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Numerical solution of linear Fredholm-Volterra integro-differential equations of fractional order*

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Abstract. We investigate the numerical solution of linear fractional Fredholm-Volterra integro-differential equations (FVIDEs) by using of Bessel polynomials of the first kind and collocation points. This method can be easily applied to many linear problems and is capable of reducing computational works. Numerical examples are presented to illustrate the efficiency and accuracy of the proposed methods.

Keywords: Bessel polynomials, fractional Fredholm-Volterra integro-differential equation, Riemann-Liouville integral, caputo fractional derivative, fractional operational matrix

1 Introduction

Many problems can be modeled by fractional integro-differential equations, which have different applications in various areas science and engineering such as thermal systems, turbulence, image processing, fluid flow, mechanics, viscoelastic, and other areas of applications[2–6, 11, 21, 22]. Also, for solution of these equations many analytical and numerical methods have been exited such as, Adomian decomposition method (ADM)[5, 13], Spline collocation method[19], Bernoulli wavelet method[9], Chebyshev wavelets method[28], Legendre wavelets method[8] and other methods who are interested to learn more about this topic could refer to [1, 14, 16, 18, 23, 26]. Yuzbasi et al.[29], Yuzbasi and Sezer[32], Yuzbasi et al.[30] have worked on the Bessel matrix and collocation methods for the numerical solutions of the neutral delay differential equations, the pantograph equations and the Lane-Emden differential equations. Recently, Yazbasi in [31] used Bessel polynomials and Bessel collocation method for solving high-order linear Fredholm-Volterra integro-differential equations.

In this section we want to discuss on the numerical solution of integro-differential equations of fractional order with initial conditions, let us consider the general form of FVIDEs:

$$\begin{aligned} &p_0(x)_*D^{\alpha}y(x)+\sum_{j=0}^{k}p_j(x)D^{\beta_j}y(x)+p_{k+1}(x)y(x)\\ &=g(x)+\lambda_1\int_0^1 k_1(x,t)y(t)dt+\lambda_2\int_0^x k_2(x,t)y(t)dt, \end{aligned} \tag{1}$$

where, $0\le x\le 1, n-1\le\alpha\le n, n\in \mathrm{N}, 0<\beta_1<\beta_2<\cdots<\beta_k<\alpha$, under the mixed conditions

$$y^{(i)}(0)=\delta_i,\quad i=0,1,\cdots,n-1, \tag{2}$$

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where $_*D^\alpha$ is Caputo fractional derivative and is a parameter describing the order of fractional derivative. Also, $y(x)$ is an unknown function, the known functions are $p_j(x), j = 0, 1, \cdots, k+1$, $g(x)$, $k_1(x,t), k_2(x,t), \lambda_1$ and λ_2 are real or complex constants. The paper is organized as follows: In Section 2, we express some necessary basic definitions of Riemann-Liouville fractional integral and Caputo fractional derivative, then describe properties of Bessel polynomial of first kind. In Section 3, we introduce the fundamental relations and method of solution. Section 4 is devoted to an estimation of the error. In Section 5, we report results of some problems which are solved by propose method. Finally, Section 6 concludes the paper.

2 Preliminaries

2.1 Basic definitions

We give some basic definitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 1. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as [17, 20],*

$$I^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)dt, \quad x \geq 0,$$
$$I^0 y(x) = y(x),$$

where $\Gamma(.)$ is Gamma function. It has the following properties

$$I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}, \quad \gamma > -1.$$

Definition 2. *The Caputo definition of fractional derivative operator is given by [25],*

$$_*D^\alpha y(x) = I^{n-\alpha}{}_*D^n y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} y^{(n)}(t)dt,$$

where $n-1 < \alpha \leq n, n \in N, x > 0$. It has the following properties

$$_*D^\alpha I^\alpha y(x) = y(x),$$
$$_*D^\alpha I^\alpha y(x) = y(x) - \sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0.$$

