ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 12 (2016) No. 1, pp. 34-47

A mathematical model to study the simultaneous effect of pollutants emitted from wood and non-wood based industries on forest resources

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(*Received June 23 2015, Accepted September 05 2015*)

Abstract. In this paper, we develop a model to study the effects of industrialization and associated pollution on forest resources. Here, we assume that wood and non-wood based industries coexist in the forest habitat and both of these industries also generate pollution and affect the forest resources. Wood-based industries directly depend on forest resources, whereas a constant rate of resources (which does not depend on the forest) is provided to non-wood industries. The proposed model is analyzed using the stability theory of differential equations and numerical simulation. From the obtained results, it is inferred that the density of forest resources decreases, not only with an increase in the density of wood- based industries, but also with increase in the density of non-wood industries. Although non-wood based industries do not depend on forest resources directly. It is also concluded that forest resources may become extinct if the pollution from wood and non-wood industries is not held in check.

Keywords: wood-based industries, non-wood industries, forest resources, stability, pollutants, simulation

1 Introduction

Jharkhand is one of the richest states in India for all kind of mineral deposits and forests. It has huge reserves of iron ores, coal, mica, bauxite, limestone, copper, uranium, zinc, manganese and other resources. Because of Jharkhand's numerous natural resources, it is a suitable place for the establishment of all types of wood and non-wood based industries. Non-wood based industries mean all types of industries such as mining industries which are based on metallurgy of different types of mineral deposits or, in general, we can say the industries that do not depend on forest directly. It is well known that forest resources play an important role for the survival of human beings and other organisms, whereas industrialization is driven by the needs of growing human population of various countries[8]. The rapid growth of industrialization in the form of wood and non-wood industries has diminished the forest habitat. Damodar Valley, Nowamundi and Saranda are typical examples where the forest habitat has been devastated significantly[19]. In particular, Damodar Valley, which once had forest cover of 65%, now stands at only 0.05%[20]. In recent years, scientists and ecologists have observed that the global temperature of the environment is slowly increasing due to the emission of toxicants (or pollutants) from industries leading to adverse effects on human being and the environment[10, 18]. The pollution associated with these industries is absorbed by the plants and damages them and thus, the growth rate of forestry biomass is affected by the pollutants emitted by different sources[3, 5, 23, 24].

Therefore, in the above scenario, it is appropriate to develop and study the behavior of systems that consider the impact of industrialization on the forest resources. For this purpose, mathematical models are one of the efficient tools available. The use of mathematical models of environmental systems is not a new endeavor. Previously, many investigators have studied the effects of industrialization and pollution on forest resources (or forestry biomass) through mathematical models. Freedman and Shukla[11] studied the effects of

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Published by World Academic Press, World Academic Union

toxicant on the growth rate of the single species and its carrying capacity in a predator-prey system. Naresh et al.[17] proposed a mathematical model on the effects of toxicant on plant biomass. In their model, they gave an intimate relationship between plant biomass and toxicant and show that as the emission rate of toxicant increases, the density of plant biomass decreases. More recently, Dubey and Hussain[9] analyzed a model in which emission rate of pollutant is assumed to be constant, zero or periodic and dependent upon industrialization. In this model, it has been shown that the resource biomass may tend to extinction due to the emission of pollutants into the environment. Dubey and Narayanan[7] discussed the effects of industrialization, population and pollution on the depletion of a renewable resource through mathematical model and found that the resource biomass decreases as the density of industrialization, population and pollution increases. Shukla et al.[21] studied the effects of primary and secondary toxicants on the natural resources and showed that the resources may tend to extinct if the emission rate of primary toxicants and formation rate of secondary toxicants are very high. Dubey et al.[8] found that the forestry resources lead to extinction on the account of population and industrialization driven by population pressure. They also discussed the control measures for maintaining the ecological stability. Agarwal et al.[2] investigated a ratio-dependent model to study the interaction among forestry biomass, wildlife population and pressure of industrialization. They concluded that the density of forestry biomass and wildlife species decreases or it may be extinct if the industrialization continues without any delay. Misra and his coauthors[15, 16, 22] have also considered the depletion and conservation of forestry resources through mathematical models. They show that advanced technological efforts can conserve the forestry resources. Chaudhary et al.[4] studied a mathematical model of forestry resources conservation considering wood based industries and synthetic industries. In their paper, they have shown that forestry resources are being depleting through wood based industries and also discussed the conservation of forestry resources through the replacement of wood industries by synthetic industries. But, they did not consider the negative effects of synthetic industries on the forestry resources.

