ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 13 (2017) No. 2, pp. 133-142

Modification of homotopy perturbation method for solving system of integro-differential equations*

Ali Tahmasebi[1†], Amir Fodian[2]

School of Mathematics and Computer Science, Damghan University of Basic Sciences, P.O.Box 36715-364, Damghan, Iran,

(*Received February 22 2016, Accepted November 11 2016)*)

Abstract. In this paper, we modified Homotopy perturbation method for approximating the solution system of nonlinear integro-differential equations. By purposed modification, we enable to control truncation error by adjusting the step size used in the numerical scheme. Finally error estimation of the purposed method is presented. Some numerica examples are provided to illustrate the accuracy of our approach.

Keywords: system of integro-differential equations, homotopy perturbation method, error estimation

1 Introduction

Mathematical modeling of real-life problems usually results in functional equations, e.g. partial differential equations, integral and integro-differential equations, stochastic equations and others. In particular, integro-differential equations arise in fluid dynamics, biological models and chemical kinetics. The solutions of integral and integro-differential equations have a major role in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc. [6, 8, 10–12, 21–23, 27, 28, 31]. The analytical solutions of some integro-differential equations cannot be found, thus the numerical methods are required, several numerical approaches have been proposed, such as, variational iteration method[30], differential transform method [5], radial basic function networks method[13], operational Tau method[1], Adomian decomposition method[33], the differential transform method[9], the Chebyshev collocation method[2], the Chebyshev cardinal functions[24], Homotopy perturbation method[3, 7], Sinc-collection method[4], Taylor polynomial solution[25, 26] and modification of the Adomian decomposition method[32].

In recent years, the application of the homotopy perturbation method[14, 16] in nonlinear problems has been developed by scientists and engineers, because this method deforms the difficult problem under study into a simple problem which is easy to solve. Most perturbation methods assume a small parameter exists, but most nonlinear problems have no small parameter at all. Many new methods, such as variational. method[19], variational iterations method[15], are proposed to eliminate the shortcomings arising in the small parameter assumption. A review of recently developed nonlinear analysis methods can be found in [17]. Recently, the applications of homotopy perturbation theory have appeared in the works of many scientists[20], which has become a powerful mathematical tool[18].

In this paper, the homotopy perturbation method is modified such that by this modification the corresponding accuracy is drastically improved. Also, the approximate method described here is very easy to implement. In this paper we solve nonlinear Volterra system of integro-differential equations in the following form

* The authors are thankful to the referees for their valuable comments and suggestions which greatly improved the paper.
† Corresponding author. *E-mail address*: tahmasbi@iust.ac.ir

$$\sum_{j=1}^{m} D_{ij}(\chi, y_j(t), \cdots, y_j^{(\alpha_{ij})})(t) + \sum_{j=1}^{m} \int_0^x k_{ij}(\chi, t)\varphi_{ij}(t, y_j(t), \cdots, y_j^{(\beta_{ij})})(y)dt = f_i(\chi), \tag{1}$$
$$\chi, t \in \mathit{\Gamma} = [0, 1].$$

For $i = 1, 2, \cdots, m$ and

$$\sum_{i=1}^{m} \sum_{j=0}^{\alpha_i - 1} B_{ijr} y_i^j(0) = c_r, \quad r = 1, 2, \cdots, \beta, \tag{2}$$

as the supplementary conditions, and

$$\alpha_j = \max_{1 \le i \le m} \alpha_{ij}, \quad \beta = \sum_{j=1}^{m} \alpha_j,$$
$$D_{ij}(\chi, y_j, \ldots, y_j^{(\alpha_{ij})}) = \sum_{l=0}^{\eta_1} p_l(\chi) \prod_{k=0}^{\alpha_{ij}} (y_j^{(k)}(\chi))^{\alpha_{ijkl}},$$
$$\varphi_{ij}(t, y_j(t), \ldots, y_j^{(\beta_{ij})}(t)) = \sum_{l=0}^{\eta_2} q_l(t) \prod_{k=0}^{\beta_{ij}} (y_j^{(k)}(t))^{\beta_{ijkl}},$$

where $\beta_{ijkl}, \alpha_{ijkl} \in N \cup (0)$. Suppose that, the functions $f_i(\chi), p_l(\chi), q_l(\chi)$ and $k_{ij}(\chi, t)$ are polynomials.

Remark 1. The following partition is used throughout of the paper

$$\mathit{\Delta} = \{x_0 = 0 < x_1 < \cdots < x_{\theta_r} = 1\},$$

is an equidistance partition on $I = [0, 1]$, where

$$x_\theta = \theta h, for \theta \in \{0, 1, \cdots, \theta_r\},$$

and h is a step size of the partition.