2.2 Bessel polynomials of first kind

The m-th degree truncated Bessel polynomials of first kind are defined by [17],

$$J_m(x) = \sum_{k=0}^{[\frac{N-m}{2}]} \frac{(-1)^k}{k!(k+m)!}(\frac{x}{2})^{2k+m}, \quad 0 \leq \mathrm{x} < \infty, \quad m \in N, \tag{3}$$

where N is chosen the positive integer so that $N \geq m$ and $m = 0, 1, \cdots, N$. We can transform the Bessel polynomials of first kind to in N-th degree Taylor basis functions. In matrix form as

$$J(x)\tilde{-}DX(x), \tag{4}$$

where

$$J(x) = [J_0(x), J_1(x), \cdots, J_N(x)]^T, X(x) = \left[1, x, x^2, \cdots, x^N\right]^T. \tag{5}$$

If N is odd

$$D=\begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!(\frac{N-1}{2})!2^{N-1}} & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!(\frac{N+1}{2})!2^{N}} \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{(\frac{N-3}{2})!(\frac{N+1}{2})!2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

If N is even

$$D=\begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{(\frac{N}{2})!(\frac{N}{2})!2^{N}} \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!(\frac{N}{2})!2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!(\frac{N+2}{2})!2^{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

2.2.1 Function approximation

A function $f \in L^2[0,1]$ may be expanded into Bessel functions as

$$f(x)\tilde{-}\sum_{n=0}^{N} c_n J_n(x) = C^T J(x), N \geq n,$$

where

$$\begin{aligned} C &= \left(\int_0^1 f(x)J(x)dx\right) Q^{-1}, \\ Q &= \int_0^1 J(t)J^T(t)dt \simeq \int_0^1 DX(t)X^T(t)D^T dt = DHD^T, \end{aligned} \tag{6}$$

and H the integration of dual operational matrix of Taylor polynomials so that

$$\begin{aligned} H &= \int_0^1 X(t)X^T(t)dt, \ \ H = [h_{ij}]\,, i,j=0,1,\cdots,N, \\ h_{ij} &= \frac{1}{i+j+1}, i,j=0,1,\cdots,N. \end{aligned}$$

In this section we can approximate the kernel function by the truncated Maclaurin series and truncated Bessel series [24], respectively

$$\begin{aligned} k(x,t) &\simeq \sum_{m=0}^{k}\sum_{n=0}^{k} {}_t k_{mn} x^m t^n, \\ k(x,t) &\simeq \sum_{m=0}^{k}\sum_{n=0}^{k} {}_b k_{mn} J_m(x) J_n(x), \end{aligned} \tag{7}$$

where

$$ {}_tk_{mn} = \frac{1}{m!n!}\frac{\partial^{m+n}k(0,0)}{\partial x^m \partial t^n}, \quad m,n = 0,1,\cdots,N. $$

We can write Eq. (7) to matrix form as

$$ k(x,t) \simeq X^T(x)k_tX(t), \quad k_t = [{}_tk_{mn}], \quad m,n = 0,1,\cdots,N, \tag{8} $$

$$ k(x,t) \simeq J^T(x)k_bJ(t), k_b = [{}_bk_{mn}], \quad m,n = 0,1,\cdots,N. \tag{9} $$

By substituting Eq. (4) in Eq. (9) and putting equal to Eq. (8) we obtain:

$$ k_t = D^T k_b D, \quad k_b = (D^T)^{-1}k_t(D)^{-1}. \tag{10} $$

2.3 Operational matrix of the fractional integration

The integration of the vector $J(x)$ defined in (5) can be obtained as

$$ \int_0^x J(t)dt \simeq LJ(x), \tag{11} $$

where L is the $(N+1)\times(N+1)$ operational matrix for integration. Our purpose is to derive the hybrid functions operational matrix of the fractional integration. For this purpose, we consider an N-set of block-pulse function as

$$ b_i(x) = \begin{cases} 1, & \frac{i}{N+1} \le x < \frac{i+1}{N+1} \\ 0, & otherwise \end{cases} \tag{12} $$

the functions $b_i(x)$ are disjoint and orthogonal. That is,

$$ b_i(x)b_j(x) = \begin{cases} 0, i \neq j, \\ b_j(x), i = j, \end{cases} \tag{13} $$

from the orthogonality of property, it is possible to expand functions into their block-pulse series. Similarly, Taylor function may be expanded into an N-set of block-pulse function as

