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The dynamics of prey-predator model with a reserved zone

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Abstract. In this paper a mathematical model has been proposed and analyzed to study the role of reserved zone on the dynamical behavior of prey-predator system in two different cases. In the first case it is assumed that there is a wholly dependent predator, while in the second case it is assumed that the predator is a partially dependent. The dynamical behaviors of the proposed systems have been investigated locally as well as globally. The conditions for the systems to be uniformly persistence have been derived. The existence of a Hopf bifurcation is discussed. Finally numerical simulation is carried out to confirm our obtained results and understand the role of each parameter.

Keywords: prey-predator, persistence, reserved zone, stability

1 Introduction

The co-existence of interacting biological species in the past few decades has been studied extensively using mathematical models by several researchers[2–9, 13]. Many biological species have been driven to extinction and man others are at the verge of extinction due to several external forces such as over-exploitation, over-predation, environmental pollution, mismanagement of the habitat, etc. In order to protect these species, appropriate measures such as restriction on harvesting, creating reserved zones, etc. should be taken that will decrease the interaction of these species with external forces. For examples about such reserved zones in the world are marine protected areas; mudumalai national park and wildlife sanctuary in India and England's marine conservation zones.

The role of reserve zones in predator-prey dynamics has received considerable attention in literatures[3]. In particular, Krivan[9] proposed and analyzed the effects of optimal anti predator behavior of prey in predator-prey system. He showed that optimal anti predator behavior of prey leads to persistence and reduction of oscillation in population densities. Chattopadhyay et al.[2] studied a prey-predator model with some cover on prey species. They observed that global stability of the system around positive equilibrium does not necessarily imply the permanence of the system. Later on, Kar[8] proposed a harvests predator-prey model incorporating a prey refuge. He showed that, it is possible to break the cyclic behavior of the system. In the above investigations, the dynamics of predator living in unreserved zone together with prey has not been studied explicitly. Naji and Intisar[13] proposed and analyzed the effects of switching and group defense on the dynamics of two prey-one predator model. They observed that adding the group defense ability to the model under consideration may not always has a stability effect on the dynamical behavior of the model. Dubey[4] proposed and analyzed the dynamics of a prey-predator model with a reserved area; it is assumed that the habitat is divided in to two disjoint zones (unreserved zone and reserved zone). The predators are not allowed to enter in to the reserved zone; however it consumes the prey in unreserved zone according to linear type (Lotka-Volterra) of functional response. He concluded that the existence of reserved zone has a stabilizing effect on prey-predator model. Recently, Mukherjee[12], proposed and analyzed a prey-predator system with a reserved area

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with Holling type-II functional response for predator, which completely depends on the prey in unreserved zone. Later on Agarwal and Pathak[1] formulated and analyzed two preys one predator model with one prey dispersal in a two homogeneous patch environment consisting of reserved area and unreserved area of prey.

In this paper, the model of Mukherjee[12] is modified so that the predator depends completely on the prey in one case and depends partially on the prey in the other case. The paper is structured in the following manner. In the next section, we present the model. In Section 3, the local stability analyses of all possible equilibrium points for wholly dependent prey-predator model are given. Global stability conditions, Hopf bifurcation and persistence of this model are derived in the same Section. However section 4 deals with local stability, global stability and persistence of partially dependent prey-predator model. The numerical simulation of the proposed model in the above two cases is investigated in section 5. Finally section 6 presents the main conclusions of the paper.

2 Mathematical models

Let $x(t)$ be the density of prey species in the unreserved zone at time t, $y(t)$ the density of prey species in the reserved zone at time t and $z(t)$ the density of a predator species at time t that consumes the prey species in the unreserved zone according to Holling type-II functional response. Then the dynamics of a prey-predator model in case of existence of a reserved area can be represented by the following system of ordinary differential equations:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\sigma_1 x+\sigma_2 y-p(x)z,\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{l}\right)+\sigma_1 x-\sigma_2 y,\\ \frac{dz}{dt} &= Q(x,z)-\beta_0 z,\end{aligned} \tag{1}$$

where $x(0)\geq 0$, $y(0)\geq 0$, $z(0)\geq 0$ and $p(x)=\frac{\beta_1 x}{\alpha+x}$.

Note that all the parameters of system (1) are assumed to be positive and can be described as follow: r and s are intrinsic growth rate coefficients of prey species in unreserved and reserved zones respectively; k and l are their respective carrying capacities; σ_1 be the migration rate coefficient of the prey species from unreserved to reserved zone and σ_2 the migration rate coefficient of the prey species from reserved to unreserved zone; β_1 and α represent the maximum attack rate and the half saturation level coefficient respectively; finally β_0 is the natural death rate coefficient of the predator species. Further, the function $p(x)$ is the Holling type-II functional response (the rate at which predator removes prey), while the function $Q(x,z)$ represents the growth rate of predator.

In this paper, system (1) is analyzed in two different cases: First when the predator is wholly dependent on the prey species, in this case we have

$$Q(x,z)=\frac{\beta_2}{\beta_1}p(x)z, \tag{2}$$

where β_2 is the conversion rate of prey to predator.

However in the second case the predator is partially dependent on the prey and hence $Q(x,z)$ can be written

$$Q(x,z)=bz\left(1-\frac{z}{m_0}\right)+\frac{\beta_2}{\beta_1}p(x)z, \tag{3}$$

here b and m_0 are their intrinsic growth rate coefficients and respective carrying capacities of predator species in unreserved zone then by substituting Eq.(3) in system (1), the third equation of system (1) can be rewritten as:

$$\frac{dz}{dt}=az\left(1-\frac{z}{m}\right)+\frac{\beta_2 xz}{\alpha+x} \tag{4}$$

where $a = b - \beta_0 > 0$, $m = m_0(b - \beta_0)/b$.

Note that according to system (1) it is easy to verify that if there is no migration from reserved to unreserved zone (i.e. $\sigma_2 = 0$) and $r - \sigma_1 < 0$, then $\frac{dx}{dt} < 0$. Similarly if there is no migration from unreserved to reserved zone (i.e. $\sigma_1 = 0$) and $s - \sigma_2 < 0$, then $\frac{dy}{dt} < 0$. Hence from now onward it is natural to assume that

$$r > \sigma_1 \text{ and } s > \sigma_2. \tag{5}$$

In the following section we will start to study system (1) in the first case, which is mean $Q(x, z)$ given by Eq. (2).

