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# Mathematical modeling on ecosystem consisting of two hosts and one commensal with mortality rate for the first and third species*

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**Abstract.** In this paper, the system comprises of two hosts $S_1$, $S_2$ and one commensal $S_3$ ie., $S_1$ and $S_2$ both benefit $S_3$, without getting themselves affected either positively or adversely. Further, $S_1$ and $S_2$ are neutral. Here all the three species posses limited resources. The model equations constitute a set of three first order non-linear simultaneous differential equations. Criteria for the asymptotic stability of all the eight equilibrium states are established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. Trajectories of the perturbations over the equilibrium states are illustrated. Further the global stability of the system is established with the aid of suitably constructed Liapunov's functions and the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme.

**Keywords:** asymptotically stable, commensal, equilibrium state, host, trajectories, stable

## 1 Introduction

Ecology, a branch evolutionary biology, deals with living species that coexist in a physical environment that sustain themselves on common resources. It is a common observation that the species of same nature can not flourish in isolation without any interaction with species of different kinds. Significant researches in the area of theoretical ecology have been discussed by Kot[4] and by [1]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and Synecology, which are described in the treatises of Arumugam [3] and Sharma[17]. Syn-ecology is an ecosystem comprised of two or more distinct species. Species interact with each other in one way or other. The Ecological interactions can be classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation, Parasitism and so on. Mathematical Modeling plays a vital role in providing insight into the mutual relationships (positive, negative) between the interacting species. The general concepts of Modeling in Biological Science have been initiated by several authors Ma[10], Murray[5] and Sze-Bi Hsu[6]. Recently the authors Papa Rao et al.[15], Shivareddy et al.[16], Srinivas[18] and Kumar et al.[9] discussed three species ecological models such as predation, completion and commensalism. Srinivas[19] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan et al.[7]studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Acharyulu et al.[2, 20] derived some productive results on various mathematical models of ecological Ammensalism with multifarious resources in the manifold directions.Further, Kumar[8] studied some mathematical models of ecological commensalism. The present author Prasad[11–14] investigated continuous and discrete models on the three species syn-ecosystems.

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Commensalism is a symbiotic interaction between two populations where one population ($S_1$) gets benefit from ($S_2$) while the other ($S_2$) is neither harmed nor benefited due to the interaction with ($S_1$). The benefited species ($S_1$) is called the commensal and the other ($S_2$) is called the host. Some real-life examples of commensalism are presented below.

i. The clownfish shelters among the tentacles of the sea anemone, while the sea anemone is not effected.

ii. A squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.

iii. A flatworm attached to the horse crab and eating the crab's food, while the crab is not put to any disadvantage.

## 2 Basic equations of the model

The model equations for the three species ecosystem is given by the following system of first order non-linear ordinary differential equations employing the following notation.

| | |
|---|---|
| $N_i\left(t\right)$ | : The population strength of $S_i$ at time $t$, $i = 1, 2, 3$; |
| $t$ | : Time instant; |
| $d_i$ | : Natural death rate of $S_i$, $i = 1, 3$; |
| $a_2$ | : Natural growth rate of $S_2$; |
| $aii$ | : Self inhibition coefficients of $S_i$, $i = 1, 2, 3$; |
| $a_{13}, a_{23}$ | : Interaction coefficients of $S_1$ due to $S_3$ and $S_2$ due to $S_3$; |
| $e_i = \frac{d_i}{a_{ii}}$ | : Extinction coefficient of $S_i$, $i = 1, 3$; |
| $k_2 = \frac{a_2}{a_{22}}$ | : Carrying capacities of $S_2$. |

Further, the variables $N_1$, $N_2$, $N_3$ are non-negative and the model parameters $d_1$, $a_2$, $d_3$, $a_{11}$, $a_{12}$, $a_{22}$, $a_{33}$, $a_{13}$, $a_{23}$, $e_1$, $k_2$, $e_3$ are assumed to be non-negative constants.