$$ X(x) = \rho B(x), \tag{14} $$

where $B(x) = [b_0(x), b_1(x),\cdots,b_N(x)]$ and ρ is an $(N+1)\times(N+1)$ product operational matrix as

$$ \rho = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{2(N+1)} & \frac{3}{2(N+1)} & \frac{5}{2(N+1)} & \cdots & \frac{(N+1)^2-N^2}{2(N+1)} \\ \frac{1}{3(N+1)^2} & \frac{7}{3(N+1)^2} & \frac{19}{3(N+1)^2} & \cdots & \frac{(N+1)^3-N^3}{3(N+1)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(N+1)^{N+1}} & \frac{2^{N+1}-1}{(N+1)^{N+1}} & \frac{3^{N+1}-2^{N+1}}{(N+1)^{N+1}} & \cdots & \frac{(N+1)^{N+1}-N^{N+1}}{(N+1)^{N+1}} \end{bmatrix}. $$

In [10], Kilicman and Al Zhour have given the block-pulse operational matrix of the fractional integration as follows:

$$ I^\alpha B(x) \simeq F^\alpha B(x), \tag{15} $$

where

$$F^{\alpha} = \frac{1}{(N+1)^{\alpha}} \frac{1}{\Gamma(\alpha+1)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{N-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{N-2} \\ 0 & 0 & 1 & \cdots & \xi_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{16}$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. Next, we find the Bessel function operational matrix of the fractional integration. Let

$$I^{\alpha} X(x) \simeq P^{\alpha} X(x), \tag{17}$$

by using Eqs. (14) and (15), we have

$$I^{\alpha} X(x) \simeq I^{\alpha} \rho B(x) = \rho I^{\alpha} B(x) \simeq \rho F^{\alpha} B(x), \tag{18}$$

from Eqs. (17) and (18), we get

$$P^{\alpha} X(x) = \rho F^{\alpha} B(x), \tag{19}$$

then, by substituting Eq. (14) in Eq. (19), we obtains

$$P^{\alpha} = \rho F^{\alpha} \rho^{-1}, \tag{20}$$

where, ρ^{-1} is inverse of matrix ρ and obtain operational matrix of fractional integration for Taylor polynomials P^{α}. Now, we get operational matrix of fractional integration for Bessel function by using Eqs. (4) and (17) as

$$I^{\alpha} J(x) \simeq I^{\alpha} D X(x) = D I^{\alpha} X(x) = D P^{\alpha} X(x) = D P^{\alpha} D^{-1} J(x), \tag{21}$$

so we have

$$I^{\alpha} J(x) \simeq \varphi^{\alpha} J(x), \tag{22}$$

where $\varphi^{\alpha} = D P^{\alpha} D^{-1}$.

3 Fundamental relations

3.1 Differential part

To solve Eq. (1) with conditions in Eq. (2), we assume

$$_{*}D^{\alpha} y(x) \simeq A^{T} J(x), \tag{23}$$

by using the initial conditions in Eq. (2) and Eqs. (22), (23) and properties of Caputo derivative, we have

$$\begin{aligned} &_{*}D^{\beta_j} y(x) = I^{\alpha-\beta_j}\, {}_{*}D^{\alpha} y(x) \simeq I^{\alpha-\beta_j} A^{T} J(x) = A^{T} I^{\alpha-\beta_j} J(x) \simeq A^{T} \varphi^{\alpha-\beta_j} J(x), \\ &j = 0, 1, \cdots, k. \end{aligned} \tag{24}$$

where $\varphi^{\alpha-\beta_j}, j = 0, 1, \cdots, k$, are operational matrix of fractional integration for Bessel function of $\alpha - \beta_j$ order and for $_{*}D^{q} y(x), q = 0, 1, \cdots, n-1$, we have

