\[
d(ln x_1) \geq (r_1 (1 – \frac{k}{k}) + \alpha x_1 – c_1 E – \gamma_1 k^2 – \frac{\sigma_1^2}{2})dt + \sigma_1 dB_1(t)
\]

or, \( d(ln x_1) \geq (r_1 – c_1 E – \gamma_1 k^2 – \frac{\sigma_1^2}{2})dt – (\frac{\gamma_1}{k} + \alpha) x_1 dt + \sigma_1 dB_1(t) \)

Integrating both sides from 0 to \( t \) and then dividing by \( t \) we get,

\[
\frac{\ln[\frac{x_1(t)}{x_1(0)}]}{t} \geq (r_1 – c_1 E – \gamma_1 k^2 – \frac{\sigma_1^2}{2}) + \frac{\sigma_1 B_1(t)}{t} – (\frac{r_1}{k} + \alpha) \langle x_1(t) \rangle
\]

Using the above lemma we get \( \langle x_1(t) \rangle \geq \left( \frac{k}{r_1 + k \alpha} \right) (r_1 – c_1 E – \gamma_1 k^2 – \frac{\sigma_1^2}{2}) \)

Therefore \( \langle x_1(t) \rangle > 0 \) whenever \( (r_1 – c_1 E – \gamma_1 k^2 – \frac{\sigma_1^2}{2}) > 0 \)

and consequently, \( \langle x_1(t) \rangle > 0 \) whenever \( (r_1 – c_1 E – \gamma_1 k^2) > \frac{\sigma_1^2}{2} \).

Since \( \langle x_1(t) \rangle > 0 \), we can find a positive number \( m_x \) such that \( x_1 \geq m_x \) for all \( t > 0 \).

Again as in case of \( x_1(t) \), we have from the second equation of (4),

\[
d(ln x_2) = (r_2 + \beta x_1 – c_2 E – \gamma_2 x_2 – \frac{\sigma_2^2}{2})dt + \sigma_2 dB_2(t)
\]

\[
d(ln x_2) \geq (r_2 + \beta m_x – c_2 E – \gamma_2 x_2 – \frac{\sigma_2^2}{2})dt + \sigma_2 dB_2(t)
\]

As earlier, integrating both sides from 0 to \( t \) and dividing by \( t \) we get,

\[
\frac{\ln[\frac{x_2(t)}{x_2(0)}]}{t} \geq \left( \beta m_x – r_2 – c_2 E – \frac{\sigma_2^2}{2} \right) + \frac{\sigma_2 B_2(t)}{t} – \gamma_2 \langle x_2(t) \rangle
\]

Therefore by using the above lemma we get \( \langle x_2(t) \rangle \geq \left( \beta m_x – r_2 – c_2 E – \frac{\sigma_2^2}{2} \right) \)

So \( \langle x_2(t) \rangle > 0 \) whenever \( (\beta m_x – r_2 – c_2 E – \frac{\sigma_2^2}{2}) > 0 \).

Hence we get the required results.

The strong persistence result of the present system in fluctuating environment greatly depends upon the intensity of environmental variation. The threshold magnitudes of environmental driving forces are given by, \( \sigma_1^* \equiv \{2 (r_1 – c_1 E – \gamma_1 k^2) \}^{\frac{1}{2}} \) and \( \sigma_2^* \equiv \{2 (\beta m_x – r_2 – c_2 E) \}^{\frac{1}{2}} \). Both the population may become extinct if forcing intensities are significantly high. It is also evident that under certain condition population neither explode nor extinct whenever forcing intensities of environmental driving forces are below the threshold values \( \sigma_1^* \) and \( \sigma_2^* \). From the expression of \( \sigma_1^* \) and \( \sigma_2^* \) it follows that, even in the presence of noise term, high growth rate of prey population can ensure long time survival of the population. But if the concentration of toxic substances are high enough then it may cause the extinction of prey population. It is also evident that ‘uncontrolled harvesting’ of both prey and predator populations could lead to the extinction of both.