Here, we propose and analyze a non-linear mathematical model to study the depletion of forest resources from industrialization-augmented pollution. It is assumed that wood-based as well as non-wood based industries coexist in the forest habitat and the pollutants are also emitted from both types of industries. Here, wood-based industries directly depend on forest resources, whereas non-wood is growing at a constant rate. The aim is to investigate the simultaneous impacts of wood and non-wood industries and the pollution caused by them on the forest resources.

2 Mathematical model

Consider a forest resources that grow logistically and simultaneously decreases due to wood-based industries as well as pollutants emitted through wood and non-wood based industries. It is assumed that the growth rate of wood-based industries is directly proportional to the density of forest resources whereas, a constant rate of resources (which does not depend on the forest resources) is provided to non-wood industries. The dynamics of the problem are governed by the following system of nonlinear ordinary differential equation.

$$\begin{aligned}
\frac{dF}{dt} &= rF\left(1-\frac{F}{L}\right) - \beta FW - \kappa_1 FP_W - \kappa_2 FP_I, \\
\frac{dW}{dt} &= \lambda F + \beta_1 FW - \pi_1 WI - \varphi_1 W, \\
\frac{dI}{dt} &= Q - \pi_2 WI - \varphi_2 I, \\
\frac{dP_W}{dt} &= \theta_1 W - \delta_1 FP_W - \lambda_W P_W, \\
\frac{dP_I}{dt} &= \theta_2 I - \delta_2 FP_I - \lambda_I P_I,
\end{aligned} \tag{1}$$

where $F(0) \geq 0$, $W(0) \geq 0$, $I(0) \geq 0$, $P_W(0) \geq 0$, $P_I(0) \geq 0$.

In model system (1), F is the density of forest resources, W is the density of wood-based industries, I is the density of non-wood based industries. P_W and P_I are the concentrations of pollutants caused by wood and non-wood industries respectively. The constants r and L are intrinsic growth rate and carrying capacity of the forest resources respectively. β represents the rate of depletion of forest resources due to wood-based industries, whereas β_1 signifies the growth rate of wood-based industries due to forest resources. The coefficients κ_1 and κ_2 are the depletion rates of forest resources caused by the pollutant generated through wood and non- wood industries respectively. Here, we consider the migration of wood-based industries to the forest region and it is represented through the growth rate λ, which directly depends on the density of forest resources[22]. π_1 and π_2 are the competition rates between wood and non-wood industries, where is the rate of competition effect of I on W and π_2 is the rate of competition effect of W on I. The constants φ_1 and φ_2 are control rate coefficients applied by government authorities to wood and non-wood based industries respectively. The proportionality constants θ_1 and θ_2 are the growth rates of pollutant generated by wood and non-wood based industries respectively. δ_1 and δ_1 are the constants representing the loss of pollutant generated by wood and non-wood industries due to forest resources respectively. Q is the constant rate of resources (which does not depend on forest resources) provided to non-wood based industries. The constants λ_W and λ_I are the natural depletion rate coefficients of pollutants emitted from wood and non-wood based industries, respectively.

3 Qualitative analysis

Here, we analyze the model (1) by using stability theory of differential equations. For this in the following lemma, first we show that all solutions of model (1) are nonnegative and bounded.