3 Prey-predator model with wholly dependent predator

In this section, system (1) can be rewritten in the following for

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \sigma_1 x + \sigma_2 y + \frac{\beta_1 xz}{\alpha + x} = xf_1(x, y, z), \\
\frac{dy}{dt} &= sy\left(1 - \frac{y}{l}\right) + \sigma_1 x - \sigma_2 y = yf_2(x, y, z), \\
\frac{dz}{dt} &= \frac{\beta_2 xz}{\alpha + x} - \beta_0 z = zf_3(x, y, z).
\end{aligned} \tag{6}$$

Obviously, the interaction functions in the right hand side of system (6) are continuously differentiable functions on $R^3_+ = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$ and hence they are Lipschitzian. Therefore the solution of system (6) exists and is unique. Further, all the solutions of system (6) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem 1. *All the solutions of system (6), which initiate in R^3_+ are uniformly bounded.*

Proof. let $(x(t), y(t), z(t))$ be any solution of the system (6) with non-negative initial condition such that $w(t) = x(t) + y(t) + z(t)$, then

$$\frac{dw}{dt} = rx - \frac{rx^2}{k} + sy - \frac{sy^2}{l} - \beta_0 z - \frac{(\beta_1 - \beta_2)xz}{\alpha + x}.$$

Since from the biological point of view the conversion rate from prey to predator can't exceed the predator's maximum attack rate, hence we get $\beta_1 \geq \beta_2$, accordingly we obtain

$$\frac{dw}{dt} + \xi w \leq rk + sl = \mu,$$

where $\xi = \min\{r, s, \beta_0\}$, hence by comparing the above differential inequality with the association linear differential equation, we obtain

$$0 < w(x(t), y(t), z(t)) \leq \frac{\mu}{\xi}(1 - e^{-\xi t}) + w(0)e^{-\xi t}.$$

Therefore $0 < w(t) \leq \frac{\mu}{\xi}$, as $t \to \infty$. Hence, all the solutions of system (6) that initiate in R^3_+ are confined in the region $\Omega = \{(x, y, z) \in R^3_+ : w = x + y + z \leq \frac{\mu}{\xi}\}$, thus these solutions are uniformly bounded, and then the proof is complete.

According to the above theorem all the population are uniformly bounded in their environment for all the time, thus system (6) is dissipative. In the following the results of stability analysis and persistent of system (6) are summarized.

The prey-predator model given by system (6) has at most three nonnegative equilibrium points. The trivial equilibrium point $E_0 = (0, 0, 0)$ always exists and it's a saddle point, however the predator free equilibrium point $E_1 = (\hat{x}, \hat{y}, 0)$ exists in the $Int.R^2_+$ of $xy-$ plane, where

$$\hat{y} = \frac{1}{\sigma_2}\left[\frac{r\hat{x}^2}{k} - (r-\sigma_1)\hat{x}\right], \tag{7}$$

which is positive provided that

$$\hat{x} > \tfrac{k}{r}(r-\sigma_1). \tag{8}$$

However, $\hat{x}$ is a positive root of the following third order algebraic polynomial

$$ax^3 + bx^2 + cx + d = 0, \tag{9}$$

here $a = \frac{sr^2}{l\sigma_2^2k^2} > 0$, $b = \frac{-2rs(r-\sigma_1)}{kl\sigma_2^2} < 0$, $c = \frac{s(r-\sigma_1)^2}{l\sigma_2{}^2} - \frac{r(s-\sigma_2)}{k\sigma_2}$ and $d = \frac{(r-\sigma_1)(s-\sigma_2)}{\sigma_2} - \sigma_1$. So by using Descartes rule of signs, Eq. (9) has a unique positive solution given by $x = \hat{x}$ if the following inequalities hold:

$$\frac{s(r-\sigma_1)^2}{l\sigma_2} < \frac{r(s-\sigma_2)}{k}, \tag{10}$$

$$(r-\sigma_1)(s-\sigma_2) < \sigma_1\sigma_2 \tag{11}$$

Consequently, conditions (8), (10) and (11) represent the necessary and sufficient conditions for the existence of $E_1 = (\hat{x}, \hat{y}, 0)$ in the $Int.R_+^2$ of plane.

The positive equilibrium point $E_2 = (x^*, y^*, z^*)$ exists uniquely in the $Int.R_+^3$ where

$$x^* = \frac{\alpha\beta_0}{\beta_2 - \beta_0}, \tag{12}$$

$$y^* = \frac{l}{2s}\left[(s-\sigma_2) + \sqrt{(s-\sigma_2)^2 + \frac{4s\alpha\beta_0\sigma_1}{l(\beta_2-\beta_0)}}\right], \tag{13}$$

$$z^* = \frac{\beta_2}{\beta_0\beta_1}\left[\sigma_2 y^* + \frac{\alpha\beta_0(r-\sigma_1)}{(\beta_2-\beta_0)} - \frac{r\beta_0^2\alpha^2}{k(\beta_2-\beta_0)^2}\right]. \tag{14}$$

Clearly E_2 is positive provided that the following conditions are satisfied.

$$\beta_2 > \beta_0, \tag{15}$$

$$\sigma_2 y^* + \frac{\alpha\beta_0(r-\sigma_1)}{(\beta_2-\beta_0)} > \frac{r\alpha^2\beta_0^2}{k(\beta_2-\beta_0)^2}. \tag{16}$$

Obviously condition (16) gives a threshold value of the carrying capacity in the free access zone at which the predator species may service.

Now straightforward computations for the local stability of system (6) near E_1 show that, the predator free equilibrium point E_1 is locally asymptotically stable in the R_+^3 if the following conditions are hold.

$$\frac{\beta_2\hat{x}}{\alpha+\hat{x}} < \beta_0, \tag{17}$$

$$\left(r - \sigma_1 - \frac{2r\hat{x}}{k}\right)\left(s - \sigma_2 - \frac{2s\hat{y}}{l}\right) > \sigma_1\sigma_2 \tag{18}$$

However, it is a saddle point with non-empty stable and unstable manifolds in R_+^3 if and only if at least one of these conditions violating.

Further, the global stability of equilibrium point E_1 in the $Int.R_+^2$ of xy-plane is investigated in the following theorem.

Theorem 2. *The predator free equilibrium point E_1 is a globally asymptotically stable in the $Int.R_+^2$ of $xy-$plane.*

Proof. Follows directly by using Bendixson-Dulic criterion and Poincare-Bendixson theorem.

Theorem 3. *Assume that the equilibrium point E_1 is locally asymptotically stable in R^3_+ with*

$$\beta_2\hat{x} < \beta_0\alpha, \tag{19}$$

then it is a globally asymptotically stable in the R^3_+.