6 Stability analysis of the system

In this section we study the effect of random increase of toxic level in ocean water. This rapid increase of toxic level may happen due to massive release of industrial waste or sudden leakage of crude petroleum or other petro oriented toxic products from cargos in a particular region of ocean. This instances are unexpected in nature and no prohibitory measure can be taken instantaneously to shield the populations from its adverse effect. Here the coefficients of toxicity (\( \gamma_1 \) and \( \gamma_2 \)) are perturbed by white noise terms \( \eta_{i}(t) \) \( (i = 1, 2) \) to
simulate the above situations. The perturbed term $\eta_i(t)$'s ($i = 1, 2$) are assumed to be Gaussian distributed white noise terms with zero mean.

$$
\begin{align*}
\frac{dx_1}{dt} &= r_1 x_1 \left(1 - \frac{x_1}{k}\right) - \alpha x_1 x_2 - c_1 E x_1 - (\gamma_1 + \eta_1(t)) x_1^3, \\
\frac{dx_2}{dt} &= -r_2 x_2 + \beta x_1 x_2 - c_2 E x_2 - (\gamma_2 + \eta_2(t)) x_2^3,
\end{align*}
$$

(9)

The mathematical expectation and correlation function of the process $\eta_i(t)$ ($i = 1, 2$) are given by

$$
\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t) \eta_j(t') \rangle = -\delta(t_1 - t_2) \quad (i, j = 1, 2),
$$

(10)

where as usual $\delta$ is the Dirac delta function and $< >$ represents the average over the ensemble of the stochastic process.

We consider solution of (6.1) in the form $x_1(t) = e^{y_1(t)}$, $x_2(t) = e^{y_2(t)}$,

(11)

Substituting eq. (11) into (9) we get,

$$
\begin{align*}
\frac{dy_1}{dt} &= r_1 \left(1 - \frac{e^{y_1}}{k}\right) - \alpha e^{y_2} - c_1 E - (\gamma_1 + \eta_1(t)) e^{2y_1}, \\
\frac{dy_2}{dt} &= -r_2 + \beta e^{y_1} - c_2 E - (\gamma_2 + \eta_2(t)) e^{2y_2}.
\end{align*}
$$

(12)

Again substituting $y_i(t) = y_i^*(t) + \xi_i(t)$ for $i = 1, 2$, and linearizing about $(x_1^*, x_2^*) = (e^{y_1*}, e^{y_2*})$ we get,

$$
\begin{align*}
\frac{d\xi_1}{dt} &= \left(-r_1 x_1^* - 2\gamma_1 x_1^* - 2x_1^* \eta_1 \right) \xi_1 - \alpha \xi_2, \\
\frac{d\xi_2}{dt} &= \beta x_1^* \xi_1 + \left(r_2 x_2^* - \gamma_2 x_2^* - \eta_2 x_2^* \right) \xi_2.
\end{align*}
$$

(13)

Here $\vec{\xi} = (\xi_1, \xi_2)$ represent stochastic perturbations around $(y_1^*, y_2^*)$. Now by the definition of Gaussian white noise, $\eta(t)$ is the derivative of Wiener process $W(t)$. Thus we write (13) in the following form,

$$
\frac{d(\vec{\xi}(t))}{dt} = M(\vec{\xi}(t)) dt + G(\vec{\xi}(t), t)dW(t),
$$

(14)

for $t \geq 0$ with initial value $\vec{\xi}(0) = \vec{\xi}_0$, where $M$ and $G$ are respectively given by,

$$
M = \begin{bmatrix}
-\frac{r_1 x_1^*}{k} - 2\gamma_1 x_1^* & -\alpha \\
\beta x_1^* & -r_2 x_2^* - \gamma_2 x_2^*
\end{bmatrix}
$$

and $G = \begin{bmatrix}
P_1 \xi_1 & 0 \\
0 & P_2 \xi_2
\end{bmatrix}$

where $P_1 = -2x_1^*$ and $P_2 = -x_2^*$.

