ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 14 (2018) No. 3, pp. 163-173

Nehari-type of solutions for a class of N-Laplacian equations in $\mathbb{R}^N$

Jing Li[1*]

College of Science, Linyi University, Linyi 276005, P.R. China

(*Received September 28 2017, Accepted March 27 2018*)

Abstract. In this paper, we focus on the existence of positive solutions to a class of quasilinear Schrödinger equations with a parameter

$$-\Delta_N u + |u|^{N-2}u - \Delta_N(|u|^{2\alpha})|u|^{2\alpha-2}u = h(u), \quad x \in \mathbb{R}^N,$$

where $\alpha > \frac{1}{2}$ is a parameter and$\Delta_N u = div(|\nabla u|^{N-2}\nabla u)$ is the $N-$Laplacian operator. since the appearance of nonlinear operator $\Delta_p(|u|^{2\alpha})|u|^{2\alpha-2}u$, we must consider suitable space and so we need more delicate estimates. By the Nehari argument and Schwarz symmetrization method, the existence of solutions for Eq.(0.1) is established.

Keywords: Quasilinear Schrödinger equations, Nehari manifold method, Schwarz symmetrization.

1 Introduction and main result

In this work, we consider the existence of positive solutions to a class of quasilinear Schrödinger equations with a parameter

$$-\Delta_N u + |u|^{N-2}u - \Delta_N(|u|^{2\alpha})|u|^{2\alpha-2}u = h(u), \quad x \in \mathbb{R}^N, \tag{1}$$

where $\Delta_N u = div(|\nabla u|^{N-2}\nabla u)$ and $\alpha > \frac{1}{2}$ is a parameter.

For the case $N = 2, \alpha = 1$, solutions of Eq.(1.1) are standing waves solutions of the following Schrödinger equation

$$iz_t = -\Delta z + W(x)z - h_1(|z|^2)z - \Delta g(|z|^2)g'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{2}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C},\ W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ is a positive constant, $h_1, g : \mathbb{R}^+ \to \mathbb{R}$ are real functions.

It is well known that the standing wave solutions of the form $z(t, x) = \exp(-i\omega t)u(x)$ satisfies (1) with $g(s) = s$ if and only if the function $u(x)$ solves the equation of elliptic type

$$-\Delta u + V(x)u - \Delta(u^2)u = h(u), \qquad x \in \mathbb{R}^N, \tag{3}$$

where $\omega \in \mathbb{R}, V(x) = W(x) - \omega$ and $h(u) \equiv h_1(|u|^2)u$.

Quasilinear Schrödinger equations of form (2) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of nonlinear term g. The case $g(s) = s$ was used for the superfluid film equation in plasma physics by Kurihura in [14] (see also [15]). Eq.(2) also appears in plasma physics and fluid mechanics [3, 13, 20], in mechanics [11] and in condensed matter theory [17].

* Corresponding author. *E-mail address*: slyphna@163.com

Eq.(1) is transformed to a semilinear one in [7, 12, 16] by a change of variables(dual approach); Nehari method is used to get the existence results of ground state solutions in [9, 19, 21]. Especially, in [16, 19], the existence of the ground state solutions for the following problem with a parameter α

$$-\Delta u+V(x)u-\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u=\lambda|u|^{p-1}u,\ \ x\in\mathbb{R}^N \tag{4}$$

was studied with $\lambda>0$ and $2<p+1<\frac{4\alpha N}{N-2}$. In [2], the uniqueness of the ground state solutions for the equation

$$-\Delta u+\lambda u-\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u=|u|^{p-1}u,\ \ x\in\mathbb{R}^N \tag{5}$$

was established via a dual approach, where $\alpha>1$ and $p\in(1,\frac{(2\alpha-1)N+2}{N-2})$.

We also refer to the recent work of Wu in [24]. By a dual approach and some special techniques, the author studied the existence of solutions for the problem (4), which the power parameter p and the potential $V(x)$ satisfy the following conditions:

$(H_1)$the parameter $\alpha\in(\frac{1}{2},1]$, $4\alpha<p<\frac{4\alpha N}{N-2}$ if $N\geq 3$ and $4\alpha<p<\infty$ if $N=1,2$;

$(H_2)$the potential $V(x)\in C(\mathbb{R}^N)$ and $0<V_0:=\inf_{x\in\mathbb{R}^N}V(x)$, and for each $M>0$, meas$(\{x\in\mathbb{R}^N:V(x)\leq M\})<\infty$.

The assumption (H_2) is essential to guarantee the embedding is compact. To the best of our knowledge, the existence of solutions to Eq.(1.1) with $p=N$ has not ever been studied by variational methods. We mainly follow the idea of [6, 18]. It is noted that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all, since the appearance of nonlinear operator $\Delta_p(|u|^{2\alpha})|u|^{2\alpha-2}u$, we must consider our problem for suitable space and so we need more delicate estimates. Our argument is different from that in the above papers. We will use the Nehari manifold method and Schwarz symmetrization to establish the existence of solutions for Eq.(1).

Throughout this paper, we let $X=W^{1,N}(\mathbb{R}^N)$ and make the following assumptions for Eq.(1).

(A_1)Function $h(t)\in C^1(\mathbb{R})$ is odd and positive for $t>0$ and there exist $b_1,\alpha_0,q>0$ such that

$$|h(t)|\leq b_1|t|^{q-1}[\exp(\alpha_0|t|^{\frac{2\alpha N}{N-1}})-S_{N-2}(\alpha_0,t)],\qquad \forall t\in\mathbb{R}=(-\infty,\infty), \tag{6}$$

where

$$S_{N-2}(\alpha_0,t)=\sum_{k=0}^{N-2}\frac{\alpha_0^k}{k!}|t|^{\frac{2k\alpha N}{N-1}},\qquad \forall t\in\mathbb{R}. \tag{7}$$

(A_2)The function $h(t)$ also satisfies

$$(q-1+2\alpha N)\frac{h(t)}{t}\leq h'(t),\quad (q+2\alpha N)H(t)\leq th(t),\quad \forall t\in\mathbb{R}. \tag{8}$$

Here and in the sequel, $H(t)=\int_0^t h(s)ds, t\in\mathbb{R}$.