The model equations for the growth rates of $S_1, S_2, S_3$ are,

$$\frac{dN_1}{dt} = -N_1\left(d_1 + a_{11}N_1\right), \tag{1}$$

$$\frac{dN_2}{dt} = N_2\left(a_2 - a_{22}N_2\right), \tag{2}$$

$$\frac{dN_3}{dt} = N_3\left(-d_3 - a_{33}N_3 + a_{13}N_1 + a_{23}N_2\right). \tag{3}$$

## 3 Equilibrium state

The system under investigation has eight equilibrium states given by

$$\frac{dN_i}{dt} = 0, \; i = 1, \; 2, \; 3, \tag{4}$$

and these are classified into four categories.

i. Fully washed out state.

$$E_1 \; : \; \bar{N}_1 = 0, \; \bar{N}_2 = 0, \; \bar{N}_3 = 0.$$

ii. States in which only two of the tree species are washed out while the other one is not.

$$\begin{aligned} E_2 \; &: \; \bar{N}_1 = -e_1, \; \bar{N}_2 = 0, \; \bar{N}_3 = 0, \\ E_3 \; &: \; \bar{N}_1 = 0, \; \bar{N}_2 = k_2, \; \bar{N}_3 = 0, \\ E_4 \; &: \; \bar{N}_1 = 0, \; \bar{N}_2 = 0, \; \bar{N}_3 = -e_3. \end{aligned}$$

iii. States in which only one of the tree species is washed out while the other two are not.

$$E_5 \; : \; \bar{N}_1 \; = \; -e_1, \; \bar{N}_2 \; = k_2, \; \bar{N}_3 \; = \; 0,$$
$$E_6 \; : \; \bar{N}_1 \; = -e_1 \; , \; \bar{N}_2 \; = \; 0, \; \bar{N}_3 \; = \; -e_3 - \frac{a_{13}e_1}{a_{33}},$$
$$E_7 \; : \; \bar{N}_1 \; = 0 \; , \; \bar{N}_2 \; = \; k_2, \; \bar{N}_3 \; = \frac{a_{23}k_2}{a_{33}} - e_3.$$

iv. The co-existent state (or) normal steady state.

$$E_8 \; : \; \bar{N}_1 \; = -e_1, \; \bar{N}_2 \; = k_2, \; \bar{N}_3 \; = \; \frac{a_{23}k_2 - a_{13}e_1}{a_{33}} - \; e_3.$$

## 4 Stability analysis of the equilibrium states

Let

$$N = (N_1, N_2, N_3) = \bar{N} \; + \; U, \tag{5}$$

where $U \; = \; (u_1, \; u_2, \; u_3)^T$ is a small perturbation over the equilibrium state $\bar{N} = \left(\bar{N}_1, \; \bar{N}_2, \bar{N}_3\right)$.

The basic Eqs. (1), (2) and (3) are quasi-linearized to obtain the equations for the perturbed state as

$$\frac{dU}{dt} \; = \; AU, \tag{6}$$

with

$$A \; = \; \begin{bmatrix} -d_1 - 2a_{11}\bar{N}_1 & 0 & 0 \\ 0 & a_2 - 2a_{22}\bar{N}_2 & 0 \\ a_{13}\bar{N}_3 & a_{23}\bar{N}_3 & -d_3 - 2a_{33}\bar{N}_3 + a_{13}\bar{N}_1 + a_{23}\bar{N}_2 \end{bmatrix}. \tag{7}$$

The characteristic equation for the system is given by

$$|A - \lambda I| = 0. \tag{8}$$

The equilibrium state is stable, if all the roots of the Eq. (8) are negative in case they are real or have negative real parts, in case they are complex.

### 4.1 Stability of $E_1$

In this state, we have

$$A(E_1) \; = \; \begin{bmatrix} -d_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}. \tag{9}$$

The characteristic equation is

$$(\lambda + d_1)\,(\lambda - a_2)\,(\lambda + d_3) \; = \; 0. \tag{10}$$

The characteristic roots of Eq. (10) are $-d_1, a_2$ and $-d_3$. Since one of these three is positive. Hence the fully washed out state is unstable and the solutions of the Eq. (6) are

$$u_i \; = \; u_{i0}\, e^{-d_i t}; u_2 \; = \; u_{20}\, e^{a_2 t}, \;\; i \; = \; 1, \; 3, \tag{11}$$

where $u_{10}, u_{20}, u_{30}$ are the initial values of $u_1, u_2, u_3$ respectively.