2 Homotopy perturbation method

To illustrate the homotopy perturbation method, He [14, 16] considered the following nonlinear differential equation

$$A(u) = f(r), \quad r \in \mathit{\Omega}, \tag{3}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \mathit{\Gamma}, \tag{4}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, $\mathit{\Gamma}$ is the boundary of the domain $\mathit{\Omega}$. Suppose the operator A can be divided into two parts: M and N. Therefore, Eq. (3) can be rewritten as follows

$$M(u) + N(u) - f(r) = 0. \tag{5}$$

By using homotopy technique, one can construct a homotopy $\upsilon(r, p) : \mathit{\Omega} \times [0, 1] \to R$, which satisfies

$$H(\nu, p) = (1 - p)[M(\nu) - M(y_0)] + p[A(\nu) - f(r)] = 0, \tag{6}$$

or

$$H(\nu, p) = M(\nu) - M(y_0) + M(y_0) + p[N(\nu) - f(r)], \tag{7}$$

where $r \in \Omega$ and $p \in (0, 1]$ is an imbedding parameter, and y_0 is an initial approximation of Eq. (3). Hence, it is easy to see

$$\begin{cases} [H(\nu, 0) = M(\nu) - M(y_0) = 0, \\ H(\nu, 1) = A(\nu) - f(r) = 0, \end{cases} \tag{8}$$

the changing process of p from zero to unity is just that of $\nu(r, p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $M(\nu) - M(y_0)$ and $A(\nu) - f(r)$ are called homotopic in topology. If, the embedding parameter $p; 0 \leq p \leq 1$ is considered as a small parameter, applying the classical perturbation technique, we can naturally assume that the solution of Eqs. (8), can be given as a power series in p, i.e.

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots, \tag{9}$$

and setting results in the approximate solution of Eq. (3) as

$$u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2. \tag{10}$$

The series (10) is convergent for most of the cases, and also the rate of convergence depends on how we choose $A(\nu)$, (see [14]).

3 Modification of the homotopy perturbation method

In this section, a procedure, for a class of system of nonlinear Volterra integro-differential equations to obtain a modified of the exact solution at $x = x_\theta$, where $x_\theta = \theta h$, for $\theta \in \{0, 1, \cdots, \theta_r\}$, is presented.

Theorem 1. *If the functions $f_i(\chi)$, $p_l(\chi)$, $q_l(\chi)$ and $k_{ij}(\chi, t)$ are polynomials and $k_{ij}(\chi, t)$, is a separable function, then there exist linear independent functions $\varphi_0(x), \varphi_1(x), \ldots, \varphi_\sigma(x)$ and the constants $c_1, c_2, \cdots ., c_\sigma$, such that $Y_1(X), Y_2(X), \ldots Y_m(X)$ is exact solution of following system of nonlinear Volterra integro-differential equations*

$$\sum_{j=1}^{m} D_{ij}(\chi, Y_j, \ldots, Y_j^{(\alpha_{ij})}) + \sum_{j=1}^{m} \int_0^\chi k_{ij}(X + x_\theta, u + x_\theta)\varphi_{ij}(u, Y_j(u), \ldots, Y_j^{(\beta_{ij})})(u)du$$
$$+ \sum_{k=0}^{\sigma} c_k \varphi_k(X) = f_i(X + x_\theta), X, u \in [0, 1],$$

with the initial conditions

$$Y_j(0) = y_j(x_\theta), Y_j'(0) = y_j'(x_\theta), \ldots, Y_j^{(\alpha_j - 1)}(0) = y_j^{(\alpha_j - 1)}(x_\theta),$$
$$j = 1, 2, \ldots, m, x_\theta = \theta h, \quad \theta \in \{0, 1, \cdots, \theta_r\},$$

where

$$D_{ij}(X, Y_j, \cdots, Y_j^{(\alpha_{ij})}) = \sum_{l=0}^{\eta_1} p_l(X + x_\theta) \prod_{k=0}^{\alpha_{ij}} (Y_j^{(k)}(X))^{\alpha_{ijkl}},$$

$$\varphi_{ij}(u, Y_j(u), \ldots, Y_j^{(\beta_{ij})}(u)) = \sum_{l=0}^{\eta_2} q_l(u + x_\theta) \prod_{k=0}^{\beta_{ij}} (Y_j^{(k)}(u))^{\beta_{ijkl}},$$