$$
\begin{aligned}
{}_*D^{n-1}y(x) &= I^{\alpha-n+1}{}_*D^{\alpha}y(x) + y^{(n-1)}(0) = I^{\alpha-n+1}A^TJ(x) + \delta_{n-1} \\
&= A^TI^{\alpha-n+1}J(x) + \delta_{n-1}E^TJ(x) = (A^T\varphi^{\alpha-n+1} + \delta_{n-1}E^T)J(x) \\
&= W_1^TJ(x), \\
{}_*D^{n-2}y(x) &= I^{\alpha-n+2}{}_*D^{\alpha}y(x) + \delta_{n-1}x + y^{(n-2)}(0) = I^{\alpha-n+2}A^TJ(x) + \delta_{n-1}x + \delta_{n-2} \\
&= A^T\varphi^{\alpha-n+2}J(x) + \delta_{n-1}C^TJ(x) + \delta_{n-2}E^TJ(x) \\
&= (A^T\varphi^{\alpha-n+2} + \delta_{n-1}C^T + \delta_{n-2}E^T)J(x) \\
&= W_2^TJ(x), \\
&\ \vdots \\
y(x) &= (A^T\varphi^{\alpha} + \delta_{n-1}C^T(\varphi^1)^{n-2} + \delta_{n-2}C^T(\varphi^1)^{n-3} + \cdots + \delta_1C^T(\varphi^1) + \delta_0E^T)J(x) \\
&= W_n^TJ(x),
\end{aligned}
\tag{25}
$$

where E and C are obtained from Eq. (6). Now, by substituting Eqs. (24) and (25) in left part of Eq. (1), we have

$$
\begin{aligned}
&p_0(x){}_*D^{\alpha}y(x) + \sum_{j=0}^{k} p_j(x){}_*D^{\beta_j}y(x) + p_{k+1}(x)y(x) \\
&= p_0(x)A^TJ(x) + \sum_{j=0}^{k} p_j(x)A^T\varphi^{\alpha-\beta_j}J(x) + p_{k+1}(x)W_n^TJ(x) \\
&= (p_0(x)A^T + \sum_{j=0}^{k} p_j(x)A^T\varphi^{\alpha-\beta_j} + p_{k+1}(x)W_n^T)J(x).
\end{aligned}
\tag{26}
$$

3.2 Method of solution

At first, we substituting Eqs. (9) and (29) in integral part of Eq. (1) as

$$
\begin{aligned}
\int_0^1 k_1(x,t)y(t)dt &\simeq \int_0^1 J^T(x)k_b^1J(t)J^T(t)W_ndt = J^T(x)k_b^1(\int_0^1 J(t)J^T(t)dt)W_n \\
&= J^T(x)k_b^1Q_1W_n = X^T(x)D^Tk_b^1Q_1W_n,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\int_0^x k_2(x,t)y(t)dt &\simeq \int_0^x J^T(x)k_b^2J(t)J^T(t)W_ndt = J^T(x)k_b^2(\int_0^x J(t)J^T(t)dt)W_n \\
&= X^T(x)D^Tk_b^2Q_2(x)W_n = X^T(x)D^Tk_b^2DH_2(x)D^TW_n \\
&= X^T(x)MH_2(x)D^TW_n,
\end{aligned}
\tag{28}
$$

where $M = D^Tk_b^2D$. Then, by substituting Eqs. (26), (27) and (28) in Eq. (1), we get

$$
\begin{aligned}
(p_0(x)A^T + \sum_{j=0}^{k} p_j(x)A^T\varphi^{\alpha-\beta_j} + p_{k+1}(x)W_n^T)DX(x) =& g(x) + \lambda_1X^T(x)D^Tk_b^1Q_1W_n \\
&+ \lambda_2X^T(x)MH_2(x)D^TW_n,
\end{aligned}
\tag{29}
$$

by using collocation points[17] defined by

$$x_i = \frac{i}{N}, \quad i = 0, 1, \cdots, N.$$

We have

$$(p_0(x_i)A^T + \sum_{j=0}^{k} p_j(x_i)A^T\varphi^{\alpha-\beta_j} + p_{k+1}(x_i)W_n^T)DX(x_i) = g(x_i) + \lambda_1 X^T(x_i)D^T k_b^1 Q_1 W_n + \lambda_2 X^T(x_i)MH_2(x_i)D^T W_n, \tag{30}$$

where $i = 0, 1, \cdots, N$. Finally, we have fundamental matrix equations as

$$(P_0A^T + \sum_{j=0}^{k} P_jA^T\varphi^{\alpha-\beta_j} + P_{k+1}W_n^T)DX = G + \lambda_1 X^T D^T k_b^1 Q_1 W_n + \lambda_2 \overline{XMHD}W_n, \tag{31}$$