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are $\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{d_1}} \; = \; \left(\frac{u_2}{u_{20}}\right)^{-\frac{1}{a_2}} \; = \; \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{d_3}}.$

### 4.2 Stability of $E_2$

In this case, we have

$$A(E_2) = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -d_3 - a_{13}e_1 \end{bmatrix}. \tag{12}$$

The characteristic roots of Eq. (12) are $d_1$,$a_2$ and $-d_3 - a_{13}e_1$. Since one of these three is positive. Hence the fully washed out state is unstable and the solutions of the Eq. (6) are

$$u_1 = u_{10}e^{d_1 t};\ u_2 = u_{20}\ e^{a_2 t};\ u_3 = u_{30}e^{-(d_3+a_{13}e_1)t}. \tag{13}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by $\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{d_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{-\frac{1}{d_3+a_{13}e_1}}$.

### 4.3 Stability of $E_3$

In this case, we have

$$A(E_3) = \begin{bmatrix} -d_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & a_{23}k_2 - d_3 \end{bmatrix}. \tag{14}$$

The characteristic roots of Eq. (14) are $-d_1$ $-a_2$ and $a_{23}k_2 - d_3$.

*Case 1.* When $a_{23}k_2 - d_3 < 0$

In this case all the three roots are negative, hence the state is stable. The Eq. (6) yield the solutions,

$$u_1 = u_{10}e^{-d_1 t};\ u_2 = u_{20}e^{-a_2 t};\ u_3 = u_{30}e^{-(a_{23}k_2+d_3)t}. \tag{15}$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$.

*Case 2.* When $a_{23}k_2 - d_3 = 0$

In this case the state is neutrally stable and the Eq. (6) yield the solutions,

$$u_1 = u_{10}e^{-d_1 t};\ u_2 = u_{20}e^{-a_2 t};\ u_3 = u_{30}. \tag{16}$$

*Case 3.* When $a_{23}k_2 - d_3 > 0$

In this case the state is unstable and the solution curves of Eq. (6) are given by,

$$u_1 = u_{10}e^{-d_1 t};\ u_2 = u_{20}e^{-a_2 t};\ u_3 = u_{30}e^{(a_{23}k_2+d_3)t}. \tag{17}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by $\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{d_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{d_3-a_{23}k_2}}$.

### 4.4 Stability of $E_4$

In this case, we get

$$A(E_4) = \begin{bmatrix} -d_1 & 0 & 0 \\ 0 & a_2 & 0 \\ -a_{13}e_3 & -a_{23}e_3 & d_3 \end{bmatrix}. \tag{18}$$

The characteristic roots are $-d_1, a_2$ and $d_3$. Since two of these three roots are positive, hence the state is unstable and the Eq. (6) yield the solutions,

$$u_1 = u_{10}e^{-d_1 t};\ u_2 = u_{20}e^{a_2 t};\ u_3 = u_{10}A_1 e^{-d_1 t} + u_{20}A_2 e^{a_2 t} + (u_{30} - u_{10}A_1 - u_{20}A_2)\, e^{d_3 t}, \tag{19}$$

where

$$A_1 = \frac{a_{13}e_3}{d_1 + d_3} > 0 \text{and} A_2 = \frac{a_{23}e_3}{d_3 - a_2}; d_3 \neq a_2. \tag{20}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{-d_1}$$

and

$$u_3 = A_1 u_{10}\left(\frac{u_2}{u_{20}}\right)^{-\frac{d_1}{a_2}} + A_2 u_2 + (u_{30} - A_1 u_{10} - A_2 u_{20})\left(\frac{u_2}{u_{20}}\right)^{\frac{d_3}{a_2}}.$$

## 4.5 Stability of $E_5$

In this case, we get

$$A(E_5) = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & a_{23}k_2 - a_{13}e_1 - d_3 \end{bmatrix}. \tag{21}$$

The characteristic roots are $d_1, -a_2$ and $a_{23}k_2 - a_{13}e_1 - d_3$. Since one of these three is positive. Hence the fully washed out state is unstable and the solutions of the Eq. (6) are

$$u_1 = u_{10}e^{d_1 t};\ u_2 = u_{20}e^{-a_2 t};\ u_3 = u_{30}e^{(a_{23}k_2 - a_{13}e_1 - d_3)t}. \tag{22}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by $\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{d_1}} = \left(\frac{u_2}{u_{20}}\right)^{-\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_{23}k_2 - a_{13}e_1 - d_3}}$.

