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The Global Stable with Bifurcation of Parasitic Diseases Model

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Abstract. In this paper a mathematical model that describes the spread of infectious disease in a population is proposed and studied. This model describes the spread of parasitic diseases, which are spread through interaction with an environment contaminated, where the disease begins in the first cycle of incubation within the body of the host and then turn into infectious diseases, for example (Bilharzias, Toxoplasmosis and rabies). In this type of models allows us to model the complete separation of fast and slow sections. In this paper, we assume that the diseases in this model in cases are latent and infected. The local and global stability of the model is studied. The occurrence of local bifurcation as well a Hopf bifurcation in the model is investigated. Finally, the global dynamics of the proposed model is studied numerically.

Keywords: epidemic disease, stability, environment contaminated, parasitic, host, local and hopf bifurcation

1 Introduction

There are many of countries that have suffered and still from the spread of parasitic diseases in both humans and animals. And the parasites are living things that use other living things, like your body, for food and a place live. You can get them from contaminated food or water, a bug bite or sexual contact. Diseases caused by parasites are known parasitic diseases. A parasite lives on or in a host organism. A host organism can be any animal or human. Billions of people are affected by parasites each year. Parasites can cause infection and disease in the host organism. Some parasitic diseases are easily treated. Others can be life threatening. One of these diseases is Bilharzia which has other names as Snail fever or Katayama fever, is a disease caused by parasitic worms of the Schistosoma type. It may infect the urinary tract or the intestines. Signs and symptoms may include abdominal pain, diarrhea, bloody stool, or blood in the urine see in [9, 10, 15, 21, 22, 31, 34, 36]. And the second type of these diseases is the Malaria disease, is caused by a parasite called Plasmodium, which is transmitted via the bites of infected mosquitoes. Approximately half of the world's population is at risk of malaria. Most malaria cases and deaths occur in Sub-Saharan Africa. In 2011, 99 countries and territories had ongoing malaria transmission[18, 20, 35]. As well as other studies about the disease among them, Ross[32], Ngwa and Shu [28], Ngwa [27] and Chitnis et al.[5, 6].

As well from above the Toxoplasma disease is one of the parasite diseases it is commonly spread through contaminated food or water and gets it from the waste of an infected cat. The parasite may be swallowed if touch mouth after cleaning a cat and the garden. On this disease there are several studies which, Dubey and Beattie [7], Bowie et al.[3] , Bahia-Oliveira et al.[11] , Dubey[8], Jones et al.[23] , De moura et al.[24] , Montoya and Liesenfeld[26] and Remington et al.[29] . And studies on the parasite and host disease see that in [1, 2, 13, 14, 16, 17, 30]. The goal of this paper is to expand this within-host approach used to study the evolution of virulence into a general approach for modeling host-parasite. The local as well as global stability

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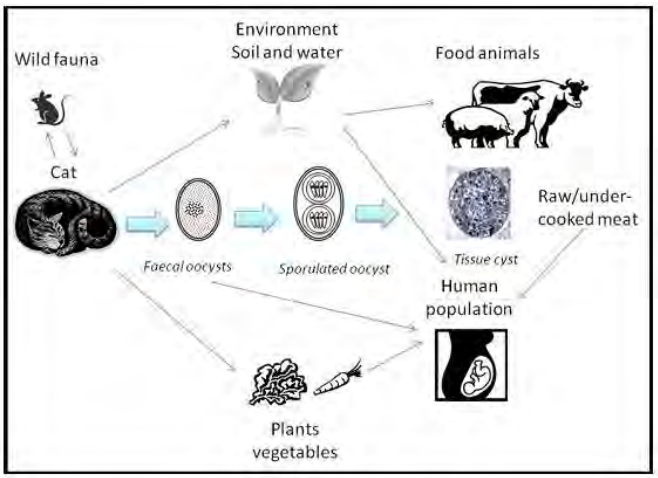

Fig. 1: Simplified life cycle of Toxoplasmosis relevant for the model considered in this paper

analysis of the modified model is investigated. Also, the local bifurcations, as well as, Hopf bifurcation are discussed.

2 The mathematical model

The coupled model considered in this paper is an extension of the model studied in [25] by including a disease induced death rate for the host. This modeling approach can be applied to environmentally-driven infectious disease such as Toxoplasmosis, which is a disease caused by the parasite Toxoplasma gondii. The main definitive hosts of the pathogen are members of the family Felidae (domestic cats and their relatives). The definitive hosts can become infected either by consumption of infected intermediate hosts or by direct ingestion of the parasites existent in a contaminated environment. The within-host system describes the cell-parasite interaction and keeps track of the densities of uninfected and infected cells as well as the average parasite load within a host (see [12] for similar models of cell-pathogen interactions). The level of environmental contamination is dependent on both the number of infected hosts and the average parasite load within a host. The parasites can remain in the environment for several months. Since infected hosts will remain infectious for life, the epidemiological model at the population level is of an SEIS type[12], and the hosts may die due to the disease. This model consists of the following seven ordinary differential equations:

$$\begin{aligned}
\dot{T} &= \Lambda - \beta VT - mT \\
\dot{T}^* &= \beta VT - (m+d)T^* \\
\dot{V} &= g(E) + pT^* - cV - \beta VT \\
\dot{S} &= \psi - \sigma SE - \mu S + \gamma I \\
\dot{L} &= \sigma SE - (\mu + \varepsilon)L \\
\dot{I} &= \varepsilon L - (\mu + \gamma)I \\
\dot{E} &= \theta V(I + \rho L)(1 - E) - \alpha E
\end{aligned} \tag{1}$$

Now, we can divide the above system into two subsystems the first three equations of system (1) called the within-host system and we can rewrite below:

$$\begin{aligned}
\dot{T} &= \Lambda - \beta VT - mT \\
\dot{T}^* &= \beta VT - (m+d)T^* \\
\dot{V} &= g(E) + pT^* - cV - \beta VT
\end{aligned} \tag{2}$$

Let T, T^* and V be the number of healthy cells, infected cells and parasites load respectively. Note that all the parameters of within-host system (2) are assumed to be positive constants and can describe as follows:

Λ is the recruitment rate of the healthy cells, β infection rate, m is the natural death rate in T and T^* classes, d, c are the disease related deaths from T^* and V respectively, p the new parasite members arriving into the parasite load class from infection cells class. The function $g(E)$ represents the rate at which an average host is inoculated. In order to link environmental contamination with the infection process at the individual, we must postulate a mechanism. In this paper we assume that environmental contamination is measured by or related to the concentration of pathogenic forms living in the environment, and that hosts acquire infection by ingesting contaminated food (which exists in the environment). The function g expresses the fact that if the environmental contamination is high then the inoculum (the average per capita concentration of pathogen infectious forms introduced into a given host) is also high. These biological considerations suggest that the function g should have the following properties:

$$g(E) \geq 0,\ g(0) = 0,\ g'(E) > 0 \tag{3}$$

Although in general the function g may reach a saturation level, we consider here a simple linear form as our aim is to illustrate framework of linking within-and between-host model who is described later. Thus, in this paper we chose the linear form:

$$g(E) = aE \tag{4}$$

here α is a positive constant. Now the other fourth equations in system (1) are called between-host system. This can be written below:

$$\begin{aligned} \dot{S} &= \psi - \sigma SE - \mu S + \gamma I \\ \dot{L} &= \sigma SE - (\mu + \varepsilon)L \\ \dot{I} &= \varepsilon L - (\mu + \gamma)I \\ \dot{E} &= \theta V(I + \rho L)(1 - E) - \alpha E \end{aligned} \tag{5}$$

Here, S, L, I and E denoted the numbers of susceptible individuals class, latent class, infected individuals and the level of environment contamination at time t, respectively. The $\psi > 0$ parameters the birth rate in susceptible class, $\sigma > 0$ the infection rate by contact, $\mu > 0$ the natural death rate, $\varepsilon > 0$ evolution rate of disease from latent class to infected class. Finally, $\gamma > 0$ the nature recovery rate form infected class to become again susceptible class. Let E represents the level of environment contamination at time t, $(0 \leq E \leq 1)$, or the concentration of the pathogen per unit area of a region being considered. Similar to the assumptions made in [4, 12], on the disease transmission at the population level, we assume that the parasite release rate of a host is likely to increase with parasite load V. Therefore, the rate environment contamination is an increasing function of and the number of infected hosts I. One of the simplest forms of a function is $\theta V I$, where the rate of contamination is, ρ the new contamination rate in environment contamination class $(i.e.,\ E)$ from latent class, α the clearance rate in the environment. Therefore, at any point of time t the total number of population becomes $N = S + L + I$. Obviously, due to the biological meaning of the variables S, L and I system (5) has the region is $\Omega = \{(S, L, I, E) : S \geq 0, L \geq 0, I \geq 0, 1 \geq E \geq 0\}$ which is positive invariant for system (5).

3 Model analysis

We assume that the above system (2) and (5) at two time scales in the analysis. For ease of reference, we will refer to systems (2) and (5) as the fast and slow systems, respectively, in the remaining sections.