where $i = 0, 1, \cdots, N$ and

$$P_k = \begin{bmatrix} p_k(x_0) & 0 & \cdots & 0 \\ 0 & p_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k(x_N) \end{bmatrix}, G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, X = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix},$$

$$\bar{M} = \begin{bmatrix} M & 0 & \cdots & 0 \\ 0 & M & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & M \end{bmatrix}, \bar{X} = \begin{bmatrix} X^T(x_0) & 0 & \cdots & 0 \\ 0 & X^T(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X(x_N) \end{bmatrix}, \bar{D} = \begin{bmatrix} D^T \\ D^T \\ \vdots \\ D^T \end{bmatrix}$$

and

$$\bar{H} = \begin{bmatrix} H_2(x_0) & 0 & \cdots & 0 \\ 0 & H_2(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_2(x_N) \end{bmatrix}.$$

We can obtain A from system of Eq. (31) and with substituting A in Eq. (25), we get approximate solution of Eq. (1).

4 Error estimation

In this section, we estimate error based on the residual function for the (FVIDEs) of fractional order. We can define the residual function of the present method as

$$r_N(x) = L[y_N(x)] - g(x), \tag{32}$$

where

$$L[y_N(x)] = p_0(x)_* D^{\alpha} y_N(x) + \sum_{j=0}^{k} p_j(x) D^{\beta_j} y_N(x) + p_{k+1}(x) y_N(x) - \lambda_1 \int_0^1 k_1(x,t) y_N(t)dt - \lambda_2 \int_0^x k_2(x,t) y_N(t)dt.$$

As regards, $y_N(x)$ is approximate solution of Eq. (1), we can define error function $e_N(x)$ as

$$e_N(x) = y(x) - y_N(x), \tag{33}$$

where $y(x)$ is the exact solution of the Eq. (1). From Eqs. (1), (32) and (33), we obtain

$$L[e_N(x)] = L[y(x)] - L[y_N(x)] = -r_N(x),$$

with the mixed conditions

$$e_N^{(i)}(0) = y^{(i)}(0) - y_N^{(i)}(0) = 0.$$

so, error problem express by

$$\begin{cases} L[e_N(x)] = -r_N(x), \\ e_N(0) = 0. \end{cases} \tag{34}$$

We can solve error problem (34), by using the technique of Section 3. Thus, we obtain approximate error as

$$e_{NM}(x) = \sum_{m=0}^{M} e_m J_m(x), (\mathrm{M} > \mathrm{N}).$$

where $e_{NM}(x)$ is approximate solution of the error problem (34). We can obtain an upper error bound for the present method

$$|e_{NM}(x_i)| \leq 10^{-d_N}, (0 \leq x_i \leq 1),$$

where d_N is positive integer. Consequently, the approximate solution is obtain

$$y_{NM}(x) = y_N(x) + \mathrm{e}_{NM}(x),$$

by means of the polynomials $y_N(x)$ and $e_{NM}(x)$. Also, by use of error function $e_N(x) = y(x) - y_N(x)$, and approximate error function $e_{NM}(x)$, we consider error of problem as

$$E_{NM}(x) = e_N(x) - e_{NM}(x) = y(x) - y_{NM}(x).$$

5 Numerical examples

We apply the present method in this section and solve some examples where given in the different papers. In addition, we express absolute error function which are define as $e_N(x) = |y(x) - y_N(x)|$ where $y(x)$ is the exact solution of Eq. (1) and $y_N(x)$ is the approximate of $y(x)$. The computations associated with the examples were performed using MATLAB.

Example 1. Let us first consider fractional linear differential equation[27]

$${}_*D^{\alpha} y(t) + y(t) = 0, 0 < \alpha \leq 2,$$

with conditions $y(0) = 1$ and $y'(0) = 0$. The exact solution of this problem when $\alpha = 1$ is $y(x) = \exp(-x)$ and when $\alpha = 2$ is $y(x) = \cos(x)$. Numerical results for $N = 8$ are given in Tab. 1 and Fig. 1 show a behavior of the numerical solution for $N = 6$. We see that, as approaches to 1 or 2, the numerical solution converges to that of integer-order differential equation.