## 4.6 Stability of $E_6$

In this case, we get

$$A(E_6) = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & a_2 & 0 \\ -a_{13}e_3 - \frac{a_{13}^2 e_1}{a_{33}} & -a_{23}e_3 - \frac{a_{13}a_{23}e_1}{a_{33}} & d_3 + a_{13}e_1 \end{bmatrix}. \tag{23}$$

The characteristic roots are $d_1, a_2$ and $d_3 + a_{13}e_1$. Since all these three roots are positive, hence the state is unstable and the Eq. (6) yield the solutions,

$$u_1 = u_{10}e^{d_1 t};\ u_2 = u_{20}e^{a_2 t};\ u_3 = u_{10}B_1 e^{d_1 t} + u_{20}B_2 e^{a_2 t} + (u_{30} - u_{10}B_1 - u_{20}B_2)\, e^{(d_3 + a_{13}e_1)t}, \tag{24}$$

where

$$B_1 = \frac{a_{13}(d_3 + a_{13}e_1)}{a_{33}(d_3 + a_{13}e_1 - d_1)}; B_2 = \frac{a_{23}(d_3 + a_{13}e_1)}{a_{33}(d_3 + a_{13}e_1 - a_2)}; d_3 + a_{13}e_1 \neq d_2; d_3 + a_{13}e_1 \neq a_2. \tag{25}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{d_1}$$

and

$$u_3 = B_1 u_{10}\left(\frac{u_2}{u_{20}}\right)^{\frac{d_1}{a_2}} + B_2 u_2 + (u_{30} - B_1 u_{10} - B_2 u_{20})\left(\frac{u_2}{u_{20}}\right)^{\frac{d_3 + a_{13}e_1}{a_2}}.$$

### 4.7 Stability of $E_7$

In this case, we get

$$A(E_7) = \begin{bmatrix} -d_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ a_{13}\left(\frac{a_{23}k_2}{a_{33}} - e_3\right) & a_{23}\left(\frac{a_{23}k_2}{a_{33}} - e_3\right) & d_3 - a_{23}k_2 \end{bmatrix}. \tag{26}$$

The characteristic roots are $-d_1, -a_2$ and $d_3 - a_{23}k_2$.

*Case 4.* When $d_3 - a_{23}k_2 < 0$

In this case all the three roots are negative, hence the state is stable. The Eq. (6) yield the solutions,

$$\begin{aligned} u_1 &= u_{10}e^{-d_1 t};\ u_2 = u_{20}e^{-a_2 t}; \\ u_3 &= u_{10}C_1e^{-d_1 t} + u_{20}C_2e^{-a_2 t} + (u_{30} - u_{10}C_1 - u_{20}C_2)\, e^{-(d_3+a_{23}k_2)t}, \end{aligned} \tag{27}$$

where

$$C_1 = \frac{a_{13}(d_3 - a_{23}k_2)}{a_{33}(d_1 + d_3 - a_{23}k_2)}; C_2 = \frac{a_{23}(d_3 - a_{23}k_2)}{a_{33}(a_2 + d_3 - a_{23}k_2)}; d_1 + d_3 \neq a_{23}k_2; a_2 + d_3 \neq a_{23}k_2. \tag{28}$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$.