Proof. Follows directly with help of the following Lyapunov function.

$$W_1(x,y,z) = \left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + \tfrac{\sigma_2\hat{y}}{\sigma_1\hat{x}}\left(y - \hat{y} - \hat{y}\ln\frac{y}{\hat{y}}\right) + \tfrac{\beta_1}{\beta_2}z.$$

Finally, the Jacobian matrix of the system (6) at the positive equilibrium point E_2 can be written as

$$J(E_2) = [a_{ij}]_{3\times3},$$

where $a_{11} = \frac{-rx^*}{k} - \frac{\sigma_2 y^*}{x^*} + \frac{\beta_1 x^* z^*}{(\alpha+x^*)^2}$, $a_{12} = \sigma_2 > 0$, $a_{13} = \frac{-\beta_1 x^*}{\alpha+x^*} < 0$, $a_{21} = \sigma_1 > 0$, $a_{22} = \frac{-sy^*}{l} - \frac{\sigma_1 x^*}{y^*} < 0$, $a_{23} = 0$, $a_{31} = \frac{\alpha\beta_2 z^*}{(\alpha+x^*)^2} > 0$ and $a_{32} = a_{33} = 0$.

Accordingly the characteristic equation of $J(E_2)$ can be written as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{20}$$

here $A_1 = -(a_{11} + a_{22})$, $A_2 = (a_{11}a_{22} - a_{12}a_{21} - a_{13}a_{31})$, and $A_3 = (a_{22}\quad a_{13}\quad a_{31})$. While $\Delta = A_1A_2 - A_3 = (a_{11} + a_{22})(a_{12}a_{21} - a_{11}a_{22}) + a_{11}a_{13}a_{31}$.

So, by substituting the value of a_{ij}, and then simplifying the resulting terms we obtain

$$A_1 = \frac{1}{klx^*y^*N_3^2}\left[N_1 ly^* + kx^*N_2N_3^2\right], \tag{21}$$

$$A_3 = \frac{\alpha\beta_1\beta_2 x^* z^* N_2}{ly^*{N_3}^2} > 0 \tag{22}$$

and

$$\begin{aligned}\Delta = \frac{1}{k^2l^2{x^*}^2{y^*}^2N_3^5}&\left[N_3(N_1N_2 - kl\sigma_1\sigma_2x^*y^*N_3^2)\right.\\ &\left.\times\,(ly^*N_1 + kx^*N_2N_3^2) + \alpha\beta_1\beta_2kl^2{x^*}^2{y^*}^2z^*N_1\right],\end{aligned} \tag{23}$$

where $N_1 = (r{x^*}^2 + k\sigma_2y^*)(\alpha+x^*)^2 - k\beta_1{x^*}^2z^*$; $N_2 = s{y^*}^2 + l\sigma_1x^* > 0$; $N_3 = \alpha + x^* > 0$.

Therefore, in the following theorem, the local stability conditions for the positive equilibrium point E_2 in the $Int.R^3_+$ is established.

Theorem 4. *Assume that E_2 exists in $Int.R^3_+$ and the following conditions are satisfied;*

$$\beta_1 < \frac{(r{x^*}^2 + k\sigma_2y^*)N_3^2}{k{x^*}^2z^*}, \tag{24}$$

$$\beta_2 > \frac{N_3(kl\sigma_1\sigma_2x^*y^*N_3^2 - N_1N_2)(ly^*N_1 + kx^*N_2N_3^2)}{\alpha\beta_1kl^2{x^*}^2{y^*}^2z^*N_1} > 0. \tag{25}$$

Then it is locally asymptotically stable in $Int.R^3_+$.

Proof. According to the Routh-Hawirtiz criteria the characteristic Eq. (20) has roots with negative real parts if and only if $A_1 > 0$, $A_3 > 0$ (always satisfied) and $\Delta > 0$.

Clearly, according to the Eqs. (21) and (23), we have $A_1 > 0$ and $\Delta > 0$ if and only if the conditions (24) and (25) are satisfied. Hence, all the eigenvalues of the $J(E_2)$ have negative real parts. Therefore E_2 is locally asymptotically stable in and hence the proof is complete.

Now, in order to investigate the Hopf bifurcation of the model system (6), we will follow the Liu approach [10]. Assume that, the local stability condition (24) holds, and let

$$\beta_2{}^* = \frac{N_3(kl\sigma_1\sigma_2 x^* y^* N_3^2 - N_1 N_2)(ly^* N_1 + kx^* N_2 N_3^2)}{\alpha\beta_1 kl^2 x^{*2} y^{*2} z^* N_1}. \tag{26}$$

Then by substituting $\beta_2 = \beta_2^*$ in the forms of A_1, A_3 and Δ we obtain that:

$$A_1(\beta_2^*) = \frac{N_1 ly^* + kx^* N_2 N_3^2}{klx^* y^* N_3^2} > 0,$$
$$A_3(\beta_2^*) = \frac{N_3(kl\sigma_1\sigma_2 x^* y^* N_3^2 - N_1 N_2)(ly^* N_1 + kx^* N_2 N_3^2)}{l^3 y^{*3} N_3^2 N_1 x^* k} > 0,$$

with

$$\begin{aligned}\Delta(\beta_2{}^*) = &\frac{1}{k^2 l^2 x^{*2} y^{*2} N_3^5}\Big[N_3(N_1 N_2 - kl\sigma_1\sigma_2 x^* y^* N_3^2) \\ &\times (ly^* N_1 + kx^* N_2 N_3^2) + (\alpha\beta_1 kl^2 x^{*^2} y^{*^2} z^* N_1) \\ &\times \frac{(N_3(kl\sigma_1\sigma_2 x^* y^* N_3^2 - N_1 N_2)(ly^* N_1 + kx^* N_2 N_3^2)}{\alpha\beta_1 kl^2 x^{*2} y^{*2} z^* N_1}\Big] = 0,\end{aligned}$$

while,

$$\left.\frac{d\Delta}{d\beta_2}\right|_{\beta_2 = \beta_2^*} = \frac{\alpha\beta_1 z^* N_1}{kN_3^5} \neq 0.$$

Consequently, the following theorem can be proved easily.

Theorem 5. *Assume that conditions (24) and (26) hold, then a simple Hopf bifurcation of the model system (6) occurs at* $\beta_2 = \beta_2^*$

Keeping the above in view, in the following the persistence condition for the system (6) is established. The system is said to be persists if and only if each species is persists. Mathematically, the system (6) is persistence if the solution of system (6) with positive initial condition does not have omega limit set on the boundary planes. In the following theorem the persistence condition of the system (6) is established.

Theorem 6. *Assume that the planar equilibrium point* E_1 *is a globally asymptotically stable in the* $Int.R_+^2$ *of* xy*-plane. Then the necessary condition for the persistence of the system (6) is*

$$\frac{\beta_2 \hat{x}}{\alpha + \hat{x}} \geq \beta_0 \tag{27}$$

and the sufficient condition for the persistence of the system (6) is

$$\frac{\beta_2 \hat{x}}{\alpha + \hat{x}} > \beta_0 \tag{28}$$

Proof. Follows directly by applying Freedman and Waltman persistence theorem[6].