The main statement is as follows.

Theorem 1. *Assume $(A_1)-(A_2)$ hold. Then, Eq.(1) admits at least a positive solution in X .* *None*

This paper is organized as follows. In Section 2, we set up the variational framework and some useful Lemmas. The proof of Theorem 1.1 is given in Section 3.

2 Preliminaries

Let Ω be an open subset of $\mathbb{R}^N$. Let $L^p(\Omega)(p \geq 1)$ be the usual Lebesgue spaces with the norm $\|u\|_p \equiv \|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p dx)^{1/p}$ and $W^{1,p}(\Omega)(W_0^{1,p}(\Omega))$ be the usual Sobolev spaces with the norm

$$\|u\|_{W^{1,p}(\Omega)} = (\int_\Omega (|\nabla u|^p + |u|^p) dx)^{1/p}. \tag{9}$$

It is well known that $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$, see [5].

In this paper, we will make use of the following Lemmas.

Lemma 1. *[1, 5, 10]. Let Ω be a open subset of $\mathbb{R}^N$ with a Lipschitz continuous boundary. Then, the following properties hold:*

(I). Embedding $W^{1,N}(\Omega)(W_0^{1,N}(\Omega)) \hookrightarrow L^q(\Omega)$ is continuous for $q \geq N$ and there exists $C_0 > 0$ such that

$$\|u\|_q \leq C_0 \|u\|_{W^{1,N}(\Omega)}. \tag{10}$$

Furthermore, let Ω be bounded.

(II). Embedding $W^{1,N}(\Omega)(W_0^{1,N}(\Omega)) \hookrightarrow L^q(\Omega)$ is compact for $q \geq N$.

In particular, for $u \in W_0^{1,N}(\Omega)$,

$$\|u\|_N \leq (\omega_N^{-1}|\Omega|)^{1/N} \|\nabla u\|_N, \tag{11}$$

where $|\Omega|$ is the $N-$dimensional volume of Ω and ω_N is the volume of unit sphere $B_1 \subset \mathbb{R}^N$, that is

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1 + N/2)}. \tag{12}$$

None

Remark 1. Obviously, $N\omega_N$ is the surface area of the unit sphere ∂B_1 in $\mathbb{R}^N$. None

Lemma 2. *[8] Let $u \in W_0^{1,N}(\Omega) \cap L^r(\Omega)$, where $r \geq 1$ and $\Omega \subseteq \mathbb{R}^N$ is an arbitrary domain. Then for $q \geq r$,*

$$\|u\|_q \leq c(N, r) q^{1-1/N} \|\nabla u\|_N^{1-r/q} \|u\|_r^{r/q}. \tag{13}$$

The exponent $1 - 1/N$ of q is the best possible. In particular,

$$c(N, N) = \frac{1}{\sqrt{\pi}} \left(\frac{\Gamma(N/2)\Gamma(2N)}{2\Gamma(N)^2} \right)^{1/N} \equiv d_N. \tag{14}$$

None

Remark 2. By Lemma 2.3, the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for every $q \geq N$ and

$$\|u\|_q \leq d_N q^{1-1/N} \|u\|_X. \tag{15}$$

None

We observe that formally Eq.(1) is the Euler-Lagrange equation associated of the natural energy functional $I : W^{1,N}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{N} \int_{\mathbb{R}^N} (1 + (2\alpha)^{N-1} |u|^{(2\alpha-1)N}) |\nabla u|^N dx + \frac{1}{N} \int_{\mathbb{R}^N} |u|^N dx - \int_{\mathbb{R}^N} H(u) dx. \tag{16}$$

It should be pointed out that the functional I is not well defined in general in X. To overcome this difficulty, we employ an argument developed by Colin, Jeanjean in [7] (see also [23]). We make the changing of variables $u = f(v)$ or $v = f^{-1}(u)$, where f is defined by

$$f'(t) = (1 + (2\alpha)^{N-1} |f(t)|^{(2\alpha-1)N})^{-1/N}, \quad t \geq 0, \quad f(0) = 0 \tag{17}$$

and by $f(t) = -f(-t)$ on $(-\infty, 0]$.

Lemma 3. *[7, 12, 23] The function $f(t)$ satisfies*

(f_1) *f is uniquely defined, C^2 and invertible in $\mathbb{R}$,*

(f_2) $0 < f'(t) \leq 1,\ \forall t \in \mathbb{R}$,

(f_3) $|f(t)| \leq |t|,\ \forall t \in \mathbb{R}$,

(f_4) $\frac{f(t)}{t} \to 1$ *as* $t \to 0$,

(f_5) $|f(t)| \leq (2\alpha)^{1/2\alpha N}|t|^{1/2\alpha},\ \forall t \in \mathbb{R}$,

(f_6) $\frac{1}{2}f(t) \leq \alpha t f'(t) \leq \alpha f(t),\ \forall t \in \mathbb{R}^+ = [0,\infty)$ *and* $\alpha f(t) \leq \alpha t f'(t) \leq \frac{1}{2}f(t),\ \forall t \in \mathbb{R}^- = (-\infty, 0]$,

(f_7) *there exists* $a \in (0,(2\alpha)^{1/2\alpha N}]$ *such that* $\frac{f(t)}{t^{1/2\alpha}} \to a$ *as* $t \to +\infty$,

(f_8) *there exists* $b_0 > 0$ *such that*

$$|f(t)| \geq \begin{cases} b_0|t| & \text{if } |t| \leq 1, \\ b_0|t|^{1/2\alpha} & \text{if } |t| \geq 1. \end{cases} \tag{18}$$

None

So, after the change of variables, we can write $I(u)$ as

$$J(v) \equiv I(f(v)) = \frac{1}{N}\int_{\mathbb{R}^N}|\nabla v|^N dx + \frac{1}{N}\int_{\mathbb{R}^N}|f(v)|^N dx - \int_{\mathbb{R}^N}H(f(v))dx, \tag{19}$$

which is well defined on X under the assumptions $(A_1) - (A_2)$.