3.1 The fast system analysis

It easy to see that the fast system (2) is in isolation always has a disease free equilibrium point (the absence of parasite load) that is mean $g(E) = 0$, $U_0 = (T_0, 0, 0)$. Now, we can calculate the within-host reproduction number ad denoted by $\Re_{f\circ}$. (here the subscript $'f'$ in $\Re_{f\circ}$ stands for within host system).

$$\Re_{f\circ} = \frac{\beta p T_\circ}{(m+d)(c+\beta T_\circ)} \tag{6}$$

Now, the fast system (2) in isolation has endemic equilibrium point, denoted by $U_1 = (T_1, T_1^*, V_1)$. The existence conditions for this equilibrium point discussed in following:

If $V \neq 0$ and $\Re_{f\circ} > 1$ then the fast system (2) in isolation has a positive equilibrium point and denoted by $U_1 = (T_1, T_1^*, V_1)$ where:

$$T_1 = \frac{\Lambda}{\beta V_1 + m} \tag{7}$$

$$T_1^* = \frac{\Lambda \beta V_1}{(m+d)(\beta V_1 + m)} \tag{8}$$

$$V_1 = \frac{-1}{2D_1}\left[D_2 + \sqrt{D_2^2 - 4D_1 D_3}\right] \tag{9}$$

Here: $D_1 = -c\beta(m+d)$; $D_2 = \beta\left[aE(m+d) + \Lambda p\right] - (m+d)(cm + \Lambda\beta)$; $D_3 = amE(m+d)$. Clearly, V_1 be positive root if and only if $D_2 > 0$ (positive).

3.2 The slow system analysis

Where the environmental contamination level equal to zero $(i.e.,\ E = 0)$ then the slow system (5) in isolation always has a disease-free equilibrium point and denoted by $U_2 = (S_2, 0, 0, 0)$. We get the between-host system (5) is in isolation, when V is constant $(i.e.,\ V = V_1)$ in Eq. (10), independent of l, I and E, the dynamics can be determined by between-host reproduction number:

$$\Re_{b\circ} = \left(\frac{\sigma S_2}{\mu+\varepsilon}\right)\left(\frac{\varepsilon}{\mu+\gamma}\right)\left(\frac{(1-\rho)\theta V}{\alpha}\right) \tag{10}$$

(here the subscript $'b'$ in $\Re_{b\circ}$ stand for the between-host system). Now, the slow system (5) in isolation has at most two biologically feasible points, namely $U_i = (S_i, L_i, I_i, E_i)$, $i = 2, 3$. The existence conditions for each of these equilibrium points are discussed in following:

- If $L = 0$, $I = 0$, $E = 0$ and $\Re_{b\circ} < 1$ then the slow system (5) in isolation has an equilibrium point called disease-free equilibrium point and denoted by $U_2 = (S_2, 0, 0, 0)$ where:

$$S_2 = \frac{\psi}{\mu} \tag{11}$$

- 2) If $L \neq 0$, $I \neq 0$, $E \neq 0$ and $\Re_{b\circ} > 1$ then the slow system (5) in isolation has a positive equilibrium point called endemic point and denoted by $U_3 = (S_3, L_3, I_3, E_3)$ where:

$$S_3 = \frac{\psi\left[(\mu+\gamma)(\mu+c)(\sigma E_3+\mu) - \sigma\gamma E_3\right] + \varepsilon\sigma\gamma\psi E_3}{(\sigma E_3+\mu)\left[(\mu+\gamma)(\mu+c)(\sigma E_3+\mu) - \sigma\gamma E_3\right]} \tag{12}$$

$$L_3 = \frac{\sigma E_3 \psi(\mu+\gamma)}{(\mu+\gamma)(\mu+c)(\sigma E_3+\mu) - \sigma\gamma E_3} \tag{13}$$

$$I_3 = \frac{\varepsilon\sigma\psi E_3}{(\mu+\gamma)(\mu+c)(\sigma E_3+\mu) - \sigma\gamma E_3} \tag{14}$$

$$E_3 = \frac{\theta\sigma\psi V_1\left[\varepsilon + \rho(\mu+\gamma)\right] - \alpha\mu(\mu+\gamma)(\mu+c)}{\sigma\left\{\gamma - \left[\theta\psi V_1(\varepsilon + \rho(\mu+\gamma)) + \alpha(\mu+\gamma)(\mu+c)\right]\right\}} \tag{15}$$

Clearly, the positive equilibrium point $U_3 = (S_3, L_3, I_3, E_3)$ of slow system (5) is existence and positive if and only if the following conditions are hold:

$$\gamma < \min.\left\{\frac{(\mu+\gamma)(\mu+c)(\sigma E_3+\mu)}{\sigma E_3},\ \theta\psi V_1(\varepsilon + \rho(\mu+\gamma)) + \alpha(\mu+\gamma)(\mu+c)\right\} \tag{16}$$

$$\theta\sigma\psi V_1\left[\varepsilon + \rho(\mu+\gamma)\right] < \alpha\mu(\mu+\gamma)(\mu+c) \tag{17}$$

4 Local stability analysis of fast and slow systems in isolation

In this section, the local stable analysis of each equilibrium points U_i, $i = 1, 2, 3$ of the fast and slow systems in isolation studding as shown in the following theorems.

Theorem 1. *The positive equilibrium point $U_1 = (T_1, T_1^*, V_1)$ of fast system in isolation is local asymptotically stable when $\Re_{f\circ} > 1$ and then the following conditions are satisfied:*

$$\beta cV_1(m+d) + m(m+d)(c+\beta T_1) > pm\beta T_1 \tag{18}$$

$$\begin{aligned}(\beta V_1 + m)(m+d)(\beta V_1 + 2m + d) + [m + c + \beta(V_1 + T_1)][mc + \beta(cV_1 + mT_1)] \\ +(m+d)(c+\beta T_1)[3m + d + c + \beta(2V_1 + T_1)] > \beta T_1[m + d + c + \beta(V_1 + T_1)]\end{aligned} \tag{19}$$

Proof. The Jacobian matrix of fast system at U_1 that denoted by $J(U_1)$ and can be written:

$$J(U_1) = \begin{pmatrix} -(\beta V_1 + m) & 0 & -\beta T_1 \\ \beta V_1 & -(m+d) & \beta T_1 \\ \beta V_1 & p & -(c+\beta T_1) \end{pmatrix}$$

Clearly, the characteristic equation of the Jacobian matrix $J(U_1)$ of the fast system (2) at the positive equilibrium point (U_1) is given by:

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$$

Here:

$$\begin{aligned} B_1 &= -[b_{11} + b_{22} + b_{33}] \\ &= c + d + 2m + \beta(V_1 + T_1) \end{aligned}$$

$$B_2 = [b_{11}b_{22} + b_{11}b_{33} - b_{13}b_{31} + b_{22}b_{33} - b_{23}b_{32}]$$

$$\begin{aligned} B_3 &= -[b_{11}b_{22}b_{33} + b_{21}b_{32}b_{13} - b_{22}b_{13}b_{31} - b_{11}b_{23}b_{32}] \\ &= \beta cV_1(m+d) + m(m+d)(c+\beta T_1) - pm\beta T \end{aligned}$$

Further:

$$\begin{aligned} \Delta &= B_1B_2 - B_3 \\ &= (\beta V_1 + m)(m+d)(\beta V_1 + 2m + d) + [m + c + \beta(V_1 + T_1)][mc + \beta(cV_1 + mT_1)] \\ &\quad +(m+d)(c+\beta T_1)[3m + d + c + \beta(2V_1 + T_1)] - p\beta T_1[m + d + c + \beta(V_1 + T_1)] \end{aligned}$$

Now according to (Routh-Hurwitz) criterion U_1 will be locally asymptotically stable provided that $B_1 > 0$; $B_3 > 0$ and $\Delta = B_1B_2 - B_3 > 0$. Clearly: $B_1 > 0$and$B_3 > 0$ provided that condition (18) holds. While $\Delta = B_1B_2 - B_3 > 0$, provided that condition (19) holds. Hence the proof is complete.