Finally, in the following, the condition of the globally asymptotically stable of the positive equilibrium point E_2 is established by using the Lyapunov function as shown in the following theorem.

Theorem 7. *Assume that, the positive equilibrium point* E_2 *is locally asymptotically stable with*

$$\frac{r}{k} > \frac{\beta_1 z^*}{\alpha(\alpha + x^*)}, \tag{29}$$

then E_2 *is a globally asymptotically stable in the* $Int.R_+^3$.

Proof. Consider the following positive definite function about (x^*, y^*, z^*),

$$W_2(x,y,z) = c_1\left(x - x^* - x^* \ln \frac{x}{x^*}\right) + c_2\left(y - y^* - y^* \ln \frac{y}{y^*}\right) + c_3\left(z - z^* - z^* \ln \frac{z}{z^*}\right),$$

where c_1, c_2 and c_3 are positive constants to be determined. Now since the derivative of W_2 along the trajectory of system (6) can be written as:

$$\begin{aligned}\frac{dW_2}{dt} =& - c_1\left(\frac{r}{k} - \frac{\beta_1 z^*}{R}\right)(x-x^*)^2 + c_1\sigma_2\left(\frac{yx^* - xy^*}{xx^*}\right)(x-x^*)\\ &-\left(\frac{c_1\beta_1(\alpha+x^*) - c_3\alpha\beta_2}{R}\right)(x-x^*)(z-z^*)\\ &-\frac{c_2 s}{l}(y-y^*)^2 + c_2\sigma_1(y-y^*)\left(\frac{xy^*-yx^*}{yy^*}\right),\end{aligned}$$

where $R = (\alpha + x)(\alpha + x^*)$. Now, by choosing the positive constants as $c_1 = 1, c_2 = \frac{\sigma_2 y^*}{\sigma_1 x^*}, c_3 = \frac{\beta_1(\alpha+x^*)}{\alpha\beta_2}$, then we obtain

$$\frac{dW_2}{dt} = -\left(\frac{r}{k} - \frac{\beta_1 z^*}{R}\right)(x-x^*)^2 - \left(\frac{\sigma_2 s y^*}{\sigma_1 l x^*}\right)(y-y^*)^2 - \left(\frac{\sigma_2}{xx^*y}\right)(xy^* - yx^*)^2.$$

Hence under the condition (29), we have $\frac{dW_2}{dt} < 0$ and then W_2 is a Lyapunv function. Therefore, E_2 is a globally asymptotically stable in the $Int.R^3_+$.

In the following section, we will study the system (1) in the second case (i.e. $Q(x,z)$ is given by Eq. (3)).

4 Prey-predator model with partially dependent predator

In this section, it is assumed that the predator is partially dependent on the prey in the unreserved zone, and hence system (1) can be written as:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \sigma_1 x + \sigma_2 y + \frac{\beta_1 xz}{\alpha + x} = xh_1(x,y,z),\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{l}\right) + \sigma_1 x - \sigma_2 y = yh_2(x,y,z),\\ \frac{dz}{dt} &= az\left(1-\frac{z}{m}\right) + \frac{\beta_2 xz}{\alpha+x} = zh_3(x,y,z).\end{aligned} \tag{30}$$

Obviously, the interaction functions in the right hand side of system (30) are continuously differentiable functions on R^3_+ and hence they are Lipschitzian. Therefore the solution of system (30) exists and is unique. Further, all solutions of system (30) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem 8. *All the solutions of system (30), which initiate in are uniformly bounded.*

Proof. Similar to proof of theorem (1).

Note that according to theorem 8, system (30) is dissipative. Moreover in the following we will presented the obtained results from the analysis of system (30).

Note that the prey predator model given in system (30) has at most four nonnegative equilibrium points, namely $F_0 = (0,0,0)$, $F_1 = (\hat{x}, \hat{y}, 0)$, $F_2 = (0,0,m)$ and $F_3 = (x^\bullet, y^\bullet, z^\bullet)$. The equilibrium points F_0 and F_2 always exist, however the equilibrium point $F_1 = (\hat{x}, \hat{y}, 0)$ in the $Int.R^2_+$ of $xy-$ plane is the same as the planar equilibrium point $E_1 = (\hat{x}, \hat{y}, 0)$ of system (6), and hence they have the same form and existence conditions. Finally, the positive equilibrium point $F_3 = (x^\bullet, y^\bullet, z^\bullet)$, where

$$z^{\bullet} = \frac{m}{a}\left[\frac{\alpha a + x^{\bullet}(a+\beta_2)}{\alpha + x^{\bullet}}\right], \tag{31}$$

$$y^{\bullet} = \frac{1}{\sigma_2}\left[\left(\frac{r}{k} + \frac{m\beta_1(a+\beta_2)}{a(\alpha+x^{\bullet})^2}\right)x^{\bullet 2} + \left(\sigma_1 - r + \frac{m\alpha\beta_1}{(\alpha+x^{\bullet})^2}\right)x^{\bullet}\right], \tag{32}$$

while, $x^{\bullet}$ is a positive root of the following third order equation

$$ax^3 + bx^2 + cx + d = 0, \tag{33}$$

here

$$a = \frac{s}{l\sigma_2^2}\left(\frac{r}{k} + \frac{m\beta_1(a+\beta_2)}{a(\alpha+x^{\bullet})^2}\right)^2 > 0,$$
$$b = \frac{2s}{l\sigma_2^2}\left(\sigma_1 - r + \frac{m\alpha\beta_1}{(\alpha+x^{\bullet})^2}\right)\left(\frac{r}{k} + \frac{m\beta_1(a+\beta_2)}{a(\alpha+x^{\bullet})^2}\right),$$
$$c = \frac{s}{l\sigma_2^2}\left(\sigma_1 - r + \frac{m\alpha\beta_1}{(\alpha+x^{\bullet})^2}\right)^2 + \frac{\sigma_2 - s}{\sigma_2}\left(\frac{r}{k} + \frac{m\beta_1(a+\beta_2)}{a(\alpha+x^{\bullet})^2}\right)$$

and

$$d = -\left(\frac{s+\sigma_2}{\sigma_2}\left(\sigma_1 - r + \frac{m\alpha\beta_1}{(\alpha+x^{\bullet})^2}\right) + \sigma_1\right).$$

Clearly the positive equilibrium point F_3 exists uniquely in the if and only if the following sufficient condition holds

$$m\alpha\beta_1 > (\sigma_1 - r)(\alpha + x^{\bullet})^2. \tag{34}$$

The results of local dynamical behavior of the system (30) around each of these equilibrium points are:

(1) The trivial equilibrium point F_0 is unstable point in the R^3_+ if and only if condition $(r-\sigma_1)(s-\sigma_2) > \sigma_1\sigma_2$ holds, while it is a saddle point with locally stable manifold of dimension one $(\dim w^s_l(F_0) = 1)$ and with unstable manifold of dimension two $(i.e. \dim w^u_l(F_0) = 2)$ provided that condition $(r-\sigma_1)(s-\sigma_2) < \sigma_1\sigma_2$ holds.