As in [23], we observe that if $v \in W^{1,N}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)(p > 1)$ is a critical point of the functional J, that is, $J'(v)\varphi = 0$ for all $\varphi \in W^{1,p}(\mathbb{R}^N)$, where

$$J'(v)\varphi = \int_{\mathbb{R}^N}|\nabla v|^{N-2}\nabla v \nabla\varphi dx + \int_{\mathbb{R}^N}|f(v)|^{N-2}f(v)f'(v)\varphi dx - \int_{\mathbb{R}^N}h(f(v))f'(v)\varphi dx, \tag{20}$$

then v is a solution of the equation

$$-\Delta_N v = g(v), \qquad x \in \mathbb{R}^N, \tag{21}$$

where

$$g(s) = -|f(s)|^{N-2}f(s)f'(s) + h(f(s))f'(s), \qquad s \in \mathbb{R}, \tag{22}$$

and then $u = f(v)$ is a weak solution of (1)is a weak solution of). By using theorem 1 in [22], we can conclude that v is locally bounded in $\mathbb{R}^N$. So, we consider the existence of solutions to Eq.(22) in Y .

We first construct the subspaces $X_r(\subset X)$. The function $u \in L^N(\mathbb{R}^N)$ is called radially nonincreasing if $u(x) \leq u(y)$ when $|x| \geq |y|$. Denote

$$X_r = \left\{u \in X :\ u \text{ is nonnegative and radially nonincreasing in } \mathbb{R}^N\right\}. \tag{23}$$

Lemma 4. *[4] If $u \in L^N(\mathbb{R}^N)$ is a nonnegative and radially nonincreasing function, then one has*

$$|u(x)| \leq |x|^{-1}\omega_N^{-1/N}\|u\|_N, \quad \forall x \in \mathbb{R}^N \setminus \{0\}. \tag{24}$$

None

Let $u \in L^N(\mathbb{R}^N)$ be such that $u(x) \geq 0$ a.e. in $\mathbb{R}^N$. For $t > 0$, set

$$\Omega(t) = \{x \in \mathbb{R}^N \mid u(x) > t\} \text{ and } \mu(t) = \text{meas}(\Omega(t)). \tag{25}$$

Here and in the sequel, meas(Ω) is Lebesgue measure of Ω. Since $u \in L^N(\mathbb{R}^N)$, we have $\mu(t) < +\infty$ for all $t > 0$. The Schwarz symmetrization constructs a radial function $u^* : \mathbb{R}^N \to \mathbb{R}$ such that

$$\{x \in \mathbb{R}^N | u^*(x) > t\} = B_{\rho(t)} \text{ with } \text{meas}(B_{\rho(t)}) = \mu(t), \tag{26}$$

where $B_{\rho(t)}$ is the sphere with the radius $\rho(t) > 0$ and the center at the origin. Thus, the sets where u and u^* are greater than t have the same Lebesgue measure. Obviously, the function u^* is radially nonincreasing. The most important properties of u^* are stated in the following result.

Lemma 5. *[18].*

(i). If $f:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and increasing function with $f(0)=0$, then

$$\int_{\mathbb{R}^N} f(u^*)dx=\int_{\mathbb{R}^N} f(u)dx.$$

(ii). Let $p\geq 1$. If $u\in W^{1,p}(\mathbb{R}^N)$, $u\geq 0$, then $u^*\in W^{1,p}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N}|\nabla u^*|^p dx\leq \int_{\mathbb{R}^N}|\nabla u|^p dx. \qquad \text{None}$$

Remark 3. Lemma 2.8 shows that if $u\in X, u\geq 0$, then $u^*\in X_r$. None

To prove the existence of nontrivial solutions for Eq. (22) in Y, we introduce the Nehari manifolds

$$\mathcal{M}=\{v\in X\setminus\{0\}:J'(v)v=0\}=\left\{v\in X\setminus\{0\}:\|\nabla v\|_N^N+\int_{\mathbb{R}^N}|f(v)|^{N-2}f(v)f'(v)vdx=\int_{\mathbb{R}^N}h(f(v))f'(v)vdx\right\}$$

and the fibering maps $\phi_v(t)=J(tv)$ for $t>0$. Clearly, we have that $v\in\mathcal{M}$ if and only if $\phi_v'(1)=0$ and, more generally, $tv\in\mathcal{M}$ if and only if $\phi_v'(t)=0$. By definition, one has

$$\begin{aligned}\phi_v(t)&=J(tv)=\frac{1}{N}\|t\nabla v\|_N^N+\frac{1}{N}\int_{\mathbb{R}^N}|f(tv)|^N dx-\int_{\mathbb{R}^N}H(f(tv))dx,\\ \phi_v'(t)&=t^{N-1}\|\nabla v\|_N^N+\int_{\mathbb{R}^N}|f(tv)|^{N-2}f(tv)f'(tv)vdx-\int_{\mathbb{R}^N}h(f(tv))f'(tv)vdx.\end{aligned}\tag{27}$$

Notice that, if $v\in\mathcal{M}$, then

$$J(v)=\frac{1}{N}\int_{\mathbb{R}^N}(h(f(v))f'(v)v-NH(f(v)))dx+\frac{1}{N}\int_{\mathbb{R}^N}|f(v)|^{N-2}(f^2(v)-f(v)f'(v)v)dx.\tag{28}$$

In the following, under the assumptions $(A_1)-(A_2)$, we derive some properties for $\mathcal{M}$.