Theorem 2. *The disease-free equilibrium point $U_2 = (S_2, 0, 0, 0)$ of slow system in isolation is local asymptotically stable when $\Re_{b\circ} < 1$ and then the following condition is satisfied:*

$$\alpha(\mu + \varepsilon) > \max\left\{ \frac{\sigma\theta S_2V_1(\varepsilon + \rho(\mu+\gamma))}{(\mu+\gamma)}, \sigma\rho\theta S_2V_1 \right\} \tag{20}$$

Proof. The Jacobian matrix of slow system at U_2 that denoted by $J(U_2)$ and can be written:

$$J(U_2) = \begin{pmatrix} -\mu & 0 & \gamma & -\sigma S_2 \\ 0 & -(\mu+\varepsilon) & 0 & \sigma S_2 \\ 0 & \varepsilon & -(\mu+\gamma) & 0 \\ 0 & \rho\theta V_1 & \theta V_1 & -\alpha \end{pmatrix}$$

Then the characteristic equation of Jacobian matrix at U_2

$$(-\mu - \lambda)\left[\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3\right] = 0 \tag{21}$$

Here:

$$C_1 = [2\mu + \varepsilon + \gamma + \alpha]$$
$$C_2 = [(\mu+\varepsilon)(\mu+\gamma) + \alpha\left((\mu+\varepsilon) + (\mu+\gamma)\right) - \sigma\rho\theta S_2 V_1]$$
$$C_3 = [\alpha(\mu+\varepsilon)(\mu+\gamma) - \sigma\theta S_2 V_1(\varepsilon + \rho(\mu+\gamma)]$$

Further

$$\begin{aligned}\Delta &= C_1C_2 - C_3 \\ &= -c_{22}c_{33}\left[c_{22}+c_{33}\right] - c_{33}c_{44}\left[2c_{22}+c_{33}+c_{44}\right] + c_{32}c_{43}c_{24} \\ &\quad + \left[c_{22}+c_{44}\right]\left[c_{24}c_{42} - c_{22}c_{44}\right] \quad = (\mu+\varepsilon)(\mu+\gamma)\left[(\mu+\varepsilon)+(\mu+\gamma)\right] + \alpha(\mu+\gamma)\left[2(\mu+\varepsilon)+(\mu+\gamma)+\alpha\right] \\ &\quad + \varepsilon\theta\sigma S_2 V_1 - \left[\mu+\varepsilon+\alpha\right]\left[\sigma\rho\theta S_2 V_1 - \alpha(\mu+\varepsilon)\right]\end{aligned}$$

Now according to (Routh-Hurwitz) criterion U_2 will be locally asymptotically stable provided that $C_1 > 0$; $C_3 > 0$ and $\Delta = C_1C_2 - C_3 > 0$. Clearly: $C_3 > 0$, $\Delta = C_1C_2 - C_3 > 0$, provided that condition (20) holds. Hence the proof is complete.

Theorem 3. *The endemic equilibrium point $U_3 = (S_3, L_3, I_3, E_3)$ of slow system is local asymptotically stable when $\Re_{b\circ} > 1$ and then the following conditions are satisfied:*

$$\mu > \rho\theta V_1(1 - E_3) \tag{22}$$

$$\alpha > 2\sigma S_3 - \theta V_1(I_3 + \rho L_3) \tag{23}$$

Proof. The Jacobian matrix of slow system in isolation at U_3 that denoted by $J(U_3)$ and can be written:

$$J(U_3) = \begin{pmatrix} -(\sigma E_3 + \mu) & 0 & \gamma & -\sigma S_3 \\ \sigma E_3 & -(\mu+\varepsilon) & 0 & \sigma S_3 \\ 0 & \varepsilon & -(\mu+\gamma) & 0 \\ 0 & \rho\theta V_1(1-E_3) & \theta V_1(1-E_3) & -\left(\theta V_1(I_3+\rho L_3) + \alpha\right) \end{pmatrix}$$

Now, according to Gersgorin theorem if the following condition holds:

Therefore, according to the given conditions (22-23) all the eigenvalues of $J(U_3)$ exists in the left half plane and hence, U_3 is locally asymptotically stable.

5 Global stability analysis of fast and slow systems in isolation

In this section, the global stability analysis of all equilibrium points of the fast and slow systems in isolation studding as shown in the following theorems.

Theorem 4. *Assume that, the positive equilibrium point of fast System (2) is locally asymptotically stable. Then it is globally asymptotically stable if satisfy the following conditions:*

$$\beta V_1^2 < (\beta V_1 + m)(m + d) \tag{24}$$
$$(p + \beta T)^2 < (c + \beta T)(m + d) \tag{25}$$

Proof. Consider the following positive definite function:

$$F_1 = \frac{(T - T_1)^2}{2} + \frac{(T^* - T_1^*)^2}{2} + \frac{(V - V_1)^2}{2}$$

Clearly, $F_1 : R_+^3 \to R$ is a continuously differentiable function such that $F_1(T_1, T_1^*, V_1) = 0$ and $F_1(T, T^*, V) > 0$, $\forall (T_1, T_1^*, V_1) \neq (T, T^*, V_1)$. Further, we have:

$$\dot{F}_1 = (T - T_1)\dot{T} + (T^* - T_1^*)\dot{T}^* + (V - V_1)\dot{V}$$

By simplifying this equation we get:

$$\begin{aligned}\dot{F}_1 = &-\tfrac{q_{11}}{2}(T - T_1)^2 - q_{13}(T - T_1)(V - V_1) - \tfrac{q_{33}}{2}(V_1 - V_1)^2 \\ &-\tfrac{q_{11}}{2}(T - T_1)^2 + q_{12}(T - T_1)(T^* - T_1^*) - \tfrac{q_{22}}{2}(T^* - T_1^*)^2 \\ &-\tfrac{q_{33}}{2}(V_1 - V_1)^2 + q_{23}(T^* - T_1^*)(V_1 - V_1) - \tfrac{q_{22}}{2}(T^* - T_1^*)^2\end{aligned}$$

With:

$$q_{11} = \beta V_1 + m \;;\; q_{13} = \beta(T + V_1)\,;\; q_{33} = c + \beta T \;;\; p_{12} = \beta V_1 \;;\; q_{22} = m + d \;;\; q_{23} = p + \beta T$$

Therefore, according to the conditions (24-25) we obtain that:

$$\begin{aligned}\dot{F}_1 \leq &-\tfrac{q_{11}}{2}(T - T_1)^2 - q_{13}(T - T_1)(V - V_1) - \tfrac{q_{33}}{2}(V_1 - V_1)^2 \\ &-\Big[\sqrt{\tfrac{q_{11}}{2}}(T - T_1) - \sqrt{\tfrac{q_{22}}{2}}(T^* - T_1^*)\Big]^2 - \Big[\sqrt{\tfrac{q_{22}}{3}}(T^* - T_1^*) + \sqrt{\tfrac{q_{33}}{2}}(V_1 - V_1)\Big]^2\end{aligned}$$

Clearly, $\dot{F}_1 < 0$, and then F_1 is a Lyapunov function provided that the given conditions (24-25) hold. Therefore, U_1 is globally asymptotically stable.

Theorem 5. *Let the disease free equilibrium point of slow System (5) is locally asymptotically stable. Then the basin of attraction of , say , it is globally asymptotically stable provided that the following condition is satisfied:*

$$-\Big[\frac{\mu}{S}(S - S_2)^2 + \frac{\gamma}{S}IS_2 + \mu(L + I) + \alpha E\Big] > [\sigma E S_2 + \theta(I + \rho L)V(1 - E)] \tag{26}$$

Proof. Consider the following positive definite function:

$$F_2 = \left(S - S_2 - S_2 \ln \frac{S}{S_2}\right) + L + I + E$$

Clearly, $F_2 : R_+^4 \rightarrow R$ is a continuously differentiable function such that $F_2(S_2\,, 0\ , 0, 0) = 0$, and $F_2(S, L, I, E) > 0\,,\ \forall (S, L, I, E) \neq (S_2\,, 0\ , 0, 0)$. Further we have:

$$\dot{F}_2 = \left(\frac{S - S_2}{S}\right)\dot{S} + \dot{L} + \dot{I} + \dot{E}$$

By simplifying this equation we get:

$$\dot{F}_2 = -\frac{\mu}{S}(S - S_2)^2 - \frac{\gamma}{S}IS_2 - \mu(L + I) - \alpha E + \sigma E S_2 + \theta(I + \rho L)V(1 - E)$$

Obviously, $\dot{F}_2 < 0$, for every initial points and then F_2 is a Lyapunov function provided that condition (26) hold. Thus U_2 is globally asymptotically stable in the interior of $B(U_2)$ which means that $B(U_2)$ is the basin of attraction and that complete the proof.