(2) The predator free equilibrium point F_1 of system (30) has the same local as well as global dynamical behavior as that of E_1 in the $Int.R^2_+$ of $xy-$ plane, however it is always unstable point in the z-direction orthogonal on $xy-$ plane.

(3) The prey free equilibrium point of system (30), represented by F_2, is locally asymptotically stable in the R^3_+ if and only if the following conditions hold:

$$r - \sigma_1 - \frac{m\beta_1}{\alpha} + s - \sigma_2 < 0, \tag{35}$$
$$\left(r - \sigma_1 - \frac{m\beta_1}{\alpha}\right)(s - \sigma_2) - \sigma_1\sigma_2 > 0. \tag{36}$$

Otherwise it is a saddle point.

(4) Finally, the Jacobian matrix of the system (30) at the positive equilibrium point F_3 can be written as $J(F_3) = [b_{ij}]_{3\times3}$, where $b_{11} = \frac{-rx^{\bullet}}{k} - \frac{\sigma_2 y^{\bullet}}{x^{\bullet}} + \frac{\beta_1 x^{\bullet} z^{\bullet}}{(\alpha+x^{\bullet})^2}$; $b_{12} = \sigma_2 > 0$; $b_{13} = \frac{-\beta_1 x^{\bullet}}{\alpha+x^{\bullet}} < 0$; $b_{21} = \sigma_1 > 0$; $b_{22} = \frac{-sy^{\bullet}}{l} - \frac{\sigma_1 x^{\bullet}}{y^{\bullet}} < 0$; $b_{23} = 0$; $b_{31} = \frac{\alpha\beta_2 z^{\bullet}}{(\alpha+x^{\bullet})^2} > 0$; $b_{32} = 0$; $b_{33} = \frac{-az^{\bullet}}{m} < 0$.

Accordingly the characteristic equation of is given by:

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0, \tag{37}$$

where

$$
\begin{aligned}
B_1 &= -(b_{11} + b_{22} + b_{33}), \\
B_2 &= (b_{11}b_{22} - b_{12}b_{21} + b_{22}b_{33} + b_{11}b_{33} - b_{13}b_{31}), \\
B_3 &= b_{33}(b_{21}b_{12} - b_{11}b_{22}) + b_{13}b_{22}b_{31}
\end{aligned}
$$

and

$$\Delta = B_1B_2 - B_3 = (b_{11} + b_{22})[b_{12}b_{21} - b_{11}b_{22}] + (b_{11} + b_{33})[b_{13}b_{31} - b_{11}b_{33}] + b_{22}b_{33}(B_1 - b_{11}).$$

So, by substituting the value of b_{ij}, and then simplifying the resulting terms we obtain:

$$B_1 = \frac{1}{klmx^\bullet y^\bullet M_3^2}\left((ly^\bullet M_1 + kx^\bullet M_2M_3^2)m + aklx^\bullet y^\bullet z^\bullet M_3^2\right), \tag{38}$$

$$B_3 = \frac{z^\bullet}{klmx^\bullet y^\bullet M_3^3}\left(aM_3(\sigma_1\sigma_2klx^\bullet y^\bullet M_3^2 - M_1M_2) + \alpha\beta_1\beta_2kmx^{\bullet^2}M_2\right) \tag{39}$$

and

$$
\begin{aligned}
\Delta =& \frac{1}{k^2l^2m^2x^{\bullet^2}y^{\bullet^2}M_3^5}\left[m^2M_3(ly^\bullet M_1 + kx^\bullet M_2M_3^2)\,(\sigma_1\sigma_2klx^\bullet y^\bullet M_3^2 - M_1M_2)\right. \\
&+ l^2y^{\bullet 2}(mM_1 + akx^\bullet z^\bullet M_3^2)(\alpha\beta_1\beta_2kmx^{\bullet 2}z^\bullet + az^\bullet M_1M_3) \\
&\left.+\, kx^\bullet M_3^3(2lmy^\bullet M_1 + kmx^\bullet M_2M_3^2 + aklx^\bullet y^\bullet z^\bullet M_3^2)\right],
\end{aligned} \tag{40}
$$

here $M_1 = (rx^{\bullet^2} + k\sigma_2y^\bullet)(\alpha + x^\bullet)^2 - k\beta_1x^{\bullet^2}z^\bullet$, $M_2 = sy^{\bullet^2} + l\sigma_1x^\bullet > 0$, $M_3 = \alpha + x^\bullet > 0$

Therefore it is easy to proof the following theorem using the Routh-Hawirtiz criterion.

Theorem 9. *Assume that the positive equilibrium point $F_3 = (x^\bullet, y^\bullet, z^\bullet)$ of system (30) exists in the $Int.R_+^3$ and the following conditions are satisfied:*

$$\beta_1 < \frac{(rx^{\bullet^2} + k\sigma_2y^\bullet)(\alpha + x^\bullet)^2}{kx^{\bullet^2}z^\bullet}, \tag{41}$$

$$\sigma_1\sigma_2klx^\bullet y^\bullet M_3^2 > M_1M_2 \tag{42}$$

Then F_3 is locally asymptotically stable.

Moreover, the global stability condition of F_3 in the $Int.R_+^3$ is presented in the following theorem.

Theorem 10. *Assume that the positive equilibrium point $F_3 = (x^\bullet, y^\bullet, z^\bullet)$ of system (30) is locally asymptotically stable in $Int.R_+^3$, and let*

$$\frac{r}{k} > \frac{\beta_1z^\bullet}{\alpha(\alpha + x^\bullet)} \tag{43}$$

holds, then is a globally asymptotically stable in the $Int.R_+^3$.

Proof. Similar to proof of theorem 7.

Furthermore, the persistence condition of the system (30) is established as shown in the following theorem.

Theorem 11. *Assume that the planar equilibrium point F_1 is a globally asymptotically stable in $Int.R_+^2$ of xy-plane, and let*

$$r - \sigma_1 - \frac{m\beta_1}{\alpha} + s - \sigma_2 > 0, \tag{44}$$

or

$$\left(r - \sigma_1 - \frac{m\beta_1}{\alpha}\right)(s - \sigma_2) - \sigma_1\sigma_2 < 0 \tag{45}$$

holds, then system (30) is persist.

Proof. Follows directly by applying the Freedman and Waltman persistence theorem[6].