Lemma 6. *The Nehari manifolds $\mathcal{M}$ are not empty sets.* *None*

Proof. Choose a nonnegative function $v_0\in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} h(f(v_0))f'(v_0)v_0dx>0\tag{29}$$

and $\|v_0\|_E\leq\rho$ for small $\rho>0$. For $t\geq 0$, set

$$\gamma(t)=J'(tv_0)tv_0=t^N\|\nabla v_0\|_N^N+\int_{\mathbb{R}^N}(f(tv_0))^{N-1}f'(tv_0)tv_0dx-\int_{\mathbb{R}^N}h(f(tv_0))f'(tv_0)tv_0dx.\tag{30}$$

We assume (A_1), it follows from (8), (16) and Lemma 2.5 that

$$\begin{aligned}0&\leq\int_{\mathbb{R}^N}h(f(tv_0))f'(tv_0)tv_0dx\leq\int_{\mathbb{R}^N}h(f(tv_0))f(tv_0)dx\\ &\leq b_1\int_{\mathbb{R}^N}|f(tv_0)|^q[\exp(\alpha_0|f(tv_0)|^{\frac{2\alpha N}{N-1}})-S_{N-2}(\alpha_0,f(tv_0))]dx\\ &\leq\sum_{k=N-1}^{\infty}\frac{b_1\alpha_0^k}{k!}\int_{\mathbb{R}^N}|f(tv_0)|^{q+\frac{2\alpha Nk}{N-1}}dx\leq\sum_{k=N-1}^{\infty}\frac{b_2\alpha_0^k}{k!}(2\alpha)^{\frac{k}{N-1}}t^{\frac{q}{2\alpha}+\frac{kN}{N-1}}\int_{\mathbb{R}^N}|v_0|^{q_k+q_0}dx\\ &\leq b_2t^{N+q_0}\sum_{k=N-1}^{\infty}\frac{\alpha_0^k}{k!}(2\alpha)^{\frac{k}{N-1}}d_N^{q_k+q_0}(q_k+q_0)^{(1-\frac{1}{N})(q_k+q_0)}\|v_0\|_X^{q_k+q_0}\\ &\leq b_2t^{q_0+N}d_N^{q_0}\|v_0\|_X^{q_0}\sum_{k=N-1}^{\infty}a_k,\end{aligned}\tag{31}$$

where d_N is given in (15) and

$$b_2=b_1(2\alpha)^{\frac{q}{2\alpha N}}, q_0=\frac{q}{2\alpha}, \beta=\frac{N+q_0}{N-1}, q_k=\frac{kN}{N-1}, a_k=\frac{((2\alpha)^{\frac{1}{N-1}}\alpha_0)^k}{k!}d_N^{q_k}\|v_0\|_X^{q_k}(\beta k)^{k+q_0(1-\frac{1}{N})}. \quad (32)$$

Since $\rho>0$ is so small that

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=e\alpha_0\beta(2\alpha)^{\frac{1}{N-1}}\|v_0\|_X^{\frac{N}{N-1}}d_N^{\frac{N}{N-1}}\leq \alpha_0\beta(2\alpha)^{\frac{1}{N-1}}\rho^{\frac{N}{N-1}}d_N^{\frac{N}{N-1}}<1, \quad (33)$$

the positive series $\sum_{k=N-1}^{\infty}a_k$ is convergent. Therefore, there is $C_1>0$ such that

$$0\leq\int_{\mathbb{R}^N}h(f(tv_0))f'(tv_0)tv_0dx\leq C_1t^{q_0+N}\|v_0\|_X^{q_0}. \quad (34)$$

Then it follows from (38) -(39) that $\gamma(t)>0$ for small $t>0$.

On the other hand, since $q>0$, it is possible to choose $s>N$ such that $q>2\alpha(s-N)$. Set

$$G_2(t)=t^{-s+1}h(f(tv_0))f'(tv_0)v_0-h(f(v_0))f'(v_0)v_0, \quad t\geq 1. \quad (35)$$

We claim that $G_2'(t)\geq 0$ for $t\geq 1$. In fact, it is from (18) and Lemma 2.5 that

$$\begin{aligned}
G_2'(t)&=t^{-s-1}\left[h'(f(tv_0))(f'(tv_0))^2t^2v_0^2-(s-1)h(f(tv_0))f'(tv_0)tv_0+h(f(tv_0))f''(tv_0)t^2v_0^2\right]\\
&\geq t^{-s-1}\frac{h(f(tv_0))}{f(tv_0)}\left[(q-1+2\alpha N)(f'(tv_0))^2t^2v_0^2+f''(tv_0)f(tv_0)t^2v_0^2-(s-1)f'(tv_0)f(tv_0)tv_0\right]\\
&\geq t^{-s-1}\frac{h(f(tv_0))}{f(tv_0)}\left[(q-1+2\alpha N)(f'(tv_0))^2t^2v_0^2+f''(tv_0)f(tv_0)t^2v_0^2-2\alpha(s-1)(f'(tv_0))^2t^2v_0^2\right]\\
&=\frac{t^{-s+1}v_0^2h(f(tv_0))f^{-1}(tv_0)}{(1+(2\alpha)^{N-1}|f(tv_0)|^{(2\alpha-1)N})^{2/N}}\left[q-1-2\alpha(s-N-1)-\frac{(2\alpha-1)(2\alpha)^{N-1}|f(tv_0)|^{(2\alpha-1)N}}{1+(2\alpha)^{N-1}|f(tv_0)|^{(2\alpha-1)N}}\right]\geq 0
\end{aligned}$$

provided that $q>2\alpha(s-N)$. Then,

$$h(f(tv_0))f'(tv_0)tv_0\geq t^sh(f(v_0))f'(v_0)v_0, \quad t\geq 1. \quad (36)$$

Moreover, the application of (42) yields

$$\gamma(t)\leq t^N(\|\nabla v_0\|_N^N+\|v_0\|_N^N)-t^s\int_{\mathbb{R}^N}h(f(v_0))f'(v_0)v_0dx\to-\infty \text{ as } t\to+\infty. \quad (37)$$