Theorem 6. *Let the endemic equilibrium point of slow system (5) is locally asymptotically stable. Then it is globally asymptotically stable provided that:*

$$(\sigma S_3)^2 < \frac{4}{9}(\sigma E + \mu) \cdot (\theta V_1 I_3 + \theta V \rho L_3 + \alpha) \tag{27}$$

$$\gamma^2 < \frac{4}{9}(\sigma E + \mu) \cdot (\mu + \gamma) \tag{28}$$

$$\sigma E^2 < \frac{4}{9}(\sigma E + \mu) \cdot (\mu + \varepsilon) \tag{29}$$

$$\varepsilon^2 < \frac{4}{9}(\mu + \gamma) \cdot (\mu + \varepsilon) \tag{30}$$

$$[\theta V_1(1 - E)]^2 < \frac{4}{9}(\mu + \gamma) \cdot (\theta V_1 I_3 + \theta V_1 \rho L_3 + \alpha) \tag{31}$$

$$[\sigma S_3 + \theta V_1 \rho(1 - E)]^2 < \frac{4}{9}(\mu + \varepsilon) \cdot (\theta V_1 I_3 + \theta V_1 \rho L_3 + \alpha) \tag{32}$$

Proof. Consider the following positive definite function:

$$F_3 = \frac{(S - S_3)^2}{2} + \frac{(L - L_3)^2}{2} + \frac{(I - I_3)^2}{2} + \frac{(E - E_3)^2}{2}$$

Clearly, $F_3 : R_+^4 \rightarrow R$ is a continuously differentiable function such that $F_3(S_3, L_3, I_3, E_3) = 0$ and $F_3(S, L, I, E) > 0, \ \forall (S, L, I, E) \neq (S_3, L_3, I_3, E_3)$. Further, we have:

$$\dot{F}_3 = (S - S_3)\dot{S} + (L - L_3)\dot{L} + (I - I_3)\dot{I} + (E - E_3)\dot{E}$$

By simplifying this equation we get:

$$\begin{array}{c} F_3 = -\frac{k_{11}}{3}(S - S_3)^2 - k_{14}(S - S_3)(E - E_3) - \frac{k_{44}}{3}(E - E_3)^2 \\ -\frac{k_{11}}{3}(S - S_3)^2 + k_{13}(S - S_3)(I - I_3) - \frac{k_{33}}{3}(I - I_3)^2 \\ -\frac{k_{11}}{3}(S - S_3)^2 + k_{12}(S - S_3)(L - L_3) - \frac{k_{22}}{3}(L - L_3)^2 \\ -\frac{k_{33}}{3}(I - I_3)^2 + k_{32}(I - I_3)(L - L_3) - \frac{k_{22}}{3}(L - L_3)^2 \\ -\frac{k_{33}}{3}(I - I_3)^2 + k_{34}(I - I_3)(E - E_3) - \frac{k_{44}}{3}(E - E_3)^2 \\ -\frac{k_{22}}{3}(L - L_3)^2 + k_{24}(L - L_3)(E - E_3) - \frac{k_{44}}{3}(E - E_3)^2 \end{array}$$

6 Local bifurcation analysis of fast and slow systems

In this section, the effect of varying parameter on the dynamical behavior of the fast and slow systems in isolation around each equilibrium points is studied. Recall that the existence of non-hyperbolic equilibrium point of the all systems (2) and (5) is the necessary but not sufficient condition for bifurcation to accurse. Therefore, in the following theorems and application to the Sotomayor's theorem for local bifurcation[33], is adapted.

6.1 Local bifurcation near (U_1)

Theorem 7. *The fast system (2) at the positive equilibrium point with the parameter* $c^* = \frac{\beta m T_1(p-m-d)}{(\beta V_1+m)(m+d)}$ *and this parameter is positive if the following condition is holds has:*

$$p > m + d \tag{33}$$

- *Saddle-node bifurcation*
- *No pitchfork bifurcation*
- *No transcritical bifurcation*

Proof. According to Jacobian matrix $J(U_1)$ of the fast system (2) at the positive equilibrium point (U_1) has zero eigenvalue $(say \ \lambda = 0)$ if and only if $\det(J(U_1)) = 0$, therefore, $c^* = \frac{\beta m T_1(p-m-d)}{(\beta V_1+m)(m+d)}$ is taken as a candidate bifurcation parameter, and the Jacobian matrix $J(U_1)$ with $c = c*$ becomes:

$$J = J_1\,(\lambda = 0) = \begin{bmatrix} -(\beta V_1 + m) & 0 & -\beta T_1 \\ \beta V_1 & -(m+d) & \beta T_1 \\ -\beta V_1 & p & -(c^* + \beta T_1) \end{bmatrix}$$

Further the eigenvector (say $K = (k_1\,, k_2\,, k_3\,)^T$) corresponding to $\lambda = 0$ satisfy the following:

$$JK = \lambda K$$

Then

$$JK = 0$$

From which we get that:

$$-(\beta V_1 + m)k_1 - \beta T_1 k_3 = 0$$
$$\beta V_1 k_1 - (m+d)k_2 + \beta T_1 k_3 = 0$$
$$-\beta V_1 k_1 + pk_2 - (c^* + \beta T_1)k_3 = 0$$

So by solving the above system of equations we get:

$$k_1 = -qk_3\,;\ k_2 = -zk_3$$

Where:

$$q = \frac{\beta T_1}{\beta V_1 + m}$$
$$z = \frac{\beta T_1 m}{(m+d)(\beta V_1 + m)}$$

Here k_3 be any non zero real number. Thus

$$K = \begin{bmatrix} -qk_3 \\ -zk_3 \\ k_3 \end{bmatrix}$$

Similarly the eigenvector $W = (w_1\ ,\ w_2\ ,\ w_3\ )^T$ that corresponding to $\lambda = 0$ of J_1^T can be written:

$$J_1^T \cdot W = \begin{bmatrix} -(\beta V_1 + m) & \beta V_1 & -\beta V_1 \\ 0 & -(m+d) & p \\ -\beta T_1 & \beta T_1 & -(c^* + \beta T_1) \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0$$

This gives:

$$W = \begin{bmatrix} xw_3 \\ yw_3 \\ w_3 \end{bmatrix}$$

Where

$$x = \frac{\beta V_1[p-(m+d)]}{(\beta V_1 + m)(m+d)}$$
$$y = \frac{p}{(m+d)}$$

Here w_3 be any non zero real number. Now, rewrite fast system (2) in a vector form as:

$$\frac{dX}{dt} = f(X)$$

Where $X = (T,\, T^*,\, V)^T$ and $f = (f_1\,, f_2\,, f_3\,)^T$ with $f_i\,,\ i = 1, 2, 3$ are given in fast system (2), and then determine $\frac{df}{dc} = f_c$ we get that:

$$f_c = \begin{bmatrix} 0 \\ 0 \\ -V \end{bmatrix}$$

Then

$$f_c(U_1\,, c^*) = \begin{bmatrix} 0 \\ 0 \\ -V_1 \end{bmatrix}$$

Therefore:

$$W^T \cdot f_c(U_1, c^*) = -w_3 V_1 \neq 0$$

Consequently, according to Sotomayor theorem, if in addition to the above, the following holds

$$W^T \cdot \left[D^2 f(U_1, c^*) \cdot (K, K)\right] \neq 0$$

Here $Df(U_1, c^*)$ is the Jacobian matrix at U_1 and $c*$, then the fast system (2) possesses a saddle-node bifurcation can occur. Now since we have that:

$$\left[D^2 f(U_1, c^*) \cdot (K, K)\right] = \begin{bmatrix} 2q\beta k_3^2 \\ -2q\beta k_3^2 \\ 2q\beta k_3^2 \end{bmatrix}$$

Therefore:

$$\begin{aligned} &W^T \cdot \left[D^2 f(U_1, c^*) \cdot (K, K)\right] \\ &= 2q\beta w_3 k_3^2 [x - y + 1] \neq 0 \end{aligned}$$

Then the fast system (2) has a saddle node bifurcation at U_1 when the parameter c passes through the bifurcation value .

6.2 Local bifurcation near (U_2)

Theorem 8. *Assume that the following condition is holds*

$$\sigma\theta S_2 V_1[\rho(\mu + \gamma) + c] = \alpha(\mu + \varepsilon)(\mu + \gamma) \tag{34}$$

Then the slow system (5) at the disease free equilibrium point (U_2) *with the parameter* $\alpha^* = \frac{\sigma\rho\theta S_2 V_1}{\mu+\varepsilon}$ *has:*

- *No saddle-node bifurcation*
- *No pitchfork bifurcation*
- *A transcritical bifurcation*

Proof. According to Jacobian matrix $\bar{J}(U_2)$ of the slow system (5) at the disease free equilibrium point (U_2) has zero eigenvalue $\left(say\ \bar{\lambda} = 0\right)$ if and only if $\det\left(\bar{J}(U_2)\right) = 0$, therefore, $\alpha^* = \frac{\sigma\rho\theta S_2 V_1}{\mu+\varepsilon}$ is taken as a candidate bifurcation parameter, and the Jacobian matrix $\bar{J}(U_2)$ with $\alpha = \alpha^*$ becomes:

$$\bar{J} = \bar{J}_2\left(\bar{\lambda} = 0\right) = \begin{bmatrix} -\mu & 0 & \gamma & -\sigma S_2 \\ 0 & -(\mu + \varepsilon) & 0 & \sigma S_2 \\ 0 & \varepsilon & -(\mu + \gamma) & 0 \\ 0 & \rho\theta V_1 & \theta V_1 & -\alpha \end{bmatrix}$$

Further the eigenvector (say $\bar{K} = \left(\bar{k}_1\,, \bar{k}_2\,, \bar{k}_3, \bar{k}_4\right)^T$) corresponding to $\bar{\lambda} = 0$ satisfy the following:

$$\bar{J}\bar{K} = \bar{\lambda}\bar{K}$$

Then

$$\bar{J}\bar{K} = 0$$

From which we get that:

$$-\mu\bar{k}_1 + \gamma\bar{k}_3 - \sigma S_2\bar{k}_4 = 0$$

$$-(\mu + \varepsilon)\bar{k}_2 + \sigma S_2\bar{k}_4 = 0$$

$$\varepsilon\bar{k}_2 - (\mu+\gamma)\bar{k}_3 = 0$$
$$\rho\theta V_1\bar{k}_2 + \theta V_1\bar{k}_3 - \alpha^*\bar{k}_4 = 0$$