Following [18] we consider a solution of equation (14) in the form

$$
\vec{\xi}(t) = e^{M(t)} \vec{\xi}_0 + \int e^{M(t-s)} G(s) dW(s).
$$

(15)

Now if $M$ has eigenvalues with negative real parts, then we can find a pair of positive constants $\beta_1$ and $\psi_1$, which satisfies,

$$
\| e^{M(t)} \|^2 \leq \beta_1 e^{\psi_1 t}.
$$

(16)
We further assume that for a pair of positive constants $\beta_2$ and $\psi_2$,

$$ | G(t) |^2 \leq \beta_2 e^{-\psi_2 t}, \quad t \geq 0. \quad (17) $$

Now

$$ E(\| \dot{\xi}(t) \|^2) \leq 3| e^{M(t)}\xi_0 \|^2 + 3 \int | e^{M(t-s)}G(s) |^2 ds $$

$$ \leq 3\beta_1 e^{-\psi_1(t)}| \xi_0 \|^2 + 3 \int \beta_1 e^{-\psi_1(t-s)}\beta_2 e^{-\psi_2 S} ds $$

$$ \leq 3\beta_1 e^{-\psi_1(t)}| \xi_0 \|^2 + 3\beta_1 \beta_2 e^{-\psi_1 \wedge \psi_2}(t). \quad (18) $$

This implies,

$$ \limsup_{t \to 0} \frac{1}{t} \log E(\| \xi(t) \|^2) \leq -(\psi_1 \wedge \psi_2), \quad (19) $$

where $(\psi_1 \wedge \psi_2)$ denotes the minimum of $\psi_1$ and $\psi_2$.

Let us chose $\varepsilon > 0$ in such a way so that $\varepsilon < \frac{\psi_1 \wedge \psi_2}{2}$.

Let $\Omega = 3\beta_1 | \xi_0 |^2 + 3\beta_1 \beta_2 \sup(t) e^{-\varepsilon t}$.

Then for $t \geq 0$,

$$ E(\| \dot{\xi}(t) \|^2) \leq \Omega e^{-\varepsilon t(\psi_1 \wedge \psi_2) - \varepsilon t}. \quad (20) $$

This proves that the system is exponentially stable in mean square.

Now since $| G(t) |^2 \leq \beta_2 e^{-\psi_2(t)}$,

it follows that $\Sigma | G(t) \xi_i |^2 \leq \beta_2 e^{-\psi_2(t)}$,

which implies $P^2 | \xi_1 |^2 + P^2 | \xi_2 |^2 \leq \beta_2 e^{-\psi_2(t)}$.

This is possible when both $P^2_1$ (i.e. $4x_1^2$) and $P^2_2$ (i.e. $x_2^2$) are small and intensities of the noise terms are also small. Therefore in this case system may remain stable in mean square for long time.

7 Numerical simulation of the model

In this section, numerical simulation results are provided using Euler-Maruyama scheme to substantiate the analytical findings for the stochastic model system reported previously. We are primarily interested to detect any change in system dynamics in presence of noise terms as the values of $\sigma_1$ and $\sigma_2$ are simultaneously varied. We have investigated the system in three particular ranges of $\sigma_1$ and $\sigma_2$. Also we have used the same values of the system parameters as used in [3]. Let $r_1 = 6.5, r_2 = 0.5, k = 300, \alpha = 0.006, \beta = 0.003, c_1 = 0.03, c_2 = 0.04, \gamma_1 = 0.00005, \gamma_2 = 0.00008, E = 1$. First we chose very low values of $\sigma_1$ and $\sigma_2$, a situation which represent low intensity variation. We observed that When $\sigma_1, \sigma_2 < 0.009$ noise terms does not affect the system significantly as the system behaves almost deterministically. Stable behaviour of the system for $\sigma_1 = \sigma_2 = 0.001$ can be shown in Fig. 1 and Fig. 2 respectively.