Then there exists $t_2>1$ such that $\gamma(t_2)=0$ and $t_2v_0\in\mathcal{M}$. Obviously, $t_2v_0\not\equiv 0$ in $\mathbb{R}^N$. We finish the proof of Lemma 2.10. None

Lemma 7. *Assume* $(A_1)-(A_2)$. *Then,*

$$d_1=\inf_{v\in\mathcal{M}}\{\|v\|_X\}>0, \quad d_2=\inf_{v\in\mathcal{M}}\{J(v)\}\geq 0. \quad \text{None}$$

Proof. Let $v\in\mathcal{M}$. Then,

$$\|\nabla v\|_N^N+\int_{\mathbb{R}^N}|f(v)|^{N-2}f(v)f'(v)vdx=\int_{\mathbb{R}^N}h(f(v))f'(v)vdx. \quad (38)$$

By (18) and (36), we derive that

$$J(v) = \frac{1}{N}\int_{\mathbb{R}^N}(h(f(v))f'(v)v - NH(f(v)))dx + \frac{1}{N}\int_{\mathbb{R}^N}|f(v)|^{N-2}(f^2(v) - f(v)f'(v)v)dx \\ \geq \frac{1}{N}\int_{\mathbb{R}^N} h(f(v))f(v)\left(\frac{f'(v)v}{f(v)} - \frac{N}{q+2\alpha N}\right)dx \geq \frac{q}{2\alpha N(q+2\alpha N)}\int_{\mathbb{R}^N} h(f(v))f(v)dx > 0. \tag{39}$$

let us show that $d_1 > 0$.

Assume, by contradiction, that there is $\{v_n\} \subset \mathcal{M}$ such that $0 < \|v_n\|_X \leq \epsilon_n \to 0$ as $n \to \infty$. As in (), we obtain

$$\|\nabla v_n\|_N^N + \int_{\mathbb{R}^N}|f(v_n)|^{N-2}f(v_n)f'(v_n)v_n dx \leq \int_{\mathbb{R}^N} h(f(v_n))f(v_n)dx \\ \leq \sum_{k=N-1}^{\infty}\frac{b_2\alpha_0^k}{k!}(2\alpha)^{\frac{k}{N-1}}\|v_n\|_{q_0+q_k}^{q_0+q_k} \leq \sum_{k=N-1}^{\infty}\frac{b_2\alpha_0^k}{k!}(2\alpha)^{\frac{k}{N-1}}d_N^{q_k+q_0}(q_k+q_0)^{(1-\frac{1}{N})(q_k+q_0)}\|v_n\|_X^{q_k+q_0} \\ \leq b_2\|v_n\|_X^{q_0+N}d_N^{q_0}\beta^{\frac{q_0(N-1)}{N}}\sum_{k=N-1}^{\infty}b_k, \tag{40}$$

where d_N is given in (15), b_2, q_0, β, q_k are in (44) and

$$b_k = \frac{\alpha_0^k}{k!}\beta^k(2\alpha)^{\frac{k}{N-1}}d_N^{q_k}\|v_n\|_X^{\frac{N(k-N+1)}{N-1}}k^{k+\frac{q_0(N-1)}{N}}, \qquad k \geq N-1. \tag{41}$$

Since

$$\lim_{k\to\infty}\frac{b_{k+1}}{b_k} = e\alpha_0\beta(2\alpha)^{\frac{1}{N-1}}d_N^{\frac{N}{N-1}}\|v_n\|_X^{\frac{N}{N-1}} \leq e\alpha_0\beta(2\alpha)^{\frac{1}{N-1}}d_N^{\frac{N}{N-1}}\epsilon_n^{\frac{N}{N-1}} < 1,$$

the positive series $\sum_{k=N-1}^{\infty} b_k$ is convergent. Denote $B_0 = \sum_{k=N-1}^{\infty} b_k$.

On the other hand, we have from (f_6) and (f_8) that

$$\|\nabla v_n\|_N^N + \int_{\mathbb{R}^N}|f(v_n)|^{N-2}f(v_n)f'(v_n)v_n dx \geq \|\nabla v_n\|_N^N + \frac{1}{2\alpha}\int_{\mathbb{R}^N}|f(v_n)|^N dx \\ \geq \|\nabla v_n\|_N^N + \frac{b_0^N}{2\alpha}\|v_n\|_N^N \geq C_3\|v_n\|_Y^N \tag{42}$$

with $C_3 = \min\{1, b_0^N/2\alpha\} > 0$. Then

$$0 < C_3 \leq b_2\|v_n\|_Y^{q_0}d_N^{q_0}\beta^{\frac{q_0(N-1)}{N}}B_0 \leq b_2\epsilon_n^{q_0}d_N^{q_0}\beta^{\frac{q_0(N-1)}{N}}B_0 \tag{43}$$

This is impossible if ϵ_n is small enough. Therefore, $d_1 > 0$ and the proof is finished. None

Lemma 8. *There exists a nonnegative function $v_0 \in \mathcal{M}$ such that $J(v_0) = \inf_{v\in\mathcal{M}}\{J(v)\} = d_2 > 0$.* *None*

Proof. we assume $(A_1) - (A_2)$. As the argument of the case $p \in (1, N)$, we let $z_n \in \mathcal{M}$ such that $J(z_n) \to d_2, J'(z_n) \to 0$ in Y^*. Similarly, let $z_n^* \in Y_r$ be the Schwarz symmetrization of z_n and $t_n \in (0, 1]$ such that $\gamma_n(t_n) = J'(t_n z_n^*)t_n z_n^* = 0$ for $n \geq 1$. Notice that the functions $G_3(t) = h(f(t))f'(t)t - NH(f(t))$ and $G_4(t) = f^N(t) - f^{N-1}(t)f'(t)t$ are increasing in $\mathbb{R}^+$. Then, as $n \to \infty$, we obtain from (36) that

$$d_2 \leq J(t_n z_n^*) = \frac{1}{N}\int_{\mathbb{R}^N}(G_3(t_n z_n^*) + G_4(t_n z_n^*))dx \leq \frac{1}{N}\int_{\mathbb{R}^N}(G_3(z_n^*) + G_4(z_n^*))dx \\ = \frac{1}{N}\int_{\mathbb{R}^N}(G_3(z_n) + G_4(z_n))dx = J(z_n) \to d_2. \tag{44}$$