Example 2. Consider fractional linear differential equation [14]

$${}_*D^{1} y(x) + {}_*D^{\alpha} y(x) + y(x) = 0, 0 \leq \alpha \leq 1,$$

with condition $y(0) = 1$ and the exact solution to this example when $\alpha = 1$ is $y(x) = \exp(\frac{-x}{2})$. We solve this problem for $N = 8$ and different$\alpha = 0.25, 0.5, 0.75, 1$. Numerical results with comparison to [14] are given in Tab. 2.

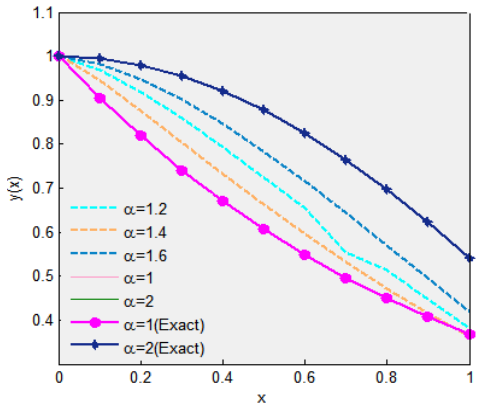

Fig. 1: Numerical and exact solution for $N = 6$

Table 1: Numerical results of Example 1

x	Exact solution of $\alpha = 1$	Present method $N = 8$ $\alpha = 1$	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$	Exact solution of $\alpha = 2$
0	1	1.0010	1.0021	1.0014	1.0006	1.0002	0.9999	1
0.1	0.9048	0.9057	0.9444	0.9684	0.9825	0.9906	0.9940	0.9950
0.2	0.8187	0.8194	0.8757	0.9180	0.9476	0.9674	0.9791	0.9801
0.3	0.7808	0.7814	0.8042	0.8583	0.9010	0.9328	0.9544	0.9553
0.4	0.6703	0.6707	0.7328	0.7930	0.8455	0.8883	0.9203	0.9211
0.5	0.6065	0.6068	0.6632	0.7244	0.7832	0.8350	0.8769	0.8776
0.6	0.5488	0.5490	0.5963	0.6543	0.7157	0.7741	0.8248	0.8253
0.7	0.4966	0.4967	0.5326	0.8539	0.6443	0.7066	0.7648	0.7648
0.8	0.4493	0.4494	0.4725	0.5143	0.5703	0.6335	0.6967	0.6967
0.9	0.4066	0.4066	0.4162	0.4463	0.4949	0.5559	0.6217	0.6217
1	0.3679	0.3679	0.3636	0.3804	0.4189	0.4747	0.5406	0.5403

Table 2: Numerical results of Example 2

x	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$	
	Bessel	Ref [14]	Bessel	Ref [14]	Bessel	Ref [14]	Bessel	Ref [14]
0	1.0031	1	1.0039	1	1.0029	1	1.0003	1
0.1	0.9156	0.7515	0.9240	0.6621	0.9364	0.5669	0.9515	0.9512
0.2	0.8474	0.6057	0.8634	0.5436	0.8832	0.4939	0.9050	0.9048
0.3	0.7921	0.4993	0.8131	0.4638	0.8364	0.4434	0.8609	0.8607
0.4	0.7462	0.4178	0.7697	0.4040	0.7940	0.4038	0.8189	0.8187
0.5	0.7076	0.3537	0.7317	0.3569	0.7553	0.3708	0.7790	0.7788
0.6	0.6749	0.3023	0.6980	0.3185	0.7196	0.3426	0.7410	0.7408
0.7	0.6467	0.2607	0.6678	0.2865	0.6866	0.3179	0.7048	0.7046
0.8	0.6224	0.2265	0.6405	0.2594	0.6559	0.2960	0.6704	0.6703
0.9	0.6012	0.1981	0.6157	0.2363	0.6273	0.2764	0.6377	0.6376
1	0.5827	0.1745	0.5933	0.2162	0.6008	0.2587	0.6066	0.6065