So by solving the above system of equations we get:

$$\bar{k}_1 = -\bar{q}\bar{k}_4\,;\ \bar{k}_2 = \bar{z}\bar{k}_4\,;\ \bar{k}_3 = \bar{h}\bar{k}_4$$

Where:

$$\bar{q} = \frac{\sigma S_2(\mu+\gamma-\varepsilon)}{(\mu+\varepsilon)(\mu+\gamma)}$$
$$\bar{z} = \frac{\sigma S_2}{\mu+\varepsilon}$$
$$\bar{h} = \frac{\sigma\varepsilon S_2}{(\mu+\varepsilon)(\mu+\gamma)}$$

Here $\bar{k}_4$ be any non zero real number. Thus

$$\bar{K} = \begin{bmatrix} -\bar{q}\bar{k}_4 \\ \bar{z}\bar{k}_4 \\ \bar{h}\bar{k}_4 \\ \bar{k}_4 \end{bmatrix}$$

Similarly the eigenvector $\bar{W} = (\bar{w}_1\ ,\ \bar{w}_2\ ,\ \bar{w}_3\ ,\ \bar{w}_4\ )^T$ that corresponding to $\bar{\lambda} = 0$ of $\bar{J}_2^T$ can be written:

$$\bar{J}_2^T \cdot \bar{W} = \begin{bmatrix} -\mu & 0 & 0 & 0 \\ 0 & -(\mu+\varepsilon) & \varepsilon & \rho\theta V_1 \\ \gamma & 0 & -(\mu+\gamma) & \theta V_1 \\ -\sigma S_2 & \sigma S_2 & 0 & -\alpha^* \end{bmatrix} \cdot \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \\ \bar{w}_4 \end{bmatrix} = 0$$

This gives:

$$\bar{W} = \begin{bmatrix} 0 \\ \bar{x}\bar{w}_4 \\ \bar{y}\bar{w}_4 \\ \bar{w}_4 \end{bmatrix}$$

Where

$$\bar{x} = \frac{\alpha^*}{\sigma S_2}$$
$$y = \frac{\theta V_1}{\mu+\gamma}$$

Here $\bar{w}_4$ be any non zero real number.
Now, rewrite slow system (5) in a vector form as:

$$\frac{dX}{dt} = f(X)$$

Where $X = (S,\ L,\ I, E)^T$ and $f = (f_1\,, f_2\,, f_3, f_4\,)^T$ with $f_i\,,\ i = 1, 2, 3, 4$ are given in slow system (5), and then determine $\frac{df}{d\alpha} = f_\alpha$ we get that:

$$f_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -E \end{bmatrix}$$

Then

$$f_\alpha(U_2\,, \alpha^*) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore:

$$\bar{W}^T \cdot f_\alpha(U_2, \alpha^*) = 0$$

Consequently, according to Sotomayor theorem [35] the slow system (5) has no saddle-node bifurcation near U_2 at α^*.

Now in order to investigate the accruing of other types of bifurcation, the derivative of f_α with respect to vector X, say $Df_\alpha(U_2, \alpha^*)$, is computed

$$Df_\alpha(U_2, \alpha^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

So

$$\bar{W}^T \cdot \left[Df_{\beta_1}(U_2, \alpha^*) \cdot \bar{K}\right] = -\bar{w}_4 \bar{k}_4 \neq 0$$

Again, according to Sotomayor theorem, if in addition to the above, the following holds

$$\bar{W}^T \cdot \left[D^2 f(U_2, \alpha^*) \cdot (\bar{K}, \bar{K})\right]$$

Here $Df(U_2, \alpha^*)$ is the Jacobian matrix at U_2 and α^*, then the slow system (5) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur. Now since we have that:

$$\left[D^2 f(U_2, \alpha^*) \cdot (\bar{K}, \bar{K})\right] = \begin{bmatrix} 2\sigma\bar{q}\bar{k}_4^2 \\ -2\sigma\bar{q}\bar{k}_4^2 \\ 0 \\ -2\theta V_1(\rho\bar{z} + \bar{h})\bar{k}_4^2 \end{bmatrix}$$

Therefore:

$$\bar{W}^T \cdot \left[D^2 f(U_2, \alpha^*) \cdot (\bar{K}, \bar{K})\right] = -2\bar{w}_4 \bar{k}_4^2 [\sigma\bar{q}\bar{x} + \theta V_1(\rho\bar{z} + \bar{h})] \neq 0$$

Then the slow system (5) has a transcritical bifurcation at U_2 when the parameter α passes through the bifurcation value α^*.

6.3 Local bifurcation near (U_3)

Theorem 9. *Assume that the following condition is holds*

$$R_1 = R_2 \tag{35}$$
$$R_3 = R_4 \tag{36}$$

Where

$$R_1 = \sigma\left\{\theta S_3 V_1(1 - E_3)(\sigma E_3 + \mu)(\mu + \lambda)(\mu + \varepsilon)[\rho(\mu + \gamma) + \varepsilon] + \varepsilon\gamma E_3[\theta V_1(I_3 + \rho L_3) + \tilde{\alpha}\right\}$$
$$R_2 = -\left\{\sigma^2\theta S_3 V_1 E_3(1 - E_3)[\rho(\mu + \gamma) + \varepsilon] + (\sigma E_3 + \mu)(\mu + \gamma)(\mu + \varepsilon)[\theta V_1(I_3 + \rho L_3) + \tilde{\alpha}]\right\}$$
$$R_3 = \sigma\mu\theta S_3 V_1(1 - E_3)[\rho(\mu + \gamma) + \varepsilon]$$
$$R_4 = [\theta V_1(I_3 + \rho L_3) + \tilde{\alpha}][(\mu + \gamma)(\mu + \varepsilon)(\sigma E_3 + \mu)]$$

Then the slow system (5) at the epidemic equilibrium point (U_3) with the parameter $\tilde{\alpha} = \frac{R_5 - R_6}{(\sigma E_3 + \mu)(\mu + \gamma)(\mu + \varepsilon)}$

Here

$$R_5 = \sigma\theta S_3 V_1(1 - E_3)[\rho(\mu + \gamma)(\sigma E_3 + \mu) + \varepsilon\sigma E_3]$$
$$R_6 = \theta V_1(I_3 + \rho L_3)(\sigma E_3 + \mu)(\mu + \varepsilon)(\mu + \gamma)$$

Clearly, α^* is positive if $R_5 > R_6$.

- *Saddle-node bifurcatio*
- *Saddle-node bifurcatio*
- *No transcritical bifurcation*

Proof. According to Jacobian matrix $\tilde{J}(U_3)$ of the slow system (5) at the disease free equilibrium point (U_3) has zero eigenvalue if and only if $\det\left(\tilde{J}(U_3)\right) = 0$, therefore, $\tilde{\alpha} = \frac{R_5 - R_6}{(\sigma E_3+\mu)(\mu+\gamma)(\mu+\varepsilon)}$ is taken as a candidate bifurcation parameter, and the Jacobian matrix $\tilde{J}(U_3)$ with $\alpha = \tilde{\alpha}$ becomes:

$$\tilde{J} = \tilde{J}_3\left(\tilde{\lambda} = 0\right) = \begin{bmatrix} -(\sigma E_3 + \mu) & 0 & \gamma & -\sigma S_3 \\ \sigma E_3 & -(\mu+\varepsilon) & 0 & \sigma S_3 \\ 0 & \varepsilon & -(\mu+\gamma) & 0 \\ 0 & \rho\theta V_1(1-E_3) & \theta V_1(1-E_3) & -[\theta V_1(I_3+\rho L_3)+\alpha] \end{bmatrix}$$

Further the eigenvector (say $\tilde{K} = \left(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4\right)^T$) corresponding to $\widetilde{\lambda} = 0$ satisfy the following:

$$\tilde{J}\tilde{K} = \tilde{\lambda}\tilde{K}$$

Then

$$\tilde{J}\tilde{K} = 0$$

From which we get that:

$$\begin{aligned} &-(\sigma E_3+\mu)\tilde{k}_1 + \gamma\tilde{k}_3 - \sigma S_3\tilde{k}_4 = 0 \\ &\sigma E_3\tilde{k}_1 - (\mu+\varepsilon)\tilde{k}_2 + \sigma S_3\tilde{k}_4 = 0 \\ &\varepsilon\tilde{k}_2 - (\mu+\gamma)\tilde{k}_3 = 0 \\ &\rho\theta V_1(1-E_3)\tilde{k}_2 + \theta V_1(1-E_3)\tilde{k}_3 - [\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]\tilde{k}_4 = 0 \end{aligned}$$