This implies that $\{t_n z_n^*\}$ is also a minimizing sequence for d_4 and $t_n z_n^* \in \mathcal{M} \cap Y_r$. Let $v_n = t_n z_n^* \geq 0$. We can assume that, up to a subsequence, $v_n \rightharpoonup v_0$ in X. By Lemma 2.7, we have $v_n \to v_0$ in $L^s(\mathbb{R}^N), s > N$, and, again up to a subsequence, $v_n(x) \to v_0(x) \geq 0$ a.e. in $\mathbb{R}^N$ and $v_0 \in X_r$. We now prove that $v_0 \in \mathcal{M}$ and $J(v_0) = d_2$. We claim that $v_0 \neq 0$ in Y. Otherwise, $v_n \to 0$ in X as $n \to \infty$. By (2.42), it is impossible. So, we have $\|v_0\|_X > 0$.

Since $v_n \in \mathcal{M}$, one has

$$\|\nabla v_n\|_N^N + \int_{\mathbb{R}^N} |f(v_n)|^{N-2} f(v_n) f'(v_n) v_n dx = \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n dx. \tag{45}$$

and

$$\|\nabla v_0\|_N^N + \int_{\mathbb{R}^N} |f(v_0)|^{N-2} f(v_0) f'(v_0) v_0 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n dx. \tag{46}$$

In what follows we make use of the following results (to be proved later)

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n dx &= \int_{\mathbb{R}^N} h(f(v_0)) f'(v_0) v_0 dx, \\ \lim_{n\to\infty} \int_{\mathbb{R}^N} H(f(v_n)) dx &= \int_{\mathbb{R}^N} H(f(v_0)) dx. \end{aligned} \tag{47}$$

If

$$\|\nabla v_0\|_N^N + \int_{\mathbb{R}^N} |f(v_0)|^{N-2} f(v_0) f'(v_0) v_0 dx = \int_{\mathbb{R}^N} h(f(v_0)) f'(v_0) v_0 dx, \tag{48}$$

then $v_0 \in \mathcal{M}$. So, arguing by contradiction, we assume that

$$\|\nabla v_0\|_N^N + \int_{\mathbb{R}^N} |f(v_0)|^{N-2} f(v_0) f'(v_0) v_0 dx < \int_{\mathbb{R}^N} h(f(v_0)) f'(v_0) v_0 dx. \tag{49}$$

Let $\gamma(t) = J'(tv_0)tv_0$. Clearly, $\gamma(t) > 0$ for small $t > 0$ and $\gamma(1) < 0$. So that there exists $t \in (0, 1)$ such that $\gamma(t) = 0$ and $tv_0 \in \mathcal{M}$. Again, since the functions $G_3(t)$ and $G_4(t)$ are increasing on $\mathbb{R}^+$, we have from (36) that

$$\begin{aligned} d_2 \leq J(tv_0) &= \frac{1}{N} \int_{\mathbb{R}^N} (G_3(tv_0) + G_4(tv_0)) dx < \frac{1}{N} \int_{\mathbb{R}^N} (G_3(v_0) + G_4(v_0)) dx \\ &\leq \frac{1}{N} \liminf_{n\to\infty} \int_{\mathbb{R}^N} (G_3(v_n) + G_4(v_n)) dx = \liminf_{n\to\infty} J(v_n) = d_2. \end{aligned}$$

This contradiction proves $v_0 \in \mathcal{M}$. Again, the application of the weakly lower semicontinuity of norms, we get $J(v_0) \leq \liminf_{n\to\infty} J(v_n) = d_2$. On the other hand, for every $v \in \mathcal{N}$, $J(v) \geq d_2$. So, $J(v_0) = d_2$. By (2.47), we have $d_2 > 0$.

Now, we prove (53). Since $\{v_n\}$ is bounded in X, we assume $\|v_n\|_X \leq M (n \geq 1)$ for some $M > 0$. Then, for any $r > 0$, it follows from Lemma 2.1 that $v_n \to v_0$ in $L^s(B_r)(s \geq N)$ and $v_n \to v_0$ a.e. in B_r. So,

$$\begin{aligned} \lim_{n\to\infty} \int_{B_r} h(f(v_n)) f'(v_n) v_n dx &= \int_{B_r} h(f(v_0)) f'(v_0) v_0 dx, \\ \lim_{n\to\infty} \int_{B_r} H(f(v_n)) dx &= \int_{B_r} H(f(v_0)) dx. \end{aligned} \tag{50}$$

In the follows we prove that, for any small $\epsilon > 0$, there is $r_0 > 1$ such that $r \geq r_0$ and

$$\begin{aligned} &\int_{B_r^c} |h(f(v_n)) f'(v_n) v_n| dx < \epsilon, \quad \int_{B_r^c} |H(f(v_n))| dx < \epsilon \quad \forall n \geq 1 \\ &\int_{B_r^c} |h(f(v_0)) f'(v_0) v_0| dx < \epsilon, \quad \int_{B_r^c} |H(f(v_0))| dx < \epsilon. \end{aligned} \tag{51}$$