Example 3. Consider fractional linear FIDEs[7]

$$
{}_*D^2y(x) + {}_*D^{\frac{1}{2}}y(x) + y(x) = \frac{9}{4} - \frac{x}{3} + \frac{2}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}} + x^2 + \int_0^1 (x-t)y(t)dt,
$$

with conditions $y(0) = y'(0) = 0$ and the exact solution of this problem is $y(x) = x^2$. Approximate solution for different N are given in Tab. 3. Now, we obtain error estimation for $N = 4$, by means of method in Section 4:

$$y_N(x) = 0.20285674\times10^{-4} + 0.32903073\times10^{-3}x + 1.000702464x^2 - 0.11823220\times10^{-2}x^3 + 0.49913563\times10^{-3}x^4,$$

where

$$r_N(x) = {}_*D^2 y_N(x) + {}_*D^{\frac{1}{2}} y_N(x) + y_N(x) - \int_0^1 (x-t)y_N(t)dt - \frac{9}{4} + \frac{x}{3} - \frac{2}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}} - x^2,$$

and

$$e_N(x) = +0.20285674\times10^{-4} + 0.32903073\times10^{-3}x + 0.000702464x^2 - 0.11823220\times10^{-2}x^3 + 0.49913563\times10^{-3}x^4.$$

Now, we solve error problem

$$\begin{cases} {}_*D^2 e_N(x) + {}_*D^{\frac{1}{2}} e_N(x) + e_N(x) - \int_0^1 (x-t)e_N(t)dt = -r_N(x), \\ e(0) = 0, \\ e'(0) = 0, \end{cases}$$

so, we get

$$e_{NM}(x) = 0.20114864\times10^{-6} - 0.23806780\times10^{-4}x + 0.77213813\times10^{-3}x^2 - 0.11024941\times10^{-2}x^3 + 0.46671429\times10^{-3}x^4.$$

We can obtain an upper bound for the error due to the x_i points and $N = 4$, so that

Table 3: Numerical results of Example 3

x	$N=2$	$N=4$	$N=6$
0	5.81×10^{-4}	2.04×10^{-5}	2.51×10^{-6}
0.1	9.12×10^{-6}	1.85×10^{-5}	6.45×10^{-6}
0.2	5.30×10^{-4}	6.49×10^{-5}	1.95×10^{-5}
0.3	1.00×10^{-3}	1.13×10^{-4}	3.28×10^{-5}
0.4	1.50×10^{-3}	1.60×10^{-4}	4.49×10^{-5}
0.5	2.00×10^{-3}	2.03×10^{-4}	5.55×10^{-5}
0.6	2.40×10^{-3}	2.39×10^{-4}	6.46×10^{-5}
0.7	2.70×10^{-3}	2.68×10^{-4}	7.22×10^{-5}
0.8	3.10×10^{-3}	2.91×10^{-4}	7.81×10^{-5}
0.9	3.40×10^{-3}	3.10×10^{-4}	8.23×10^{-5}
1	3.70×10^{-3}	3.28×10^{-4}	8.58×10^{-5}

Example 4. Consider fractional linear FIDEs [1]

$${}_*D^1 y(x) = g(x) + \int_0^1 x^2t^2({}_*D^{\frac{1}{4}} y(t))dt,$$

with condition $y(0) = 0$ and $g(x) = 8x^3 - \frac{3}{2}x^{\frac{1}{2}} - (\frac{48}{6.25\Gamma(4.75)} - \frac{\Gamma(2.5)}{4.25\Gamma(2.25)})x^2$. The exact solution of this problem is $y(x) = 2x^4 - x^{\frac{3}{2}}$. Approximate solution and results of [1] for different N are given in Tab. 4.

Example 5. Consider fractional linear VIDEs [12]

$${}_*D^\alpha y(x) = 1 + 2x - y(x) + \int_0^x x(1+2x)e^{t(x-t)}y(t)dt,$$

with condition $y(0) = 1$. The exact solution of this problem is $y(x) = \exp(x^2)$ and approximate solution for $N = 12$ are given in Tab. 5. The results in Tab. 5 show as $\alpha \to 1$ numerical results tend to exact solution of $\alpha = 1$.