So by solving the above system of equations we get:

$$\tilde{k}_1 = \tilde{q}\tilde{k}_4\,;\ \tilde{k}_2 = \tilde{z}\tilde{k}_4\,;\ \tilde{k}_3 = \tilde{h}\tilde{k}_4$$

Where:

$$\begin{aligned} &\tilde{q} = \tfrac{R_7 - R_8}{\sigma\theta V_1 E_3(1-E_3)[\rho(\mu+\gamma)+\varepsilon]} \\ &here \\ &R_7 = (\mu+\gamma)(\mu+\varepsilon)[\theta V_1(I_3+\rho L_3)+\tilde{\alpha} \\ &R_8 = \sigma\theta V_1 S_3(1-E_3)[\rho(\mu+\gamma)+\varepsilon] \end{aligned}$$

$$\tilde{z} = \tfrac{(\mu+\gamma)[\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]}{\theta V_1(1-E_3)[\rho(\mu+\gamma)+\varepsilon]}$$

$$\tilde{h} = \tfrac{\varepsilon[\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]}{\theta V_1(1-E_3)[\rho(\mu+\gamma)+\varepsilon]}$$

Here $\widetilde{k}_4$ be any non zero real number. Thus

$$\tilde{K} = \begin{bmatrix} \tilde{q}\tilde{k}_4 \\ \tilde{z}\tilde{k}_4 \\ \tilde{h}\tilde{k}_4 \\ \tilde{k}_4 \end{bmatrix}$$

Similarly the eigenvector $\tilde{W} = (\tilde{w}_1\ ,\ \tilde{w}_2\ ,\ \tilde{w}_3\ ,\ \tilde{w}_4\ )^T$ that corresponding to $\tilde{\lambda} = 0$ of $\tilde{J}_3^T$ can be written:

$$\tilde{J}_3^T \cdot \tilde{W} = \begin{bmatrix} -(\sigma E_3+\mu) & \sigma E_3 & 0 & 0 \\ 0 & -(\mu+\varepsilon) & \varepsilon & \rho\theta V_1(1-E_3) \\ \gamma & 0 & -(\mu+\gamma) & \theta V_1(1-E_3) \\ -\sigma S_3 & \sigma S_3 & 0 & -[\theta V_1(I_3+\rho L_3)+\tilde{\alpha} \end{bmatrix} \cdot \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \end{bmatrix} = 0$$

This gives:

$$\tilde{W} = \begin{bmatrix} \tilde{x}\tilde{w}_4 \\ \tilde{y}\tilde{w}_4 \\ \tilde{g}\tilde{w}_4 \\ \tilde{w}_4 \end{bmatrix}$$

Where

$$\tilde{x} = \frac{E_3[\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]}{\mu S_3}$$
$$\tilde{y} = \frac{(\sigma E_3+\mu)[\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]}{\sigma\mu S_3}$$
$$\tilde{g} = \frac{(\mu+\varepsilon)(\sigma E_3+\mu)[\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]-\rho\sigma\theta\mu V_1 S_3(1-E_3)}{\varepsilon\sigma\mu S_3}$$

Here $\widetilde{w}_4$ be any non zero real number. Now, rewrite slow system (5) in a vector form as:

$$\frac{dX}{dt} = f(X)$$

Where $X = (S,\, L,\, I, E)^T$ and $f = (f_1\,, f_2\,, f_3, f_4\,)^T$ with $f_i\,,\ i = 1, 2, 3, 4$ are given in slow system (5), and then determine $\frac{df}{d\alpha} = f_\alpha$ we get that:

$$f_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -E \end{bmatrix}$$

Then

$$f_\alpha(U_3\,, \tilde{\alpha}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -E_3 \end{bmatrix}$$

Similarly the eigenvector $\tilde{W} = (\tilde{w}_1\ ,\ \tilde{w}_2\ ,\ \tilde{w}_3\,,\ \tilde{w}_4\,)^T$ that corresponding to $\tilde{\lambda} = 0$ of $\tilde{J}_3^T$ can be written:

$$\tilde{J}_3^T \cdot \tilde{W} = \begin{bmatrix} -(\sigma E_3+\mu) & \sigma E_3 & 0 & 0 \\ 0 & -(\mu+\varepsilon) & \varepsilon & \rho\theta V_1(1-E_3) \\ \gamma & 0 & -(\mu+\gamma) & \theta V_1(1-E_3) \\ -\sigma S_3 & \sigma S_3 & 0 & -[\theta V_1(I_3+\rho L_3)+\tilde{\alpha} \end{bmatrix} \cdot \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \end{bmatrix} = 0$$

This gives:

$$\tilde{W} = \begin{bmatrix} \tilde{x}\tilde{w}_4 \\ \tilde{y}\tilde{w}_4 \\ \tilde{g}\tilde{w}_4 \\ \tilde{w}_4 \end{bmatrix}$$

Where

$$\tilde{x} = \frac{E_3[\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]}{\mu S_3}$$
$$\tilde{y} = \frac{(\sigma E_3+\mu)[\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]}{\sigma\mu S_3}$$
$$\tilde{g} = \frac{(\mu+\varepsilon)(\sigma E_3+\mu)[\theta V_1(I_3+\rho L_3)+\tilde{\alpha}]-\rho\sigma\theta\mu V_1 S_3(1-E_3)}{\varepsilon\sigma\mu S_3}$$

Here $\tilde{w}_4$ be any non zero real number. Now, rewrite slow system (5) in a vector form as:

$$\frac{dX}{dt} = f(X)$$

Where $X = (S,\, L,\, I, E)^T$ and $f = (f_1\,, f_2\,, f_3, f_4\,)^T$ with $f_i\,,\ i = 1, 2, 3, 4$ are given in slow system (5), and then determine $\frac{df}{d\alpha} = f_\alpha$ we get that:

$$f_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -E \end{bmatrix}$$

Then

$$f_\alpha(U_3\,, \tilde{\alpha}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -E_3 \end{bmatrix}$$

Therefore:

$$\tilde{W}^T \cdot f_\alpha(U_3, \tilde{\alpha}) = -\tilde{w}_4 E_3 \neq 0$$

Consequently, according to Sotomayor theorem, if in addition to the above, the following holds

$$\tilde{W}^T \cdot \left[D^2 f(U_3, \tilde{\alpha}) \cdot (\tilde{K}, \tilde{K})\right]$$

Here $Df(U_3, \tilde{\alpha})$ is the Jacobian matrix at U_3 and $\widetilde{\alpha}$, then the slow system (5) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur.

Now since we have that:

$$\left[D^2 f(U_3, \tilde{\alpha}) \cdot (\tilde{K}, \tilde{K})\right] = \begin{bmatrix} -2\sigma\tilde{q}\tilde{k}_4^2 \\ 2\sigma\tilde{q}\tilde{k}_4^2 \\ 0 \\ -2\theta V_1(\rho\tilde{z} + \tilde{h})\tilde{k}_4^2 \end{bmatrix}$$

Therefore:

$$\tilde{W}^T \cdot \left[D^2 f(U_3, \tilde{\alpha}) \cdot (\tilde{K}, \tilde{K})\right] = -2\tilde{w}_4\tilde{k}_4^2[\sigma\tilde{q}(\tilde{x} - \tilde{y}) + \theta V_1(\rho\tilde{z} + \tilde{h})] \neq 0$$

Then the slow system (5) has a transcritical bifurcation at U_3 when the parameter α passes through the bifurcation value $\widetilde{\alpha}$.

7 Hopf bifurcation analysis of fast system

In order to, investigate the Hopf bifurcation of the model fast system (2), we will follow the Liu approach[19] as shown in the following theorem.

Theorem 10. *Assume that the positive equilibrium point of fast system (2) exists and if the following condition is holds:*

$$\beta c V_1(m + d) + m(m + d)(c + \beta T_1) > pm\beta T_1 \tag{37}$$

Then a simple Hopf bifurcation of the fast system (2) occurs at $p^* = \frac{R_9}{R_{10}}$, where

$$\begin{aligned} R_9 &= (\beta V_1 + m)(m + d)(\beta V_1 + 2m + d) + [m + c + \beta(V_1 + T_1)][mc + \beta(cV_1 + mT_1)] \\ &\quad + (m + d)(c + \beta T_1)[3m + d + c + \beta(2V_1 + T_1)] \\ R_{10} &= \beta T_1[m + d + c + \beta(V_1 + T_1)] \end{aligned}$$

Proof. Clearly, the characteristic equation of the Jacobian matrix $J(U_1)$ of the fast system (2) at the positive equilibrium point (U_1) is given by:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$$

Here:

$$\begin{aligned} A_1 &= -[a_{11} + a_{22} + a_{33}] \\ &= c + d + 2m + \beta(V_1 + T_1) \\ A_2 &= [a_{11}a_{22} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}] \\ A_3 &= -[a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} - a_{22}a_{13}a_{31} - a_{11}a_{23}a_{32}] \\ &= \beta cV_1(m + d) + m(m + d)(c + \beta T_1) - pm\beta T \end{aligned}$$