As in (43), we obtain

$$\int_{B_r^c}|h(f(v_n))f'(v_n)v_n|dx \le \int_{B_r^c}h(f(v_n))f(v_n)dx \le \sum_{k=N-1}^{\infty}\frac{b_2\alpha_0^k}{k!}(2\alpha)^{\frac{k}{N-1}}\int_{B_r^c}|v_n(x)|^{q_0+q_k}dx \tag{52}$$

where b_2, q_0, q_k are given in (47). By (26), we have

$$|v_n(x)|\le |x|^{-1}\omega_N^{-\frac{1}{N}}\|v_n\|_N\le M\omega_N^{-\frac{1}{N}}|x|^{-1}, \;\; x\in\mathbb{R}^N\setminus\{0\} \tag{53}$$

and

$$\int_{B_r^c}|v_n(x)|^{q_k+q_0}dx \le M^{q_k+q_0}\omega_N^{-\frac{q_k+q_0}{N}}\int_r^{\infty}\int_{|\omega|=1}\rho^{N-1-q_k-q_0}d\omega d\rho$$
$$= M^{q_k+q_0}N\omega_N^{1-\frac{q_k+q_0}{N}}\frac{r^{N-q_k-q_0}}{q_k+q_0-N} \le \frac{r^{-q_0}}{q_0}M^{q_0}N\omega_N^{\frac{N-q_0}{N}}(M^{\frac{N}{N-1}}\omega_N^{-\frac{1}{N-1}})^k \equiv A^k\frac{r^{-q_0}}{q_0}M^{q_0}N\omega_N^{\frac{N-q_0}{N}}$$

and then

$$\int_{B_r^c}|h(f(u_n))f'(v_n)v_n|dx \le \frac{r^{-q_0}}{q_0}b_2NM^{q_0}\omega_N^{\frac{N-q_0}{N}}\sum_{k=N-1}^{\infty}\frac{((2\alpha)^{1/(N-1)}A\alpha_0)^k}{k!} \to 0 \;\text{ as }\; r\to\infty.$$

This shows that for any $\epsilon > 0$, there is $r_0 > 1$ such that $r > r_0$ and

$$\int_{B_r^c}|h(f(v_n))f'(v_n)v_n|dx < \epsilon, \;\; \forall n\ge 1. \tag{54}$$

The other three inequalities can be proved similarly. Now, the application of (51) and (54) yields

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}h(f(v_n))f'(v_n)v_ndx = \int_{\mathbb{R}^N}h(f(v_0))f'(v_0)v_0dx. \tag{55}$$

Similarly, we can prove

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}H(f(v_n))dx = \int_{\mathbb{R}^N}H(f(v_0))dx \tag{56}$$

Then we finish the proof of Lemma 2.12. None

3 Proof of theorem 1.1

We now can prove the main results in this paper by dint of Lemmas in Section 2.

Proof of Theorem 1.1 First, we assume (A_1). Clearly, it is sufficient to prove that v_0 is a critical point for J in X, that is , $J'(v_0)v = 0$ for all $v\in X$ and thus $J'(v_0) = 0$ in X^*, where v_0 is in the position of Lemma 2.12. For every $v\in X$, we choose $\varepsilon > 0$ such that $v_0 + sv \ne 0$ for all $s\in(-\varepsilon,\varepsilon)$. Define a function $\varphi : (-\varepsilon,\varepsilon)\times(0,\infty)\to\mathbb{R}$ by

$$\varphi(s,t) = J'(t(v_0+sv))t(v_0+sv) = \int_{R^N}|f(t(v_0+sv))|^{N-2}f(t(v_0+sv))f'(t(v_0+sv))t(v_0+sv)dx$$
$$+\|\nabla t(v_0+sv)\|_N^N - \int_{\mathbb{R}^N}h(f(t(v_0+sv)))f'(t(v_0+sv))t(v_0+sv)dx.$$

Then, $\varphi(0,1) = J'(v_0)v_0 = 0$ and

$$\frac{\partial\varphi}{\partial t}(0,1) = (N-1)\int_{\mathbb{R}^N}|f(v_0)|^{N-2}((f'(v_0))^2v_0^2 - f(v_0)f'(v_0)v_0)dx + \int_{\mathbb{R}^N}|f(v_0)|^{N-1}f''(v_0)v_0^2dx + \int_{\mathbb{R}^N}G(v_0)dx$$

with $(f'(v_0))^2v_0^2 - f(v_0)f'(v_0)v_0 \le 0, f''(v_0) \le 0$ in $\mathbb{R}^N$ and

$$\begin{aligned} G(v_0) &= (N-1)h(f(v_0))f'(v_0)v_0 - h'(f(v_0))(f'(v_0))^2v_0^2 - h(f(v_0))f''(v_0)v_0^2 \\ &\le (N-1)h(f(v_0))f'(v_0)v_0 - (q-1+2\alpha N)\frac{h(f(v_0))}{f(v_0)}(f'(v_0))^2v_0^2 - h(f(v_0))f''(v_0)v_0^2 \\ &= \frac{h(f(v_0))}{f(v_0)}\left[(N-1)f(v_0)f'(v_0)v_0 - (q-1+2\alpha N)(f'(v_0))^2v_0^2 - f(v_0)f''(v_0)v_0^2\right] \\ &\le \frac{h(f(v_0))}{f(v_0)}\left[(2\alpha(N-1)-(q-1+2\alpha N))(f'(v_0))^2v_0^2 - f(v_0)f''(v_0)v_0^2\right] \\ &= \frac{h(f(v_0))v_0^2}{f(v_0)\left[1+(2\alpha)^{N-1}f^{(2\alpha-1)N}(v_0)\right]^{2/N}}\left[1-q-2\alpha+\frac{(2\alpha-1)(2\alpha)^{N-1}f^{(2\alpha-1)N}(v_0)}{1+(2\alpha)^{N-1}f^{(2\alpha-1)N}(v_0)}\right] \le 0 \end{aligned}$$

provided that $q > 0$. Therefore, $\frac{\partial\varphi}{\partial t}(0,1) < 0$.