3.1 Boundedness of solutions

Lemma 1. *The set*

$$\Omega = \left\{ (F,\ W,\ I,\ P_W,\ P_I) : 0 \le F \le L,\ 0 \le W \le W_m,\ 0 \le I \le \frac{Q}{\varphi_2},\ 0 \le P_W \le P_{W_m},\ 0 \le P_I \le P_{I_m} \right\},$$

where $W_m = \dfrac{\lambda L}{(\varphi_1 - \beta_1 L)}$, $P_{W_m} = \dfrac{\theta_1 \lambda L}{\lambda_W (\varphi_1 - \beta_1 L)}$, $P_{I_m} = \dfrac{\theta_2 Q}{\lambda_I \varphi_2}$ *attracts all solutions initiating in the interior of the positive octant, where* $\varphi_1 - \beta_1 L > 0$.

Proof. By using comparison theorem found in [12], From the first equation of the model (1), we have

$$\frac{dF}{dt} \le rF\left(1 - \frac{F}{L}\right).$$

This gives $0 \le F \le L$. From the second equation of the model, we get

$$\frac{dW}{dt} \le \lambda F + \beta_1 F W - \varphi_1 L.$$

This gives, $0 \le W \le \dfrac{\lambda L}{(\varphi_1 - \beta_1 L)}$, $\varphi_1 - \beta_1 L > 0$, or $L < \dfrac{\varphi_1}{\beta_1}$.

This condition implies that the carrying capacity of forest resources is dependent on control rate φ_1 of wood industries and the growth rate β_1 of wood industries through forest resources. From the third equation of the model, we have

$$\frac{dI}{dt} \le Q - \varphi_2 I.$$

That gives $0 \le I \le \frac{Q}{\varphi_2}$.Similarly from the fourth and last equation of the model, we have

$$\frac{dP_W}{dt} \le \theta_1 W - \lambda_W P_W.$$

This gives

$$0 \leq P_W \leq \frac{\theta_1 \lambda L}{\lambda_W \left(\varphi_1 - \beta_1 L\right)},$$

$$\frac{dP_I}{dt} \leq \theta_2 I - \lambda_I P_I.$$

This gives

$$0 \leq P_I \leq \frac{\theta_2 Q}{\lambda_I \varphi_2}.$$

Hence the lemma follows.

3.2 Equilibrium points and their existence

There are three equilibrium points.

(1) $E_1\left(0, 0, \frac{Q}{\varphi_2} 0, \frac{\theta_2 Q}{\lambda_I \varphi_2}\right)$ exists without any condition.

(2) $E_2(F_2^*,\ 0,\ I_2^*,\ 0,\ P_{I_2}^*)$ exists provided the following condition is satisfied:

$$(r\varphi_2\lambda_I - \kappa_2\theta_2 Q) > 0.$$

(3) $E_3\left(F^*,\ W^*,\ I^*,\ P_W^*,\ P_I^*\right)$ exists provided the following conditions are satisfied:

$$(r\varphi_2\lambda_I - \kappa_2\theta_2 Q) > 0,$$

$$(\varphi_1 - \beta_1 F^*) > 0.$$

Proof. The equilibrium points of the model (1) may be obtained by solving the following algebraic equations:

$$rF\left(1 - \frac{F}{L}\right) - \beta FW - \kappa_1 F P_W - \kappa_2 F P_I = 0, \tag{2}$$

$$\lambda F + \beta_1 FW - \pi_1 WI - \varphi_1 W = 0, \tag{3}$$

$$Q - \pi_2 WI - \varphi_2 I = 0, \tag{4}$$

$$\theta_1 W - \delta_1 F P_W - \lambda_W P_W = 0, \tag{5}$$

$$\theta_2 I - \delta_2 F P_I - \lambda_I P_I = 0. \tag{6}$$

(1) $E_1\left(0, 0, \frac{Q}{\varphi_2}, 0, \dfrac{\theta_2 Q}{\lambda_I \varphi_2}\right)$ is obvious.