$$Y_j(X) = y_j(X + x_\theta), j = 1, 2, \ldots, m, X = \chi - x_\theta.$$

Proof. See [29]. Let

$$c_i = \tau_i, \quad i = 0, 1, \cdots, \delta, \tag{11}$$

where τ_i are known functions respec to x_θ and $y_j^{(\alpha_j)}(x)$ and $Y^{(i)}(0) = y^{(i)}(x_\theta)$,, $i = 0, 1, \cdots, \delta, j = 1, 2, \cdots, m$ and

$$y_j(x) = \Psi_j, \quad j = 1, 2, \cdots, m, \tag{12}$$

where ψ_j is a known function respect to, $x - x_\theta, x_\theta, c_0, \cdots, c_\delta, y_j^{(\alpha_j)}, j = 1, 2, \cdots, m$, which is Taylor expansion of solution Eq. (1) for $y_j(x)$ and its derivations up to $m + \alpha_j - 1$ at $x = x_\theta$. Note that, in the Eq. (12), in the any step of proposed method c_i, $i = 0, 1, \cdots, \delta$, and $y^{(i)}(0)$, $i = 0, 1, \cdots, \alpha_i - 1$ are known values by using Eq. (11) and Eq. (12).

4 Algorithm of the approach

In this section, we try to propose an algorithm on the basis of the above discussions and suppose that we face with system of the nonlinear Volterra integro-differential Eq. (1), where its kernel satisfies the conditions of Theorem 1. This algorithm is presented in two stages such as initialization step and main steps.
Initialization step:

Choose step size $h > 0$ for equidistance partition Δ on I. set $c_j = 0$, $j = 0, 1, \cdots, \alpha_j$, $Y_j^i(0) = y_j^i(0)$, $j = 1, \cdots m, i = 0, 1, \cdots, \alpha_i - 1$. Set $\theta = 0$ and go to main steps.
Main steps:

Step 1. Compute by Eq. (12), the following approximate solution

$$y_j^{(i)}(x) = \Psi_j^{(i)}, i = 0, 1, \cdots, m + \alpha_j - 1 \tag{13}$$

, which is Taylor expansion approximate Eq. (1) for $y(x)$ and its derivations up to $m + \alpha_j - 1$ at $x = x_\theta$. Go to next Step.

Step 2. Set $\theta = \theta + 1$, if $\theta > \theta_r - 1$, stop, otherwise, using Eq. (13) compute the approximate values, therefore, let

$$Y_j^{(i)}(0) = y_j^{(i)}(x_\theta), i = 0, 1, \cdots, m + \alpha_j - 1 \tag{14}$$

,
which are the initial conditions Eq. (1) at $x = x_\theta$. Go to next Step.

Step 3. By Eq. (14) compute the following approximate values

$$c_i = \tau_i, i = 0, 1, \cdots, \delta,$$

and go to Step 1.

5 Estimation of error function

In this section we are going to obtain an error function for the approximate solution of purposed method. Let $e_j(\chi) = y_j(\chi) - \tilde{y}_j(\chi)$, where $y_j(\chi)$ and $\tilde{y}_j(\chi)$ are the exact and approximate solution of Eq. (1) and Eq. (2) respectively. Substituting $y_j(\chi) = \tilde{y}_j(\chi) + e_j(\chi)$. in Eq. (1), for $\chi \in [0, b]$, then we have

$$\sum_{j=1}^{m} \sum_{l=0}^{r} p_l(\chi)\psi_{ijkl}(\chi) + \sum_{j=1}^{m} \int_0^x k_{ij}(\chi, t)\psi'_{ijkl}(x)dt = H_i(\chi), i = 0, 1, \cdots, m, \tag{15}$$

with initial conditions

$$\sum_{i=1}^{m}\sum_{j=0}^{\alpha_i-1} B_{ijr}e_i^{(j)}(0) = 0r = 1,2,\cdots,\beta, \tag{16}$$

where

$$\psi_{ijkl}(x) = \prod_{k=h}^{\alpha_{ij}}\sum_{\nu=1}^{\alpha_{ijkl}} \begin{pmatrix} \alpha_{ijkl} \\ \nu \end{pmatrix} (\tilde{y}_j^{(k)}(\chi))^{\alpha_{ijkl}-\nu}(e_j^{(k)}(\chi))^{\nu},$$

$$\psi'_{ijkl}(x) = \sum_{l=0}^{s} q_l(t)\prod_{k=p}^{\beta_{ij}}\sum_{\nu=1}^{\beta_{ijkl}} \begin{pmatrix} \beta_{ijkl} \\ \nu \end{pmatrix} (\tilde{y}_j^{(k)}(\chi))^{\beta_{ijkl}-\nu}(e_j^{(k)}(\chi))^{\nu}$$