*Case 5.* When $d_3 - a_{23}k_2 = 0$

In this case the state is neutrally stable and the Eq. (6) yield the solutions,

$$u_1 = u_{10}e^{-d_1 t};\ u_2 = u_{20}e^{-a_2 t};\ u_3 = u_{30}. \tag{29}$$

*Case 6.* When $d_3 - a_{23}k_2 > 0$

In this case the state is unstable and the solution curves of (4.2) are given by

$$\begin{aligned} u_1 &= u_{10}e^{-d_1 t};\ u_2 = u_{20}e^{-a_2 t}; \\ u_3 &= u_{10}C_1e^{-d_1 t} + u_{20}C_2e^{-a_2 t} + (u_{30} - u_{10}C_1 - u_{20}C_2)\, e^{(d_3+a_{23}k_2)t}. \end{aligned} \tag{30}$$

The trajectories in $u_1 - u_3$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{d_1}$$

and

$$u_3 = C_1u_{10}\left(\frac{u_2}{u_{20}}\right)^{\frac{d_1}{a_2}} + C_2u_2 + (u_{30} - C_1u_{10} - C_2u_{20})\left(\frac{u_2}{u_{20}}\right)^{\frac{a_{23}k_2 - d_3}{a_2}}.$$

### 4.8 Stability of $E_8$

In this case, we get

$$A(E_8) = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ \frac{a_{13}}{a_{33}}(a_{23}k_2 - a_{13}e_1 - d_3) & \frac{a_{23}}{a_{33}}(a_{23}k_2 - a_{13}e_1 - d_3) & d_3 + a_{13}e_1 - a_{23}k_2 \end{bmatrix}. \tag{31}$$

The characteristic roots are $d_1, -a_2$ and $d_3 + a_{13}e_1 - a_{23}k_2$. Since one of these three is positive. Hence the fully washed out state is unstable and the solutions of the Eq. (6) are

$$u_1 = u_{10}e^{d_1 t};\ u_2 = u_{20}e^{-a_2 t};\ u_3 = u_{10}D_1e^{d_1 t} + u_{20}D_2e^{-a_2 t} + (u_{30} - u_{10}D_1 - u_{20}D_2)\, e^{(d_3+a_{13}e_1-a_{23}k_2)t}, \tag{32}$$

where

$$D_1 = \frac{a_{13}(d_3 + a_{13}e_1 - a_{23}k_2)}{a_{33}(d_3 + a_{13}e_1 - d_1 - a_{23}k_2)}; D_2 = \frac{a_{23}(d_3 + a_{13}e_1 - a_{23}k_2)}{a_{33}(d_3 + a_{13}e_1 + a_2 - a_{23}k_2)}, \tag{33}$$

with

$$d_3 + a_{13}e_1 \neq d_1 + a_{23}k_2; a_2 + d_3 + a_{13}e_1 \neq a_{23}k_2. \tag{34}$$

Trajectories in $u_1 - u_3$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{-a_2} = \left(\frac{u_2}{u_{20}}\right)^{d_1}$$

and

$$u_3 = D_1 u_{10} \left(\frac{u_2}{u_{20}}\right)^{-\frac{d_1}{a_2}} + D_2 u_2 + (u_{30} - D_1 u_{10} - D_2 u_{20}) \left(\frac{u_2}{u_{20}}\right)^{\frac{a_{23}k_2 - a_{13}e_1 - d_3}{a_2}}.$$

## 5 Liapunov's function for global stability

In section 4, we discussed the local stability of all eight equilibrium states. From which only the states $E_3$ and $E_7$ are stable. We now examine the global stability of dynamical system (1), (2) and (3) at these two states by suitable Liapunov's functions.

**Theorem 1.** *The equilibrium state $E_3\,(0, k_2, 0)$ is globally asymptotically stable.*

*Proof.* Let us consider the following Liapunov's function

$$V(N_2) = N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right). \tag{35}$$

Now, the time derivative of V, along with solution of (2) can be written as

$$\frac{dV}{dt} = \left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt}, \frac{dV}{dt} = -a_{22}\big(N_2 - \bar{N}_2\big)^2. \tag{36}$$

Which is negative definite. Hence, the state is globally asymptotically stable.

**Theorem 2.** *The equilibrium state $E_7\,\left(0, \bar{N}_2, \bar{N}_3\right)$ is globally asymptotically stable.*

*Proof.* Let us consider the following Liapunov's function

$$V\,(N_2, N_3) = N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right) + l_1\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right], \tag{37}$$

where $l_1$ is a suitable constant to be determined as in the subsequent steps.