Now the values of toxic parameters $\gamma_1, \gamma_2$ and noise parameters $\sigma_1, \sigma_2$ are increased simultaneously. This situation resembles the situation where significant amount of toxic substance is present in the water with high variation of ocean environment. It is observed that as the values of $\sigma_1$ and $\sigma_2$ are increased beyond 0.02 oscillation in the system increases and system fluctuate within a certain range. It is further observed that biomass level of the $x_2$ population decreases very rapidly. But possibility of extinction does not arise as long as values

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Fig. 1: Stable dynamics of $x_1$ for $\gamma_1 = 0.00005$, $\gamma_2 = 0.00008$ and $\sigma_1 = \sigma_2 = 0.001$

Fig. 2: Stable dynamics of $x_2$ for $\gamma_1 = 0.00005$, $\gamma_2 = 0.00008$ and $\sigma_1 = \sigma_2 = 0.001$

Fig. 3: Fluctuation of $x_1$ for $\gamma_1 = 0.00005$, $\gamma_2 = 0.008$ and $\sigma_1 = \sigma_2 = 0.08$

Fig. 4: Fluctuation of $x_2$ for $\gamma_1 = 0.00005$, $\gamma_2 = 0.008$ and $\sigma_1 = \sigma_2 = 0.08$
of \( \sigma_1 \) and \( \sigma_2 \) do not cross 0.1. Fluctuation of the populations for \( \sigma_1 = \sigma_2 = 0.08 \) can be found in Fig. 3 and Fig. 4.

Now the value of \( \sigma_1 \) and \( \sigma_2 \) are further increased. It is found that if the values of \( \sigma_1 \) and \( \sigma_2 \) are increased above 0.5 \( x_2 \) population biomass decreases very rapidly and ultimately goes into extinction. On the other hand \( x_1 \) population decays significantly but survives ultimately. Extinction of \( x_2 \) population for \( \sigma_1 = \sigma_2 = 0.5 \) is given in Fig. 5.

![Fig. 5: Extinction of \( x_2 \) for \( \gamma_1 = 0.0005, \gamma_2 = 0.008 \) and \( \sigma_1 = \sigma_2 = 0.5 \)](image)

8 Concluding remarks

In this paper we have analyzed a two species fisheries model in fluctuating natural environment. Existence and uniqueness of the positive global solution of the present stochastic model is established. We have also derived the conditions of stochastic persistence for population in the system and obtained the threshold values \( \sigma^*_1 \) and \( \sigma^*_2 \) of environmental driving forces. It can be concluded from the expressions of \( \sigma^*_1 \) and \( \sigma^*_2 \) that, intrinsic growth rate of prey population and nature of harvesting of predator population play a vital role regarding survival of the populations in fluctuating environmental condition. It is observed that, even in the presence of noise term, high growth rate of prey population can ensure long time survival of the population. Increase in toxic level may cause the extinction of the populations. It is further observed that ‘uncontrolled harvesting’ of the population could also lead to the extinction of predator population, whereas ‘controlled harvesting’ could be served as a deterrent for unwanted population explosion. We have also studied the effect of rapid increase of toxic level in ocean in case of some unexpected instances. Ecologically speaking it may represent the scenario when vast amount of toxic materials are accumulated in a small portion of ocean due to sudden leakage of crude petroleum or other petro oriented toxic products from cargos. These instances though rare, may be observed time to time in various parts of the ocean. It should be noted that no effective prohibitory measure can be taken in this regard to shield the populations from its adverse effect. In this investigation it is observed that if the intensities of the noise terms are moderate or small then system may remain stable in mean square for long time. On the other hand if the intensities are extremely high system may loose its stability and become unstable.

Understanding of the environmental factors that control variability in fish populations and the effect of multiple stresses on these stocks over their entire geographical range is crucial for improvement of our capacity to predict the effects of pollution on fisheries. Increased predictive capability can be achieved effectively through the closer integration of the both disciplines of population dynamics and toxicology. Normally chronic and sublethal changes take place very slowly and it may be safely concluded that most of the organisms are quite capable of adapt themselves in these changing environment. However this may not happen if the fluctuation is rapid and drastic due to extreme pollution in marine environment. Because of the inability to adapt themselves to these rapid changes, such changes might become catastrophic for many population. Coastal estuarine pollution can affect the fish in any stage of its life cycle. But it is during their preliminary stages of life and more specifically during their first few months of life, that fish can be particularly sensitive to toxic contaminants.

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Therefore more interdisciplinary research collaboration is necessary to protect the marine community from the hazardous effects of toxic pollutants.

References


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