5 Numerical analysis

In this section the global dynamics of system (1), in case 1 and 2 is studied numerically. In both the cases, system (1) is solved numerically for different sets of parameters and different sets of initial conditions, using predictor-corrector method with six order Runge-Kutta method, and then the attracting sets and their time series are drown. For the following set of parameters

$$\begin{aligned} &r = 4, k = 40, \sigma_1 = 2.5, \sigma_2 = 1.5, \beta_1 = 2, \alpha = 2, \\ &s = 3.5, l = 50, \beta_2 = 1.25, a = 3, m = 30, \beta_0 = 1. \end{aligned} \tag{46}$$

The attracting set along with their time series of system (1) are drown in Fig. 1 and Fig. 2 starting from different sets of initial conditions in case 1 and 2 respectively. Clearly, these figures show that the trajectories of system (1) approach to a global asymptotically stable point in both the cases. In order to investigate the effect of the migration rate coefficient of the prey species from unreserved to reserved zone (σ_1), the trajectories of system (1) in both the cases are drown in the Figs. 3-6.

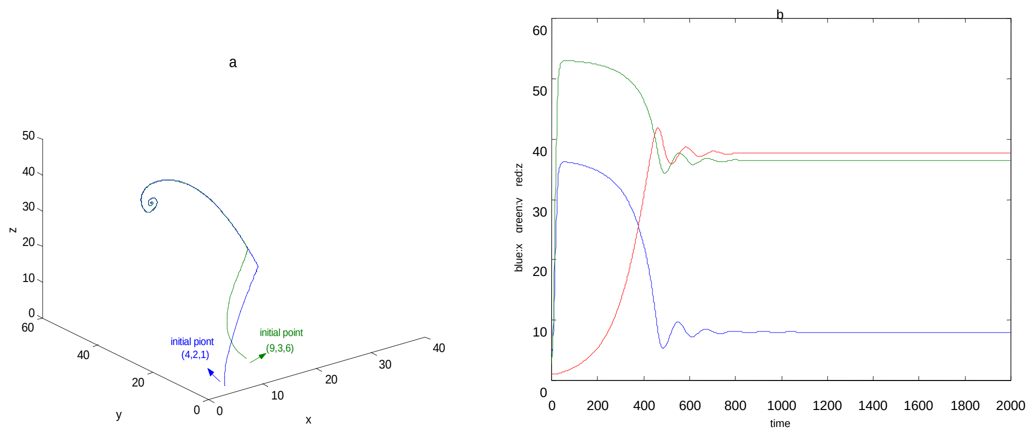

Fig. 1: (a) globally stable point $E_2 = (8.0, 36.41, 37.64)$ of system (1) in case 1, for data given in Eq. (46) with different initial points. (b) Time series of Fig. 1(a) for the initial point (4, 2, 1)

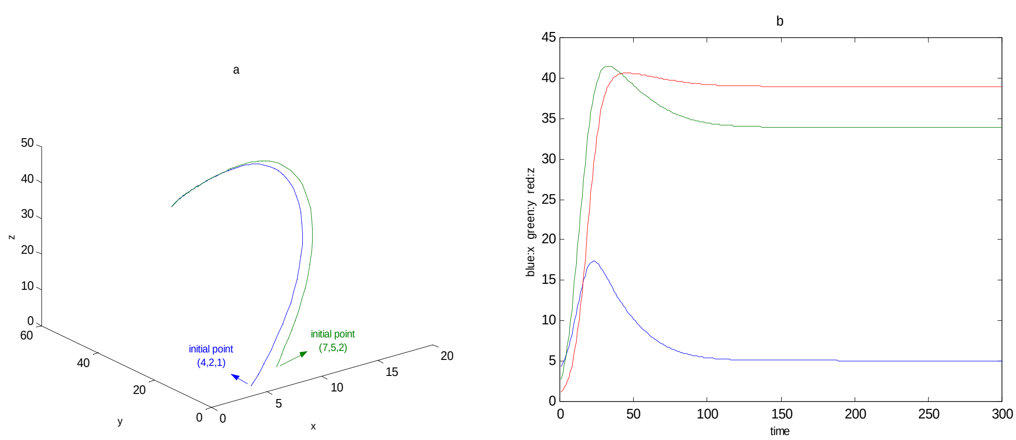

Fig. 2: (a) globally Stable point $F_3 = (5.07, 33.91, 38.96)$ of system (1) in case 2, for data given in Eq. (46) with different initial points. (b) Time series of Fig. 2(a) for the initial point (4, 2, 1)

From these figures we note that for case 1 when $\sigma_1 < 1.53$, the trajectory of system (1) approaches to periodic dynamics as shown in Fig. 3, however its approaches to a globally asymptotically stable point for

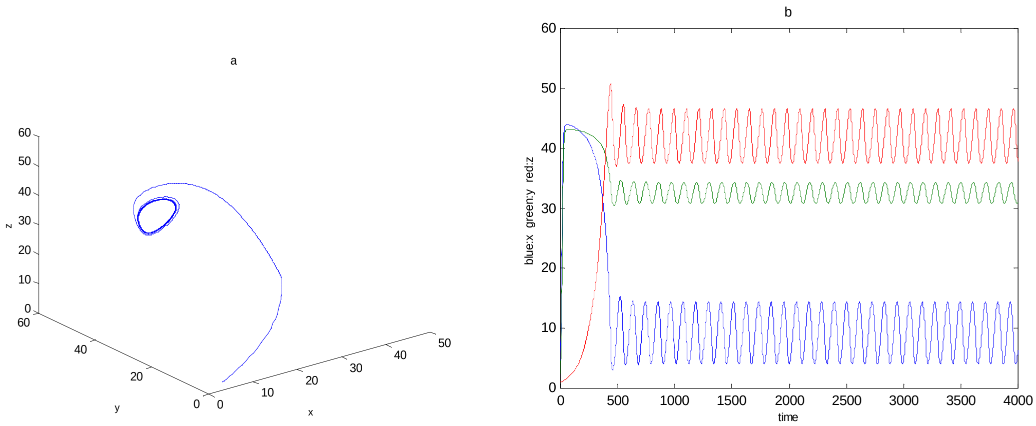

Fig. 3: (a) Periodic attractor of system (1) in case 1, for data given in Eq. (46) with $\sigma_1 = 1$. (b) Time series of Fig. 3(a)