Table 4: Numerical results of Example 4

x	Present method			Ref [1]	
	$N=6$	$N=9$	$N=15$	$N=15$	$N=50$
0	8.80×10^{-4}	6.10×10^{-5}	3.52×10^{-6}		
0.1	1.50×10^{-6}	1.40×10^{-5}	6.87×10^{-6}		
0.2	5.90×10^{-4}	2.60×10^{-5}	1.10×10^{-5}		
0.3	8.21×10^{-3}	3.70×10^{-4}	1.12×10^{-5}		
0.4	1.09×10^{-3}	4.80×10^{-4}	1.09×10^{-5}		
0.5	1.43×10^{-3}	6.10×10^{-4}	1.65×10^{-5}	1.24×10^{-1}	2.00×10^{-3}
0.6	1.83×10^{-3}	7.50×10^{-4}	1.55×10^{-5}		
0.7	2.26×10^{-3}	8.80×10^{-4}	1.11×10^{-5}		
0.8	2.75×10^{-3}	1.01×10^{-4}	6.53×10^{-5}		
0.9	3.30×10^{-3}	1.13×10^{-4}	2.11×10^{-5}		
1	3.36×10^{-3}	1.30×10^{-4}	7.38×10^{-5}	7.24×10^{-2}	7.32×10^{-3}

Table 5: Numerical results of Example 5

x	Present method $N=12$				Ref [12]			Exact solution
	$\alpha=0.25$	$\alpha=0.5$	$\alpha=0.75$	$\alpha=1$	$\alpha=0.25$	$\alpha=0.5$	$\alpha=0.75$	$\alpha=1$
0.2	1.1511	1.1039	1.0654	1.0533	1.17030	1.11535	1.07139	1.04081
0.4	1.3495	1.2732	1.2030	1.1823	1.46437	1.35298	1.25464	1.17351
0.6	1.6123	1.5004	1.3992	1.4223	1.97637	1.76817	1.58370	1.43330
0.8	2.0191	1.8312	1.8759	1.8891	2.93641	2.51824	2.17645	1.89648
1	2.7664	2.3852	2.1039	1.9787	4.88142	3.95283	3.25252	1.71828

Example 6. Consider fractional linear VIDEs [24]

$$_{*}D^{0.5}y(x) = (\cos(x) - \sin(x))y(x) + g(x) + \int_0^x x\sin(t)y(t)dt,$$

with condition $y(0) = 0$ and $g(x) = \frac{2}{\Gamma(2.5)}x^{1.5} + \frac{1}{\Gamma(1.5)}x^{0.5} + x(2 - 3\cos(x) - x\sin(x) + x^2\cos(x))$. The exact solution of this problem is $y(x) = x^2 + x$ and approximate solution for different N are given in Tab. 6.

Table 6: Numerical results of Example 6

x	Present method			Ref [24]
	$N=5$	$N=8$	$N=10$	(ADM)
0.1	2.8×10^{-3}	3.5×10^{-3}	3.0×10^{-3}	7.98×10^{-4}
0.2	7.5×10^{-3}	3.1×10^{-3}	1.8×10^{-3}	3.64×10^{-3}
0.3	6.1×10^{-3}	1.9×10^{-3}	1.3×10^{-3}	7.80×10^{-3}
0.4	3.9×10^{-3}	1.5×10^{-3}	9.5×10^{-4}	1.19×10^{-2}
0.5	2.8×10^{-3}	1.1×10^{-3}	7.1×10^{-4}	1.50×10^{-2}
0.6	2.6×10^{-3}	8.2×10^{-4}	5.3×10^{-4}	1.64×10^{-2}
0.7	2.1×10^{-3}	6.2×10^{-4}	3.9×10^{-4}	1.65×10^{-2}
0.8	8.4×10^{-4}	3.9×10^{-4}	2.6×10^{-4}	1.58×10^{-2}
0.9	5.5×10^{-4}	4.3×10^{-4}	3.0×10^{-4}	1.51×10^{-2}
1	6.3×10^{-4}	2.1×10^{-4}	1.5×10^{-4}	1.45×10^{-2}

6 Conclusion

In this paper we have given a scheme for the numerical solution of linear (FVIDEs) of fractional order based on Bessel polynomials. The examples which have exact solutions have been used to show the efficiency of results of method. Graphics and numerical results show that this method is extremely effective and practical for this sort of approximate solutions of integro-differential equations.

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