, and a perturbation term associated with $\tilde{y}_j$, is

$$H_i(\chi) = f_i(\chi) - \sum_{j=1}^{m}\sum_{l=0}^{r} p_l(\chi)\prod_{k=0}^{\alpha_{ij}}(y_j^{(k)}(\chi))^{\alpha_{ijkl}} - \sum_{j=1}^{m}\sum_{l=0}^{s} q_l(t)\prod_{k=0}^{\beta_{ij}}(y_j^{(k)}(\chi))^{\beta_{ijkl}}.$$

Solving Eq. (15) and Eq. (16) in the same manner in the previus Sections 2, 3 and 4 can obtain an approximation of $\tilde{e}_j(\chi)$.

6 Numerical examples

In order to illustrate the performance and accuracy of the proposed methods in the solution of nonlinear volterra integro-differential equations and also system of nonlinear integro-differential equations, we consider the following Examples. In Example 1, 2, 3 we applied our modified Homotopy perturbation method. In Example 4 we applied purposed error estimation.

Example 1. Consider the first order linear integro-differential equation

$$\begin{cases} y'(x) = 1 + \int_0^x y(t)dt, \\ y(0) = 1. \end{cases} \tag{17}$$

By using Theorem 1, we have

$$\begin{cases} Y'(X) = 1 + C + \int_0^X Y(u)du, \\ Y(0) = y(x_\theta), \end{cases} \tag{18}$$

where $C = y'(x_\theta) - 1$. By using Homotopy perturbation method, we have

$$(1-P)(Y'(X) - \sum_{n=0}^{\infty} a_nX^n) + P(Y'(X) - 1 - C - \int_0^X Y(u)du) = Y'(X) - \sum_{n=0}^{\infty} a_nX^n$$

$$- PY'(X) + P\sum_{n=0}^{\infty} a_nX^n + P(Y'(X) - 1 - C - \int_0^X Y(u)du),$$

$$Y(X) = y(x_\theta) + \int_0^x \sum_{n=0}^{\infty} a_nX^n dX - P(\int_0^x \sum_{n=0}^{\infty} a_nX^n dX - X - CX - \int_0^x\int_0^\varepsilon Y(u)dud\varepsilon),$$

$$Y_0(X) = y(x_\theta) + \int_0^x \sum_{n=0}^{\infty} a_nX^n dX,$$

$$Y_1(X) = -\int_0^x \sum_{n=0}^{\infty} a_nX^n dX + X + CX + \int_0^x\int_0^\varepsilon Y(u)dud\varepsilon.$$

We assume that

$$Y_1(X) = 0$$

, then we have

$$(-a_0 + 1 + c)X + (-\frac{a_1}{2} + \frac{y(x_\theta)}{2})x^2 + (-\frac{a_2}{3} - \frac{a_0}{6})x^3 + (-\frac{a_3}{4} + \frac{a_1}{12})x^4 + (-\frac{a_4}{5} - \frac{a_2}{6})x^5 + \ldots = 0.$$

It easily follows that

$$a_0 = 1 + C, a_1 = y(x_\theta), a_2 = \frac{C+1}{2}, a_3 = \frac{y(x_\theta)}{6}, a_4 = \frac{C+1}{5},$$

therefore, the exact solution of the system of integral equation can be expressed as

$$\begin{aligned} Y(X) =& y(x_\theta) + (1 + C)X + \frac{y(x_\theta)}{2}X^2 + \frac{C+1}{6}X^3 + \frac{y(x_\theta)}{24}X^4 + \frac{C+1}{120}X^5 + \frac{y(x_\theta)}{720}X^6 \\ &+ \frac{C+1}{5040}X^7 + \frac{y(x_\theta)}{40320}X^8 + \frac{C+1}{362880}X^9, \end{aligned} \tag{19}$$

therefore, we can rewrite Eq. (19) as follows

$$\begin{aligned} y(x) =& y(x_\theta) + (1 + C)(x - x_\theta) + \frac{y(x_\theta)}{2}(x - x_\theta)^2 + \frac{C+1}{6}(x - x_\theta)^3 + \frac{y(x_\theta)}{24}(x - h)^4 + \frac{C+1}{120}(x - x_\theta)^5 \\ &+ \frac{y(x_\theta)}{720}(x - x_\theta)^6 + \frac{C+1}{5040}(x - x_\theta)^7 + \frac{y(x_\theta)}{40320}(x - x_\theta)^8 + \frac{C+1}{362880}(x - x_\theta)^9. \end{aligned} \tag{20}$$

Step 1. we assume that $h = 0.1$, $\theta = 0$, $c = 0$ and $y(0) = 1$. Then we have

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880}, \tag{21}$$

the Eq. (21) is approximation solution of Eq. (17) at $x = 0$.