(2) $E_2(F_2^*,\ 0,\ I_2^*,\ 0,\ P_{I_2}^*)$

From first, third and fifth equation of the model (1), we have

$$rF\left(1 - \frac{F}{L}\right) - \kappa_2 F P_I = 0, \tag{7}$$

$$Q - \varphi_2 I = 0, \tag{8}$$

$$\theta_2 I - \delta_2 F P_I - \lambda_I P_I = 0. \tag{9}$$

From Eqs. (8) and (9), we have

$$I = \frac{Q}{\varphi_2}, \tag{10}$$

$$P_I = \frac{\theta_2 I}{\delta_2 F + \lambda_I} = \frac{\theta_2 Q}{\varphi_2 \left(\delta_2 F + \lambda_I\right)}. \tag{11}$$

From the Eqs. (7) and (11), we get an equation in F,

$$R_1\left(F\right) = r\left(1 - \frac{F}{L}\right) - \frac{\kappa_2 \theta_2 Q}{\varphi_2 \left(\delta_2 F + \lambda_I\right)}. \tag{12}$$

From the Eq. (12), we may easily conclude the following results.

(a) $R_1(0) = r - \dfrac{\kappa_2\theta_2 Q}{\varphi_2\lambda_I} = \dfrac{(r\varphi_2\lambda_I - \kappa_2\theta_2 Q)}{\varphi_2\lambda_I}$, which is positive if

$$(r\varphi_2\lambda_I - \kappa_2\theta_2 Q) > 0. \tag{13}$$

(b) $R_1(L) = -\dfrac{\kappa_2\theta_2 Q}{\varphi_2(\delta_2 L + \lambda_I)}$ which is negative.

The above points (a) and (b) together imply that there exists a positive root ($F = F_2^*$) of Eq. (12) in the interval $(0, L)$. This root will be unique provided $R'(F)$ is negative in $(0, L)$. With the help of this unique positive value of $F = F_2^*$ in Eqs. (8) and (9), we get a unique positive value of $P_I = P_{I_2}^*$ and $I = I_2^*$.

(3) $E_3(F^*, W^*, I^*, P_W^*, P_I^*)$

From the third equation of the model (1), we have

$$I = \frac{Q}{\pi_2 W + \varphi_2}. \tag{14}$$

From the second equation of the model (1) and Eq. (14), we have

$$\pi_2(\varphi_1 - \beta_1 F)W^2 - (\lambda\pi_2 F - \varphi_2(\varphi_1 - \beta_1 F) - \pi_1 Q)W - \lambda\varphi_2 F = 0, \tag{15}$$

or

$$W = \frac{(\lambda\pi_2 F - \varphi_2(\varphi_1 - \beta_1 F) - \pi_1 Q) + \sqrt{(\lambda\pi_2 F - \varphi_2(\varphi_1 - \beta_1 F) - \pi_1 Q)^2 + 4(\varphi_1 - \beta_1 F)\lambda\varphi_2\pi_2 F}}{2\pi_2(\varphi_1 - \beta_1 F)}$$
$$= f(F) \ \text{(Say)}. \tag{16}$$

W is positive if

$$\varphi_1 - \beta_1 F > 0. \tag{17}$$

From the third, fourth and fifth equations of the model (1), we have

$$I = \frac{Q}{\pi_2 W + \varphi_2} = \frac{Q}{\pi_2 f(F) + \varphi_2} = u(F), \tag{18}$$

$$P_W = \frac{\theta_1 W}{\delta_1 F + \lambda_W} = \frac{\theta_1 f(F)}{\delta_1 F + \lambda_W} = g(F), \tag{19}$$

$$P_I = \frac{\theta_2 I}{\delta_2 F + \lambda_I} = \frac{\theta_2 u(F)}{\delta_2 F + \lambda_I} = h(F). \tag{20}$$

From Eqs. (16), (19), (20) and first equation of the model (1), we get an equation in F,

$$R_2(F) = r\left(1 - \frac{F}{L}\right) - \beta f(F) - \kappa_1 g(F) - \kappa_2 h(F). \tag{21}$$