So, by the implicit function theorem, there exists a C^1 function $t : (-\varepsilon_0, \varepsilon_0)(\subseteq (-\varepsilon, \varepsilon)) \to \mathbb{R}$ such that $t(0) = 1$ and $\varphi(s, t(s)) = 0$ for all $s \in (-\varepsilon_0, \varepsilon_0)$. This also shows that $t(s) \ne 0$, at least for ε_0 very small. Therefore, $t(s)(v_0 + sv) \in \mathcal{M}$. Denote $t = t(s)$ and

$$\phi(s) = J(t(v_0+sv)) = \frac{1}{N}\|\nabla t(v_0+sv)\|_N^N + \frac{1}{N}\int_{\mathbb{R}^N}|f(t(v_0+sv))|^N dx - \int_{\mathbb{R}^N}H(f(t(v_0+sv)))dx. \quad (57)$$

We see that the function $\phi(s)$ is differentiable and has a minimum point at $s = 0$. Thus,

$$0 = \phi'(0) = t'(0)J'(v_0)v_0 + J'(v_0)v. \quad (58)$$

Since $v_0 \in \mathcal{M}$ and $J'(v_0)v_0 = 0$, it follows from (3.2) that $J'(v_0)v = 0$ for every $v \in Y$ and thus $J'(v_0) = 0$ in Y^*. So, v_0 is a critical point for J and then v_0 is a weak solution of (2.14) in Y, and so $u_0 = f(v_0)$ is a weak solution of (1.1). Since $J(v_0) = J(|v_0|) = d_2 > 0$, we can assume $v_0 \ge 0$ a.e. in $\mathbb{R}^N$. Furthermore, as consequence of Harnack's inequality, we have $v_0(x) > 0$ in $\mathbb{R}^N$. Thus $u_0(x) = f(v_0(x)) > 0$ in $\mathbb{R}^N$. Then the proof of Theorem 1.1 is completed.

References

[1] S. Adachi, K. Tanaka. Trudinger type inequalities in rnand their best exponents. *Proceedings of the American Mathematical Society*, 2000, **128**(7): 2051–2057.

[2] S. Adachi, T. Watanabe. Uniqueness of the ground state solutions of quasilinear schrödinger equations. *Nonlinear Analysis Theory Methods & Applications*, 2012, **75**(2): 819–833.

[3] F. Bass, N. Nasonov. Nonlinear electromagnetic-spin waves. *Physics Reports*, 1990, **189**(4): 165–223.

[4] H. Berestycki, P. L. Lions. Nonlinear scalar field equations, i existence of a ground state. *Archive for Rational Mechanics & Analysis*, 1983, **82**(4): 313–345.

[5] T. Cazenave. Semilinear schrödinger equations. **in:** *Courant Lecture Notes in Mathematics, 10, NYU, CIMS, AMS*, 2003, 635C637.

[6] C. Chen, Q. Zhu. Existence of positive solutions to -kirchhoff-type problem without compactness conditions. *Applied Mathematics Letters*, 2014, **28**(1): 82–87.

[7] M. Colin, L. Jeanjean. Solutions for a quasilinear schrödinger equation: a dual approach. *Nonlinear Analysis Theory Methods & Applications*, 2004, **56**(2): 213–226.

[8] D. E. Edmunds, A. A. Ilyin. Asymptotically sharp multiplicative inequlities. *Bulletin of the London Mathematical Society*, 1995, **27**(1): 71–74.

[9] X. D. Fang, A. Szulkin. Multiple solutions for a quasilinear schrÖdinger equation. *Journal of Differential Equations*, 2013, **254**(4): 2015–2032.

[10] D. Gilbarg, N. S. Trudinger. *Trudinger N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin*. 2001.

[11] R. W. Hasse. A general method for the solution of nonlinear soliton and kink schrÖdinger equations. *Zeitschrift Fr Physik B Condensed Matter*, 1980, **37**(1): 83–87.

[12] M. d. O. João, U. Severo. Solitary waves for a class of quasilinear schrödinger equations in dimension two. *Calculus of Variations & Partial Differential Equations*, 2010, **38**(3-4): 275–315.

[13] A. M. Kosevich, B. Ivanov, A. S. Kovalev. Magnetic solitons. *Physics Reports*, 1990, **194**(3): 117–238.

[14] S. Kurihara. Large-amplitude quasi-solitons in superfluid films. *Journal of the Physical Society of Japan*, 1981, **50**(50): 3262–3267.

[15] E. W. Laedke, K. H. Spatschek, L. Stenflo. Evolution theorem for a class of perturbed envelope soliton solutions. *Journal of Mathematical Physics*, 1983, **24**(12): 2764–2769.

[16] Liu, Jia-quan, Wang, Ya-qi, Wang, Zhi-Qiang. Soliton solutions for quasilinear schrödinger equations, ii. *Proceedings of the American Mathematical Society*, 2003, **131**(2): 441–448.

[17] V. G. Makhan'kov, V. K. Fedyanin. Nonlinear effects in quasi-one-dimensional models and condensed matter theory. *Physics Reports*, 1984, **104**(1): 1–86.

[18] J. a. Marcos, B. Do Ó. n-laplacian equations in $boldrspn$ with critical growth. *Abstract & Applied Analysis*, 1997, **2**(3-4): 301–315.

[19] J. quan Liu, Y. qi Wang, Z.-Q. Wang. Solutions for quasilinear schrödinger equations via the nehari method. *Communications in Partial Differential Equations*, 2004, **29**(5-6): 879–901.

[20] G. R. W. Quispel, H. W. Capel. Equation of motion for the heisenberg spin chain. *Physics Letters A*, 1982, **85**(4): 248–250.

[21] S. G. Ruiz D. Existence of ground states for a modified nonlinear schrödinger equation. *Nonlinearity*, 2010, **23**: 1221–1233.

[22] J. Serrin. Local behavior of solutions of quasi-linear equations. *Acta Mathematica*, 1964, **111**(1): 247–302.

[23] U. Severo. Existence of weak solutions for quasilinear elliptic equations involving the p-laplacian. *Electronic Journal of Differential Equations*, 2008, **2008**(56): 359–370.

[24] X. Wu. Multiple solutions for quasilinear schrödinger equations with a parameter. *Journal of Differential Equations*, 2014, **256**(7): 2619–2632.