Step 2. We obtain Taylor expansion approximate $x = 0.1$. Therefore, let $\theta = 1$, from (21), we have $y(0.1) = 1.1051709$.

$$y(0.1) = 1.1051709.$$

Now, by using Eq. (21), and $c = 1 - y'(0.1)$. we can write

$$C = 0.1051641.$$

Now, substituting C and x_θ into Eq. (21), gives

$$y(x) = 1.10517 + (1.10516)(x - 0.1) + 0.55258(x - 0.1)^2 + 0.18419(x - 0.1)^3 \tag{22}$$

$$+ 0.04604(x - 0.1)^4 + 0.0087636(x - 0.1)^5, \tag{23}$$

where is approximation solution of Eq. (17) at $x = 0.1$. We obtain Taylor expansion approximate value around point $x = 0.2$. By Eq. (22),

$$\begin{aligned} &y(0.2) = 1.22140, \\ &C = y'(h) - 1 \Rightarrow y'(0.2) - 1 = 0.22138. \end{aligned}$$

Now, substitutins C and h into Eq. (19),

$$\begin{aligned} y(x) =& 1.22140 + (1.22138)(x - 0.2) + 0.1107(x - 0.2)^2 + 0.20356(x - 0.2)^3 + 0.05089(x - 0.2)^4 \\ &+ 0.010178(x - 0.2)^5, \end{aligned}$$

and this method will continue to point $x = 1$. Tab. 1. compare numerical results for Homotopy perturbation method and purposed modification.

Table 1

Homotopy perturbation method	Modification of the Homotopy	perturbation method
0.1	0	0
0.2	2.8644×10^{-14}	2.2204×10^{-16}
0.3	1.6729×10^{-12}	8.8818×10^{-16}
0.4	2.9983×10^{-11}	1.7764×10^{-15}
0.5	2.8188×10^{-10}	3.3307×10^{-15}
0.6	1.7764×10^{-15}	5.1070×10^{-15}
0.7	8.3102×10^{-9}	7.9936×10^{-15}
0.8	3.1894×10^{-8}	1.1102×10^{-14}
0.9	1.0458×10^{-7}	1.4211×10^{-14}
1	3.0289×10^{-7}	1.8652×10^{-14}

Example 2. Consider the following nonlinear system of integro-differential equations

$$f''(x) = 1 - \frac{1}{3}x^3 - \frac{1}{2}g'^2(x) + \frac{1}{2}\int_0^x (f^2(t) + g^2(t))dt,$$
$$g''(x) = -1 + x^2 - xf(x) + \frac{1}{4}\int_0^x (f^2(t) - g^2(t))dt,$$

with the initial conditions

$$f(0) = 1,\ f'(0) = 2,\ g(0) = 1,\ g'(0) = 0.$$

The exact solution of this problem is given in [4], as $f(x) = x + e^x$ and $g(x) = x - e^x$. We are solving this problem by using the purposed method and compared with Homotopy perturbation method.

Table 2

x_i	Homotopy perturbation method		Modification of the Homotopy perturbation method	
0.1	2.2204×10^{-16}	0	2.2204×10^{-16}	0
0.2	2.8644×10^{-14}	2.8644×10^{-14}	1.5543×10^{-15}	1.5543×10^{-15}
0.3	1.6729×10^{-12}	1.6729×10^{-12}	7.5495×10^{-15}	7.3275×10^{-15}
0.4	2.9983×10^{-11}	2.9983×10^{-11}	1.9984×10^{-14}	2.0428×10^{-14}
0.5	2.8188×10^{-10}	4.2188×10^{-14}	4.3743×10^{-14}	
0.5	2.8188×10^{-10}	2.8188×10^{-10}	4.2188×10^{-14}	4.3743×10^{-14}
0.6	1.7619×10^{-9}	1.7619×10^{-9}	7.6383×10^{-14}	8.1268×10^{-14}
0.7	8.3102×10^{-9}	8.3102×10^{-9}	1.2568×10^{-13}	1.3745×10^{-13}
0.8	3.1894×10^{-8}	3.1894×10^{-8}	1.9185×10^{-13}	2.1694×10^{-13}
0.9	1.0458×10^{-7}	1.0458×10^{-7}	2.7400×10^{-13}	3.2463×10^{-13}
1	3.0289×10^{-7}	3.0289×10^{-7}	3.7392×10^{-13}	4.6718×10^{-13}