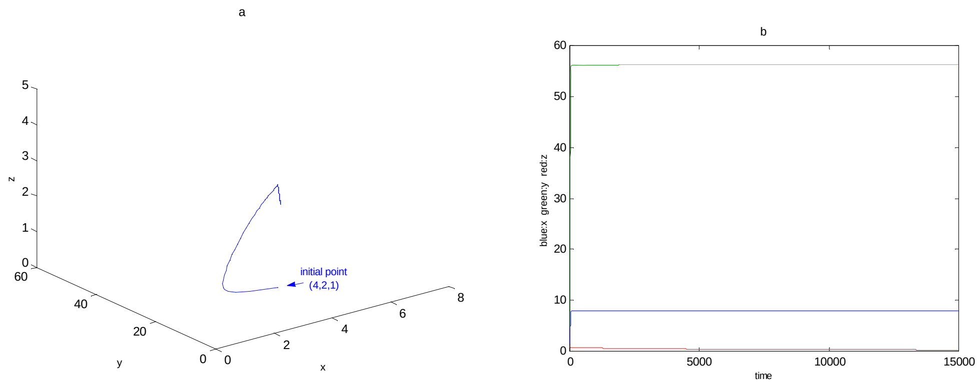

Fig. 4: (a) globally stable point $E_1 = (8.0, 56.43, 0.0)$ of system (1) in case 1, for data given in Eq. (46) with $\sigma_1 = 13.76$. (b) Time series of Fig. 4(a))

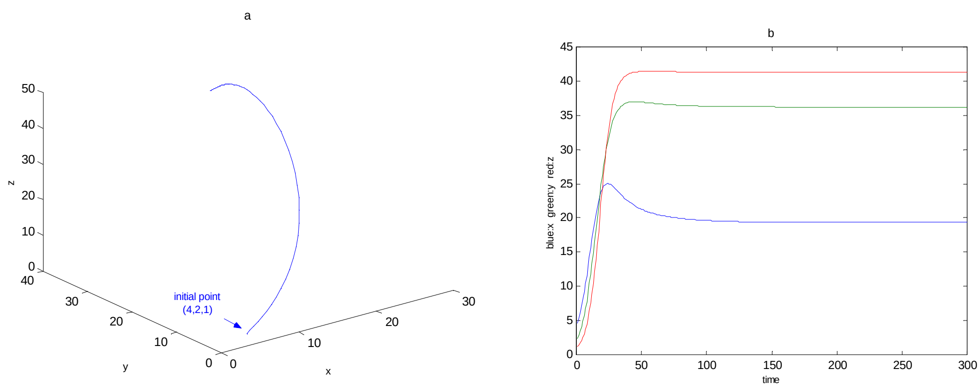

Fig. 5: (a) globally stable point $F_3 = (19.35, 36.20, 41.32)$ of system (1) in case 2, for data given in Eq. (46) with $\sigma_1 = 1$. (b) Time series of Fig. 5(a)

$1.53 < \sigma_1 < 13.76$ as shown in Fig. 1. Moreover as σ_1 increases further (i.e. $\sigma_1 \geq 13.8$) system (1) losses the persistence and the trajectory approaches to the equilibrium point $E_1 = (8.0, 56.43, 0.0)$ in the xy-plane, see Fig. 4. On the other hand in case 2, it is observed that the trajectory of system (1) approaches to a globally asymptotically stable point for $\sigma_1 > 0$, see Figs. 2, 5 and 6.

Now, for different values of σ_2 (the migration rate coefficient of the prey species from reserved to unreserved zone) the attracting sets along with their time series of system (1) are drown in both the cases, see Figs.

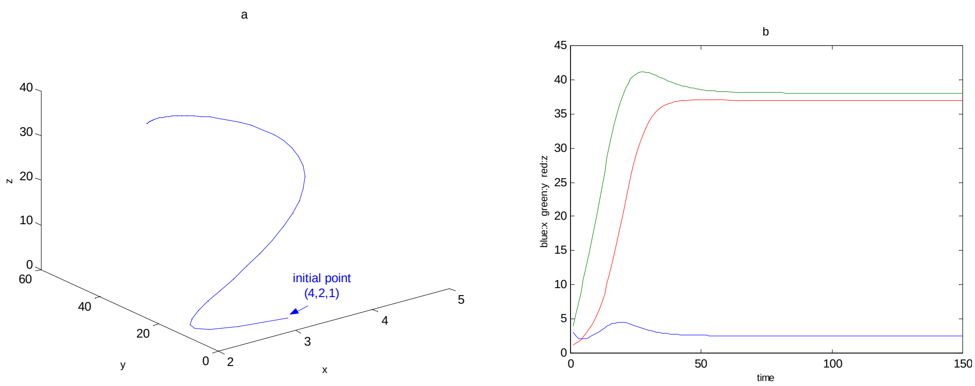

Fig. 6: (a) globally stable point $F_3 = (2.52, 38.05, 36.97)$ of system (1) in case 2, for data given in Eq. (46) with $\sigma_1 = 10$. (b) Time series of Fig. 5(a)

7-8. In case 1 it is observed that the trajectory of system (1) approaches to a global asymptotically stable point for $\sigma_2 < 2.23$, see Fig. 1, while approaches to periodic dynamics for $\sigma_2 \geq 2.23$, as shown in Fig. 7.

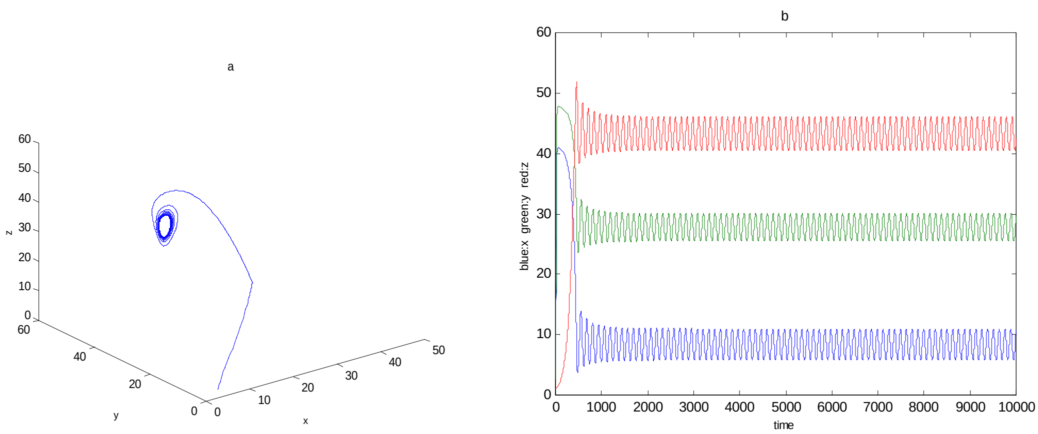

Fig. 7: (a) Periodic attractor of system (1) in case 1, for data given in Eq. (46) with $\sigma_2 = 2.3$. (b) Time series of Fig. 7(a)

In case 2 the trajectory of system (1) approaches to a global asymptotically stable point for $\sigma_2 \geq 6.2$, see Fig. 2. However for $\sigma_2 \geq 6.2$, system (1) losses the persistence and the trajectory approach to the equilibrium point $F_2 = (0, 0, m) = (0, 0, 30)$ on the z-axis, as shown in Fig. 8.