Now, according to the Liu approach a simple Hopf bifurcation occurs if and only if $A_1(\mu^*) > 0$, $A_3(\mu^*) > 0$, $\Delta(\mu^*) = 0$ and $\frac{d\Delta}{d\mu}\Big|_{\mu=\mu^*} \neq 0$, where, μ^* is a critical value of the key parameter. Now, by substituting the value of in these equations we obtain:

$$A_1(p^*) = c + d + 2m + \beta(V_1 + T_1)$$

$$A_3(p^*) = \beta cV_1(m + d) + m(m + d)(c + \beta T_1) - p^* m\beta T$$

Clearly, $A_3(p^*)$ is positive under the conditions (37), now to check the $\Delta(p^*)$ in the following

$$\begin{aligned} \Delta(p^*) &= A_1A_2 - A_3 \\ &= (\beta V_1 + m)(m + d)(\beta V_1 + 2m + d) + [m + c + \beta(V_1 + T_1)][mc + \beta(cV_1 + mT_1)] \\ &\quad +(m + d)(c + \beta T_1)[3m + d + c + \beta(2V_1 + T_1)] - p^*\beta T_1[m + d + c + \beta(V_1 + T_1)] \end{aligned}$$

Hence, it easyto verify that $\Delta(p^*) = 0$. Finally, since $\frac{d\Delta}{dp}\Big|_{p=p^*} = -\beta T_1[m + d + c + \beta(V_1 + T_1)] \neq 0$, Thus, a simple Hopf bifurcation occurs in fast system (2) at $p = p^*$.

8 Numerical simulation of fast and slow systems:

In this section, the global dynamics of systems (2) and (5) is studied numerically. The objectives of this study are confirming our obtained analytical results and understand the effects of some parameters on the dynamics of fast and slow systems. Consequently, first system (2) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions (24-25) of positive equilibrium point , system (2) has a globally asymptotically stable positive equilibrium point as shown in following figure.

$$\Lambda = 750\ ;\ \beta = 0.001\ ;\ d = 0.01\ ;\ p = 0.3\ ;\ m = 0.2\ ;\ c = 0.1\ ;\ a = 0.1\ ;\ E = 0.2 \tag{38}$$

Obviously, Fig. 2, shows that the solution of system (2) approaches asymptotically to the positive equilibrium point has a globally asymptotically stable $U_1 = (233, 3349, 3014)$ starting from three different initial points and this is confirming our obtained analytical results.

Now, Fig. 3 as the infected incidence rate of disease resulting from contact between health cells with parasite increases (through increasing β), the trajectory of system (2) approaches asymptotically to the positive equilibrium point U_1. In fact as β increases it is observed that the number of health cells decrease and the number of infected cells and parasite load increases.

However, Fig. 4 it is clear that as the new parasite members increases the system approaches asymptotically to the endemic equilibrium point of system (2), but in case decreasing in this rate then trajectory of system (2) approaches to the disease free equilibrium point.

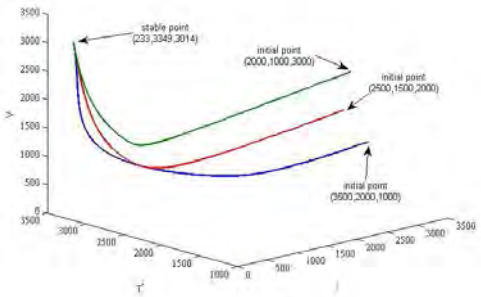

Fig. 2: phase plot of the solution of system (2), starting from three different initial points

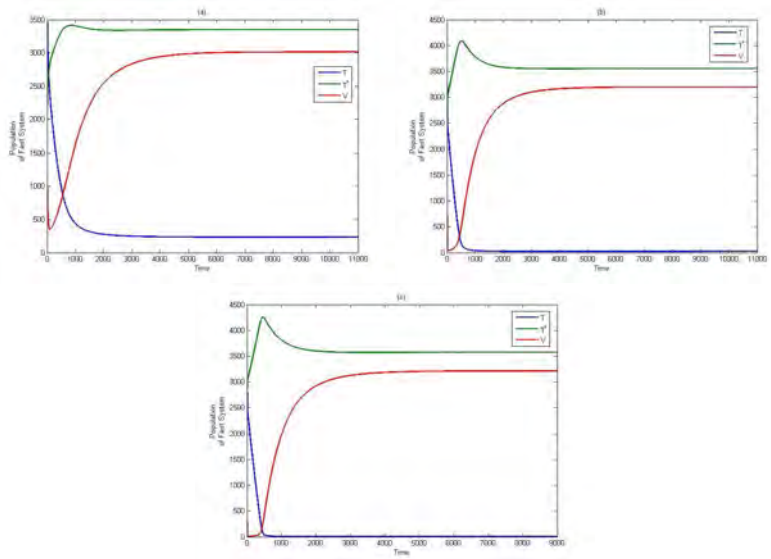

Fig. 3: Time series of the solution of system (2). (a) for $\beta = 0.001$, (b) for $\beta = 0.01$, (c) for $\beta = 0.005$

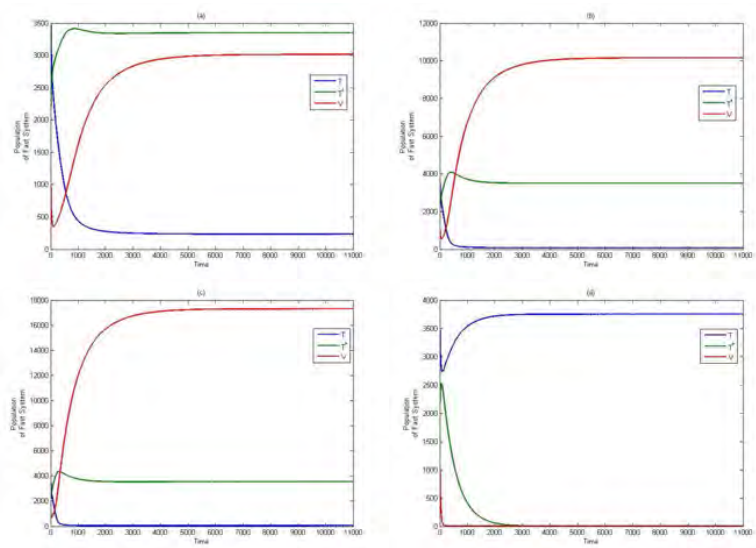

Fig. 4: Time series of the solution of system (2). (a) for $p = 0.3$, (b) for $p = 0.5$, (c) for $p = 0.7$

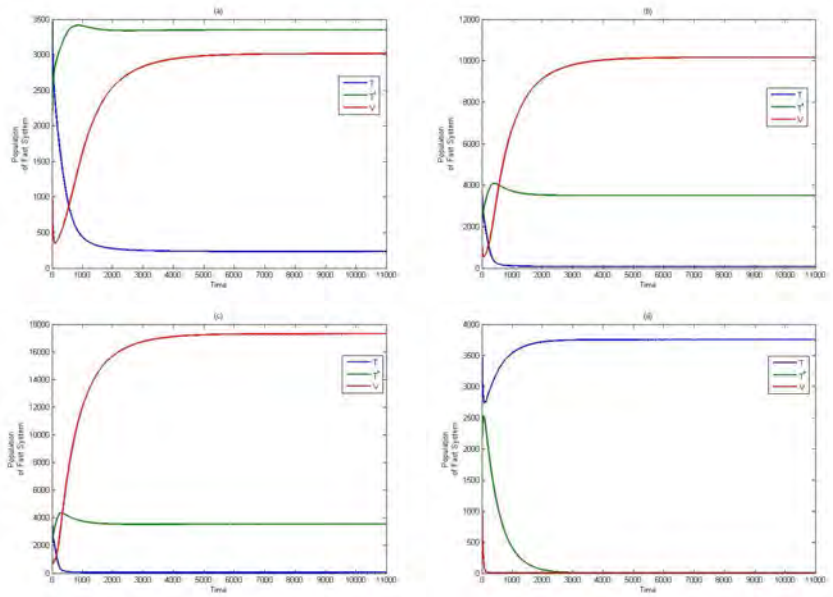

Fig. 5: Time series of the solution of system (5). (a) trajectories of S, (b) trajectories of L and (c) trajectories of I, (d) trajectories of E, the blue starting at (0.2,0.4,0.5,500) and green starting at (1000,2000, 2000,1000)

In the following the global dynamics of slow system (5) for disease free equilibrium point is carried out. Slow system (5) is solved numerically for the following different set of parameters, which satisfies condition (26), and then the trajectories are drawn in Fig. 5.

$$\psi = 500,\ \sigma = 0.01\ ;\ \gamma = 0.01\ ;\ \varepsilon = 0.01\ ;\ \mu = 0.1\ ;\ \theta = 0.002\ ;\ \rho = 0.01,\ \alpha = 0.6\ ;\ V_1 = 0.4 \quad (39)$$

Clearly, Fig. 5, shows that the solution of system (5) approaches asymptotically to the disease free equilibrium point has a globally asymptotically stable starting from two different initial points and this is confirming our obtained analytical results.