Example 3. Consider the following nonlinear system of integro-differential equations

$$f'''(x) = x - f'(x) - \int_0^x \left(f''^2(t) + g''^2(t)\right)dt,$$
$$g''(x) = \sin x + \frac{1}{2}\sin^2(x) + \int_0^x f''(t)g(t)dt,$$

with the initial conditions

$$g''(0) = -1,\ g'(0) = 0,\ g(0) = 1,\ f'(0) = 1,\ f''(0) = 0,\ f(0) = 1.$$

The exact solution of this problem is given in [4], as $f(x) = \sin x$ and $g(x) = \cos x$. We are solving this problem by using the above and compared with normal homotopy.

Table 3

x_i	Homotopy perturbation method		Modification of the Homotopy perturbation method	
0.1	0	1.1102×10^{-16}	0	1.1102×10^{-16}
0.2	5.2736×10^{-16}	2.8200×10^{-14}	0	4.8850×10^{-15}
0.3	4.4353×10^{-14}	1.6261×10^{-12}	3.8580×10^{-16}	3.6859×10^{-14}
0.4	1.0497×10^{-12}	2.8861×10^{-11}	3.1697×10^{-15}	1.3844×10^{-13}
0.5	1.2213×10^{-11}	2.6861×10^{-10}	1.3402×10^{-14}	3.7370×10^{-13}
0.6	9.0619×10^{-11}	1.6618×10^{-9}	4.0411×10^{-14}	8.3278×10^{-13}
0.7	4.9381×10^{-10}	7.7554×10^{-9}	9.8913×10^{-14}	1.6381×10^{-12}
0.8	2.1432×10^{-9}	2.9446×10^{-8}	2.0997×10^{-13}	2.9561×10^{-12}
0.9	7.8210×10^{-9}	9.5499×10^{-8}	4.0172×10^{-13}	5.0081×10^{-12}
1	2.4892×10^{-8}	2.7350×10^{-7}	7.1001×10^{-13}	8.0864×10^{-12}

Example 4. Consider the following nonlinear system of IDEs

$$f''(x) = 1 - \frac{1}{3}x^3 - \frac{1}{2}g'^2(x) + \frac{1}{2}\int_0^x (f^2(t) + g^2(t))dt, \tag{24}$$

$$g''(x) = -1 + x^2 - xf(x) + \frac{1}{4}\int_0^x (f^2(t) - g^2(t))dt, \tag{25}$$

with the initial conditions $f(0) = 1,\ f'(0) = 2,\ g(0) = 1$ and $g'(0) = 0$. The exact solution of this problem is given in Ref. [3], as $f(x) = x + e^x$ and $g(x) = x - e^x$.. By Eq. (22), we have

$$\begin{aligned} e''_f(x) =& L_1(x) - \frac{1}{2}e'^2_g(x) + xe'_g(x) + \frac{1}{2}x^2e'_g(x) + \frac{1}{6}x^3e'_g(x) + \frac{1}{2}\int_0^x (e^2_f(t) + 2e_f(t) + 4te_f(t) \\ &+ t^2e_f(t) + \frac{1}{3}t^3e_f(t) + \frac{1}{12}t^4e_f(t) + e^2_g(t) - 2e_g(t) - t^2e_g(t) - \frac{1}{3}t^3e_g(t) - \frac{1}{12}t^4e_g(t) \\ &+ 2 + 4t + 6t^2 + \frac{8}{3}t^3 + \frac{4}{3}t^4 + \frac{1}{2}t^5 + \frac{5}{36}t^6 + \frac{1}{36}t^7 + \frac{1}{72}t^8)dt, \\ e''_g(x) =& L_2(x) - xe_f(x) + \frac{1}{4}\int_0^x (e^2_f(t) + 2e_f(t) + 4te_f(t) + t^2e_f(t) + \frac{1}{3}t^3e_f(t) + \frac{1}{12}t^4e_f(t) \\ &- e^2_g(t) + 2e_g(t) + t^2e_g(t) + \frac{1}{3}t^3e_g(t) + \frac{1}{12}t^4e_g(t) + 4t{+}4t^2 + 2t^3 + \frac{2}{3}t^4 + \frac{1}{6}t^5)dt, , \\ L_1(x) =& -x - x^2 - \frac{5}{6}x^3 - \frac{7}{24}x^4 - \frac{1}{12}x^5 - \frac{1}{72}x^6, \\ L_2(x) =& -\frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{6}x^4 - \frac{1}{24}x^5. \end{aligned}$$