Now, the time derivative of V, along with solutions of Eq. (2) and Eq. (3) can be written as

$$\begin{aligned}
\frac{dV}{dt} &= \left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + l_1\left(\frac{N_3 - \bar{N}_3}{N_3}\right)\frac{dN_3}{dt},\\
\frac{dV}{dt} &= -a_{22}\big(N_2 - \bar{N}_2\big)^2 + l_1 a_{23}\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right) + l_1\left[-a_{33}\big(N_3 - \bar{N}_3\big)^2\right],\\
\frac{dV}{dt} &= -\left[\sqrt{a_{22}}\left(N_2 - \bar{N}_2\right) + \sqrt{l_1 a_{33}}\left(N_3 - \bar{N}_3\right)\right]^2\\
&\quad + \left(2\sqrt{l_1 a_{22} a_{33}} + l_1 a_{23}\right)\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right).
\end{aligned} \tag{38}$$

The positive constant $l_1$ as so chosen that, the coefficient of $\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right)$ in Eq. (38) vanish. Then we have, $l_1 = \frac{4a_{22}a_{33}}{a_{23}^2} > 0$ and, with this choice of the constant $l_1$,

$$V\left(N_2, N_3\right) = N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right) + \frac{4a_{22}a_{33}}{a_{23}^2}\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right], \tag{39}$$

$$\frac{dV}{dt} = -\sqrt{a_{22}}\left[\left(N_2 - \bar{N}_2\right) + \frac{2a_{33}}{a_{23}}\left(N_3 - \bar{N}_3\right)\right]^2, \tag{40}$$

which is negative definite. Hence, the steady state is globally asymptotically stable.

## 6 A numerical approach of the growth rate equations

The numerical solutions of the growth rate Eq. (1), (2) and (3) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Fig. 1, 2, 3 and 4.

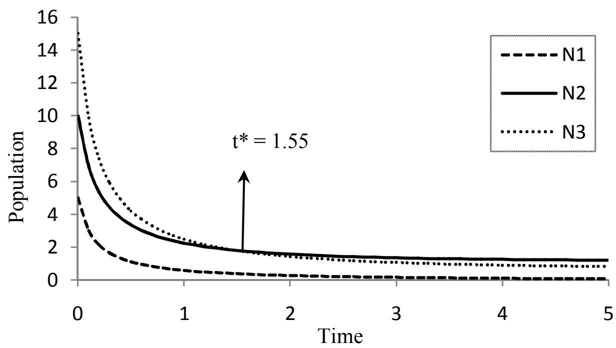


Fig. 1: Variation of population against time for $d_1 = 0.26$, $a_{11} = 1.26$, $a_{13} = 1.72$, $a_2 = 0.6$, $a_{22} = 0.52$, $a_{23} = 6.3$, $d_3 = 3.78$, $a_{33} = 4.9$, $N_1 = 5$, $N_2 = 10$, $N_3 = 15$

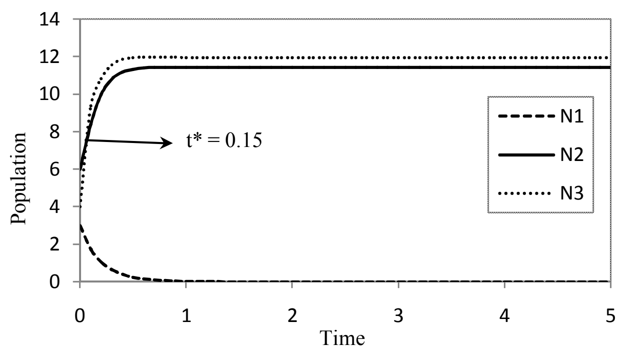


Fig. 2: Variation of population against time for $d_1 = 4.56$, $a_{11} = 0.4$, $a_{13} = 4.56$, $a_2 = 9.14$, $a_{22} = 0.8$, $a_{23} = 9.1$, $d_3 = 2.78$, $a_{33} = 8.48$, $N_1 = 3$, $N_2 = 6$, $N_3 = 4$