Finally, Figs. 9 and 10 show the effect of the intrinsic growth rate coefficient of the prey species in unreserved zone (i.e. r) on the dynamics behavior of system (1) in both the cases.

From these Figures we note that in case 1 the trajectory of system (1) approaches to a global stable point for $r < 5.22$, see Fig. 1, however it approaches to periodic dynamics for $r > 5.22$, see Fig. 9. While in case 2 the trajectory of system (1) approaches to a global stable point for all the values of r see Figs. 2 and 10

6 Conclusions

In this paper, a mathematical model has been proposed and analyzed to study the role of reserved zone on the dynamics behavior of prey-predator system in two cases. In both the cases, the dynamical behavior of system (1) has been investigated locally as well as globally. The conditions for system (1) to be uniformly persistence have been derived.

Now, we shall discuss the effects of changing the parameters on the dynamics of system (1) according to the numerical results in section 5. For case 1, it is observed that:

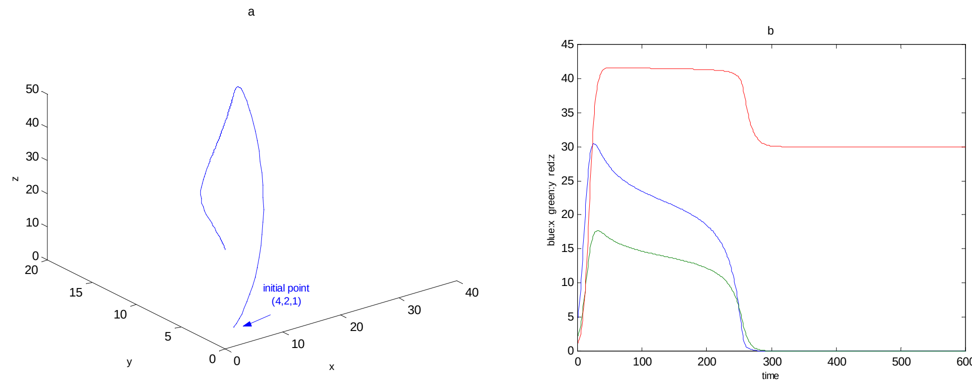

Fig. 8: (a) globally stable point $F_2 = (0, 0, 30)$ of system (1) in case 2, for data given in Eq. (46) with $\sigma_2 = 6.2$. (b) Time series of Fig. 8(a)

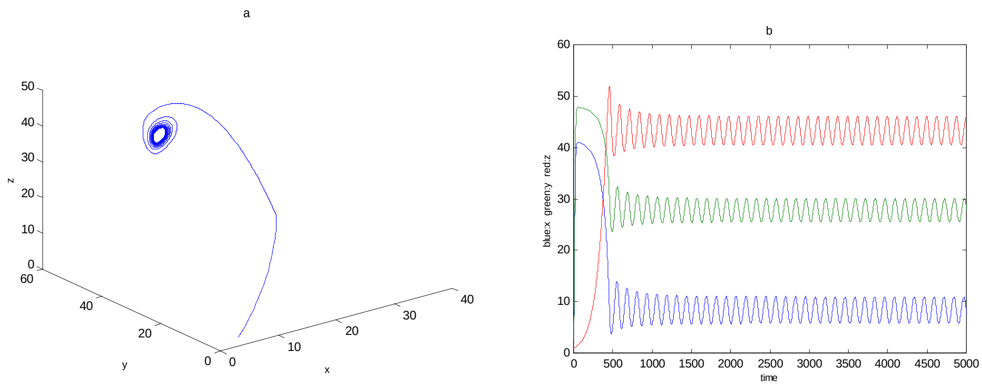

Fig. 9: (a) Periodic attractor of system (1) in case 1, for data given in Eq. (46) with $r = 5.3$. (b) Time series of Fig. 9(a)

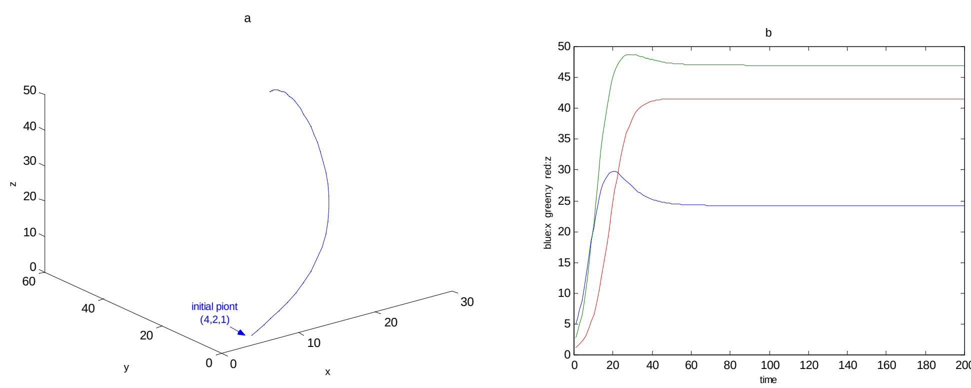

Fig. 10: (a) globally stable point $F_3 = (24.23, 46.99, 41.54)$ of system (1) in case 2, for data given in Eq. (46) with $r = 7$. (b) Time series of Fig. 10(a)

(1) For small value of σ_1 (for example $\sigma_1 = 1$), the trajectory of system (1) approaches to a periodic attractor. However, its approaches to a global asymptotically stable point for $1.53 \leq \sigma_1 < 13.76$ and finally its loss the persistence for $\sigma_1 \geq 13.76$.

(2) For $\sigma_2 < 2.23$ and $r < 5.22$ the trajectory of system (1) approaches to a global asymptotically stable point in the $Int.\ R^3_+$ while it approaches to periodic dynamics in the $Int.\ R^3_+$ for $\sigma_2 \geq 2.23$ and $r \geq 5.22$.

However, in case 2 it is observed that:

(1) The trajectory of system (1) always approaches to a global asymptotically stable point in the $Int.\ R^3_+$ for different values of σ_1 as well as r.

(2) The trajectory of system (1) approaches to a global asymptotically stable point in the $Int.\ R^3_+$ for $\sigma_2 < 6.2$, while system (1) losses the persistence for $\sigma_2 \geq 6.2$.

Finally the following two conclusions can be drawn; first, it is well known that, the prey-predator system has a periodic dynamics[11]. However the existence of reserved zone, as shown in this paper, plays an important role in stabilizing the dynamics of prey-predator system for some ranges of parameters sets. Second, adding food recourses to the predator species (as given in case 2) makes the system stable for the ranges of parameters sets.

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