But, the following set of the global dynamics of slow system (5) for endemic equilibrium point . Slow system (5) is solved numerically for the following different set of parameters with different initial conditions, and then the trajectories are drawn in Fig 6.

$$\psi = 500,\ \sigma = 0.1\ ;\ \gamma = 0.01\ ;\ \varepsilon = 0.01\ ;\ \mu = 0.1\ ;\ \theta = 0.6\ ;\ \rho = 0.01,\ \alpha = 0.1\ ;\ V_1 = 0.6 \quad (40)$$

From Fig. 6, shows that the solution of system (5) approaches asymptotically to the endemic equilibrium point has a globally asymptotically stable $U_3 = (2511, 2280, 207, 0.9)$ starting from two different initial points.

Now, Fig. 7 that, as the infection rate increases from 0.01 to 0.3 the trajectory of system (5) approaches asymptotically to the endemic equilibrium point. But the number of susceptible individuals decreases while the number of the latent, infected and environment contamination increases.

Obviously from these figures, increases the parameter ε with keeping other parameters fixed as in (40) causes still in the stability of system (5) to endemic equilibrium point and the number of susceptible individuals no effect in this change while the number of the latent, infected and environment contamination increases.

Now, Fig. 9 Therefore, the recovery rate increases then the system (5) still approaches to endemic equilibrium point and the number of susceptible individuals increases but not effect of the number of the latent, while the infected and environment contamination decreases.

Finally, the effect of the parasite load rate increases the disease free equilibrium point of system (5) becomes unstable point and the trajectory of system (5) approaches asymptotically to the endemic equilibrium point, and then the number of susceptible individuals decrease while the number of the latent, infected and environment contamination increases.

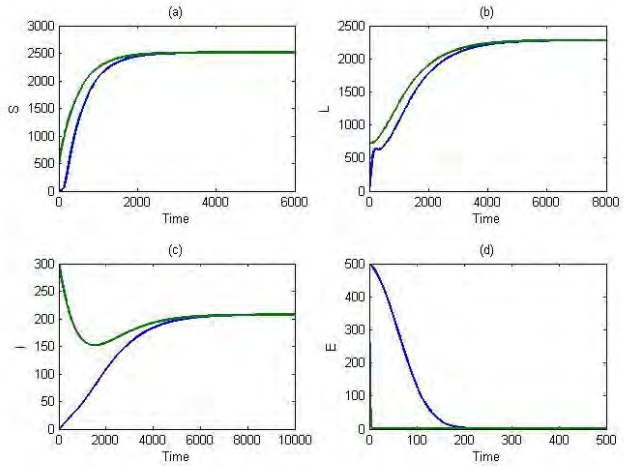

Fig. 6: Time series of the solution of system (5). (a) trajectories of S, (b) trajectories of L and (c) trajectories of I, (d) trajectories of E, the blue starting at (0.2,0.4,0.5,500) and green starting at (1500,2000, 2000,1000)

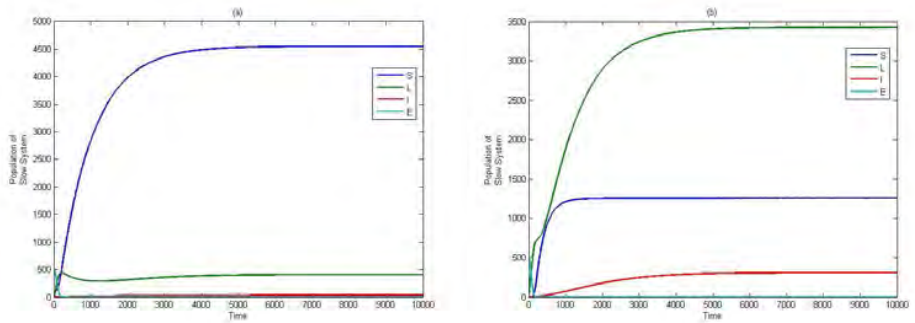

Fig. 7: Time series of the solution of system (5). (a) for $\sigma = 0.01$, (b) for $\sigma = 0.3$

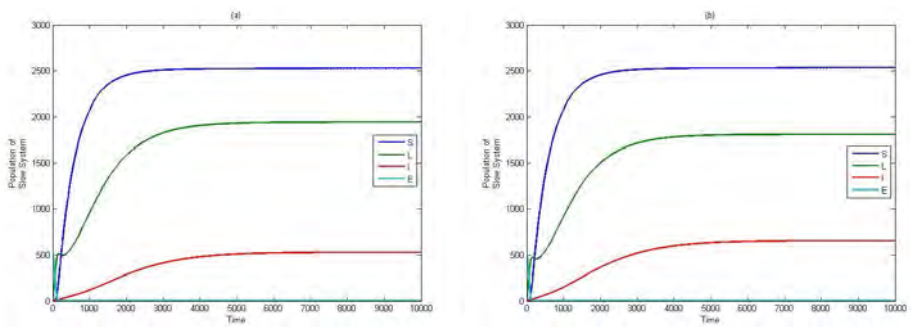

Fig. 8: Time series of the solution of system (5). (a) for $\varepsilon = 0.05$, (b) for $\varepsilon = 0.2$

9 Discussion

In this paper, a mathematical model has been studied and analyzed to study the effect of parasite load and environment contamination on the dynamical behavior of SLI epidemic model. The existence and the stability analysis of all possible equilibrium point are studied analytically as well as numerically. It is observed that the system (2) has a saddle node bifurcation near the positive equilibrium point, but neither transcritical nor pitchfork bifurcation can accrue. While, the system (5) has a transcritical bifurcation near the disease free point and has a saddle node bifurcation near the endemic equilibrium point. Further the system (2) has a Hopf bifurcation near the positive equilibrium point. Finally according to the numerically simulation the following results are obtained:

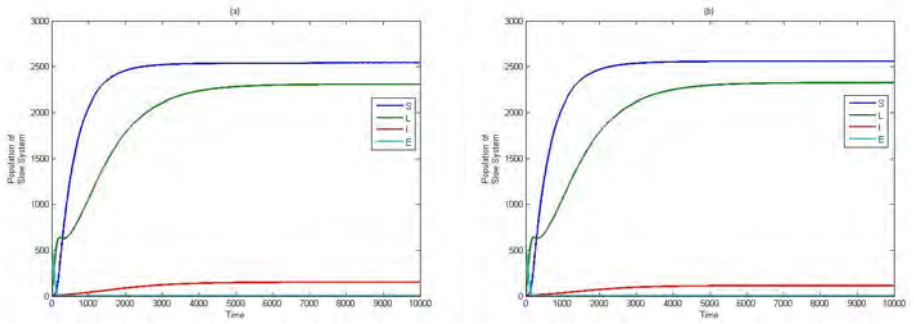

Fig. 9: Time series of the solution of system (5). (a) for $\gamma = 0.02$, (b) for $\gamma = 0.1$

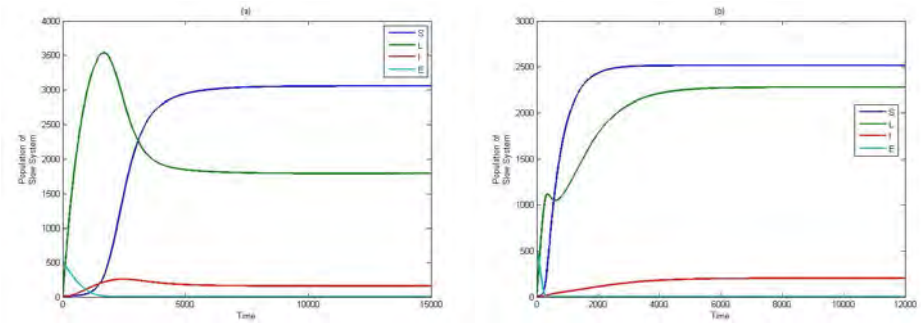

Fig. 10: Time series of the solution of system (5). (a) for $V_1 = 0.01$, (b) for $V_1 = 0.9$

- As the infected incidence rate of disease resulting (β), the trajectory of system (2) approaches asymptotically to the positive equilibrium point U_1. In fact as β increases it is observed that the number of health cells decrease and the number of infected cells and parasite load increases. Also if the new parasite members increase (p) we get same result.
- As infection rate increases the trajectory of system (5) still approaches to the endemic equilibrium point. But the number of susceptible individuals decreases while the number of the latent, infected and environment contamination increases.
- The evolution rate of disease (ε) increases in this causes the solution of system (5) still approaches to endemic equilibrium point and this change not effect on the number of susceptible individuals while the number of the latent, infected and environment contamination increases.
- The recovery rate increases then the system (5) still approaches to endemic equilibrium point and the number of susceptible individuals increases but not effect of the number of the latent, while the infected and environment contamination decreases.
- The parasite load rate increases the trajectory of system (5) approaches asymptotically to the endemic equilibrium point, and then the number of susceptible individuals decrease while the number of the latent, infected and environment contamination increases.

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