## 7 Observations of the above graphs

*Case 7.* In this case the initial values of $S_1$, $S_2$, $S_3$ are in increasing order. The natural death rate of $S_1$ is greater than of $S_3$. Initially the third species dominates over the second species till the time instant $t^* = 1.55$

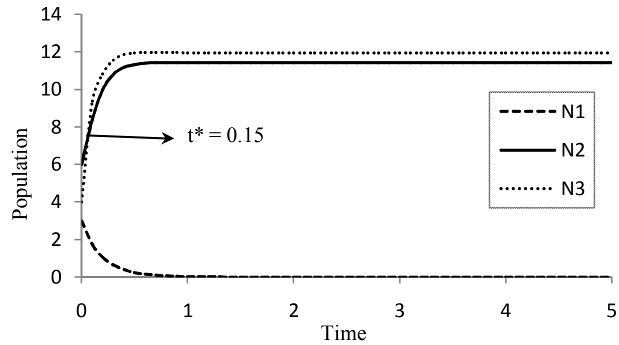


Fig. 3: Variation of population against time for $d_1 = 0.3$, $a_{11} = 1.4$, $a_{13} = 1.4$, $a_2 = 2.52$, $a_{22} = 4.1$, $a_{23} = 7.48$, $d_3 = 3.98$, $a_{33} = 2.52$, $N_1 = 5$, $N_2 = 2$, $N_3 = 8$

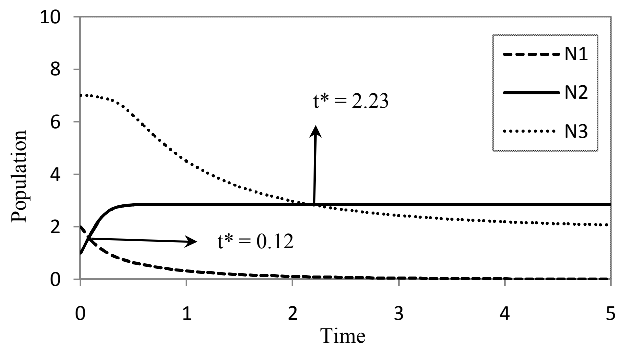


Fig. 4: Variation of population against time for $d_1 = 0.72$, $a_{11} = 1.4$, $a_{13} = 4.44$, $a_2 = 11$, $a_{22} = 3.84$, $a_{23} = 2.4$, $d_3 = 5.3$, $a_{33} = 0.8$, $N_1 = 2$, $N_2 = 1$, $N_3 = 7$

and thereafter the dominance is reversed. Further it is evident that all the three species asymptotically converge to the equilibrium point as shown in Fig. 1.

*Case 8.* In this case the first species has least initial value and the self inhibition coefficient of the third species is highest. Initially the second species dominates over the third species till the time instant $t^* = 0.15$ and the dominance gets reversed thereafter. This is illustrated in Fig. 2.

*Case 9.* In this case the first species has the least natural death rate. The self inhibition coefficient of $S_1$ and interaction coefficient of $S_1$ due to $S_3$ are identical. Initially the first and third species dominates over the second till the time instant $t^* = 0.8$ and $t^* = 1.75$ respectively and thereafter the dominance is reversed. In course of time we notice a steady variation with no appreciable growth rate in all the three species. (Fig. 3).

*Case 10.* Initially the first and third species dominates over the second species till the time instant $t^* = 0.12$ and $t^* = 2.23$ respectively and thereafter the dominance is reversed. In this case the natural birth rate of the second species is highest and the third species has the least self inhibition coefficient (Fig. 4).

## 8 Conclusion

The present paper deals with an investigation on the stability of a syn eco-system consisting of two hosts and one commensal with mortality rate for the first and third species. In this paper we established all possible equilibrium states. It is conclude that, in all eight equilibrium states, only the two states $E_3$ and $E_7$ are stable. Further the global stability is established with the help of suitable Liapunov's function and the numerical solutions are computed using Runge-Kutta fourth order method.

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