Therefore, the exact solutions of the above system of integral equation can be expressed as

$$\begin{cases} e_f(x) = \frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + \frac{1}{20}a_3x^5 + \frac{1}{30}a_4x^6 + \ldots, \\ e_g(x) = \frac{-1}{2}b_0x^2 + \frac{1}{6}b_1x^3 + \frac{1}{12}b_2x^4 + \frac{1}{20}b_3x^5 + \frac{1}{30}b_4x^6 + \ldots. \end{cases}$$

7 Conclusion

In this paper, an adapted Numero-type method based on Homotopy perturbation method of the classical Taylor expansion has been modified. purposed is an applicable method with high accuracy for solving a large variety system of integro-differential equations. The proposed approach has been applied to the solution system of integro-differential equations, the numerical results indicate that the new adapted method is much more efficient than standard Homotopy perturbation method. Also we can improve the accuracy of the solution by selecting the large value of and by decreasing step size .

Table 4

x_i	Exact error		Error estimation	
0.01	8.3472×10^{-13}	8.3472×10^{-13}	8.3467×10^{-13}	8.3467×10^{-13}
0.02	2.6756×10^{-11}	2.6756×10^{-11}	2.6756×10^{-11}	2.6756×10^{-11}
0.03	2.0351×10^{-10}	2.0351×10^{-10}	2.0352×10^{-10}	2.0352×10^{-10}
0.04	8.5902×10^{-10}	8.5902×10^{-10}	8.5905×10^{-10}	8.5905×10^{-10}
0.05	2.6259×10^{-9}	2.6259×10^{-9}	2.6260×10^{-9}	2.6260×10^{-9}
0.06	6.5448×10^{-9}	6.5448×10^{-9}	6.5454×10^{-9}	6.5454×10^{-9}
0.07	1.4169×10^{-8}	1.4169×10^{-8}	1.4171×10^{-8}	1.4171×10^{-8}
0.08	2.7671×10^{-8}	2.7671×10^{-8}	2.7675×10^{-8}	2.7675×10^{-8}
0.09	4.9946×10^{-8}	4.9946×10^{-8}	4.9946×10^{-8}	4.9946×10^{-8}
0.1	8.4722×10^{-8}	8.4722×10^{-8}	8.4722×10^{-8}	8.4722×10^{-8}

References

[1] S. Abbasbandy, A. Taati. Numerical solution of the system of nonlinear volterra integro-differential equations with nonlinear differential part by the operational tau method and error estimation. *Journal of Computational & Applied Mathematics*, 2009, **231**(1): 106–113.

[2] A. Akyüz-Dacyoolu, M. Sezer. Chebyshev polynomial solutions of systems of higher-order linear fredholm-volterra integro-differential equations. *Journal of the Franklin Institute*, 2005, **342**(6): 688–701.

[3] H. Aminikhah, M. Salahi. A new homotopy perturbation method for system of nonlinear integro-differential equations. *International Journal of Computer Mathematics*, 2010, **87**(5): 1186–1194.

[4] H. Aminikhah, M. Salahi. A new homotopy perturbation method for system of nonlinear integro-differential equations. *International Journal of Computer Mathematics*, 2010, **87**(5): 1186–1194.

[5] A. Arikoglu, I. Ozkol. Solutions of integral and integro-differential equation systems by using differential transform method. *Computers & Mathematics with Applications*, 2008, **56**(9): 2411–2417.

[6] D. Bainov, M. Dimitrova, A. Dishliev. Oscillation of the bounded solutions of impulsive differential-difference equations of second order. *Applied Mathematics & Computation*, 2000, **114**: 61–68.

[7] J. Biazar, M. Eslami. A new homotopy perturbation method for solving systems of partial differential equations. *Computers & Mathematics with Applications*, 2011, **62**(1): 225–234.

[8] J. Cao, J. Wang. Delay-dependent robust stability of uncertain nonlinear systems with time delay. *Applied Mathematics & Computation*, 2004, **154**(1): 289–297.

[9] P. Darania, A. Ebadian. A method for the numerical solution of the integro-differential equations. *Applied Mathematics & Computation*, 2007, **188**(1): 657–668.

[10] L. M. Delves, J. L. Mohamed. *Computational Methods for Integral Equations*. CAMBRIDGE UNIV. PR, 1985.

[11] C. E. Elmer, E. S. V. Vleck. Traveling wave solutions for bistable differential-difference equations with periodic diffusion. *Siam Journal on Applied Mathematics*, 1999, **61**(5): 1648–1679.

[12] C. E. Elmer, E. S. V. Vleck. A variant of newton's method for the computation of traveling waves of bistable differential-difference equations. *Journal of Dynamics and Differential Equations*, 2002, **14**(3): 493–517.

[13] A. Golbabai, M. Mammadov, S. Seifollahi. Solving a system of nonlinear integral equations by an RBF network. *Computers & Mathematics with Applications*, 2009, **57**(10): 1651–1658.

[14] J. H. He. Homotopy perturbation technique. *Computer Methods in Applied Mechanics & Engineering*, 1999, **178**(3C4): 257–262.

[15] J. H. He. Variational iteration method - a kind of non-linear analytical technique: some examples. *International Journal of Non-Linear Mechanics*, 1999, **34**(4): 699–708.

[16] J. H. He. A coupling method of a homotopy technique and a perturbation technique for non-linear problems. *International Journal of Non-Linear Mechanics*, 2000, **35**(1): 37–43.

[17] J. H. He. A review on some new recently developed nonlinear analytical techniques. *International Journal of Nonlinear Sciences & Numerical Simulation*, 2000, **1**(1): 51–70.

[18] J. H. He. The homotopy perturbation method for nonlinear oscillators with discontinuities. *Applied Mathematics & Computation*, 2004, **151**(1): 287–292.

[19] J. H. He. Variational principles for some nonlinear partial differential equations with variable coefficients. *Chaos Solitons & Fractals*, 2004, **19**(4): 847–851.

[20] J. H. He. Application of homotopy perturbation method to nonlinear wave equations. *Chaos Solitons & Fractals*, 2005, **26**(3): 695–700.

[21] Z. S. Hu. Boundedness of solutions to functional integro-differential equations. *Proceedings of the American Mathematical Society*, 1992, **114**(2): 519–526.

[22] M. K. Kadalbajoo, K. K. Sharma. Numerical analysis of boundary-value problems for singularly perturbed differential-difference equations: small shifts of mixed type with rapid oscillations. *Communications in Numerical Methods in Engineering*, 2004, **20**(3): 167–182.

[23] M. K. Kadalbajoo, K. K. Sharma. Numerical analysis of singularly perturbed delay differential equations with layer behavior. *Applied Mathematics & Computation*, 2004, **157**(1): 11–28.

[24] M. Lakestani, M. Dehghan. Numerical solution of fourth-order integro-differential equations using chebyshev cardinal functions. *Progress of Theoretical Physics*, 2010, **87**(6): 1389–1394.

[25] K. Maleknejad, Y. Mahmoudi. Taylor polynomial solution of high-order nonlinear volterra-fredholm integro-differential equations. *Applied Mathematics & Computation*, 2003, **145**(2C3): 641–653.

[26] K. Maleknejad, Y. Mahmoudi. Taylor polynomial solution of high-order nonlinear volterra-fredholm integro-differential equations. *Applied Mathematics & Computation*, 2003, **145**(2C3): 641–653.

[27] K. Maleknejad, Y. Mahmoudi. Numerical solution of linear fredholm integral equation by using hybrid taylor and block-pulse functions. *Applied Mathematics & Computation*, 2004, **149**(3): 799–806.

[28] M. T. Rashed. Numerical solution of functional differential, integral and integro-differential equations. *Applied Mathematics & Computation*, 2004, **156**(2): 485–492.

[29] J. Rashidinia, A. Tahmasebi. Systems of nonlinear volterra integro-differential equations. *Numerical Algorithms*, 2012, **59**(2): 197–212.

[30] J. Saberi-Nadjafi, M. Tamamgar. The variational iteration method: A highly promising method for solving the system of integro-differential equations. *Computers & Mathematics with Applications*, 2008, **56**(2): 346–351.

[31] W. Wang, C. Lin. A new algorithm for integral of trigonometric functions with mechanization. *Applied Mathematics & Computation*, 2005, **164**(1): 71–82.

[32] A. M. Wazwaz. A reliable modification of adomian decomposition method. *Applied Mathematics & Computation*, 1999, **102**(1): 77–86.

[33] A. M. Wazwaz. The combined laplace transform-adomian decomposition method for handling nonlinear volterra integro-differential equations. *Applied Mathematics & Computation*, 2010, **216**(4): 1304–1309.