(a) $R_2(0) = r - \dfrac{\kappa_2\theta_2 Q}{\varphi_2\lambda_I} = \dfrac{(r\varphi_2\lambda_I - \kappa_2\theta_2 Q)}{\varphi_2\lambda_I}$, which is positive if

$$(r\varphi_2\lambda_I - \kappa_2\theta_2 Q) > 0, \tag{22}$$

or

$$Q < \frac{r\varphi_2\lambda_I}{\kappa_2\theta_2},$$

this gives a threshold value of Q.

(b) $R_2(L) = -\beta f(L) - \kappa_1 g(L) - \kappa_2 h(L) < 0$, which is negative as $f(L)$, $g(L)$ and $h(L)$ all are positive. The above points (a) and (b) together imply that there exists a positive root ($F = F^*$) of Eq. (21) in the interval $(0, L)$. This root will be unique provided $R_2'(F)$ is negative in $(0, L)$. With the help of this unique positive value of $F = F^*$ in Eqs. (16), (18), (19) and (20), we get a unique positive value of $W = W^*$, $I = I^*$, $P_W = P_W^*$ and $P_I = P_I^*$.

Remark 1. From Eqs. (16), (18), (19) and (20), we get the following results.

(1) $\frac{dW}{d\pi_1} < 0$, this implies that if the value of π_1 increases, then the density of non-wood industries increases and the density of wood-based industries decreases.

(2) $\frac{dI}{d\pi_2} < 0$, this implies that if the value of π_2 increases, then the density of wood-based industries increases and the density of non-wood industries decreases.

(3) $\frac{dP_W}{dF} > 0$, this implies that pollutants of wood-based industries increases as the density of forest resources increases, because growth of wood industries is dependent on the density of forest resources and the concentration of pollutants emitted from them is directly proportional to the density of wood industries.

(4) $\frac{dP_I}{dF} < 0$, this implies that pollutants of non-wood industries decrease as the density of forest resources increases.

3.3 Stability analysis

The local stability behavior of equilibrium points may be determined by eigenvalues of the corresponding Jacobian matrix M. The general Jacobian matrix M for the model (1) is given as,

$$M = \begin{bmatrix} a_{11} & -\beta F & 0 & -\kappa_1 F & -\kappa_2 F \\ \lambda + \beta_1 W & a_{22} & -\pi_1 W & 0 & 0 \\ 0 & -\pi_2 I & -\pi_2 W - \varphi_2 & 0 & 0 \\ -\delta_1 P_W & \theta_1 & 0 & -\delta_1 F - \lambda_W & 0 \\ -\delta_2 P_I & 0 & \theta_2 & 0 & -\delta_2 F - \lambda_I \end{bmatrix},$$

where $a_{11} = r - \frac{2r}{L}F - \beta W - \kappa_1 P_W - \kappa_2 P_I,\ a_{22} = \beta_1 F - \pi_1 I - \varphi_1$. Let M_i be the matrix M evaluated at the equilibrium $E_i (i = 1, 2, 3)$ then

$$M_1 = \begin{bmatrix} \frac{r\varphi_2\lambda_I - \kappa_2\theta_2 Q}{\varphi_2\lambda_I} & 0 & 0 & 0 & 0 \\ \lambda & -\frac{\pi_1 Q}{\varphi_2} - \varphi_1 & 0 & 0 & 0 \\ 0 & -\frac{\pi_2 Q}{\varphi_2} & -\varphi_2 & 0 & 0 \\ 0 & \theta_1 & 0 & -\lambda_W & 0 \\ -\frac{\delta_2\theta_2 Q}{\varphi_2\lambda_I} & 0 & \theta_2 & 0 & -\lambda_I \end{bmatrix}.$$

As we know that at equilibrium E_2, F_2^* has a unique positive value. For this, we have a condition from equation (13), $r\varphi_2\lambda_I - \kappa_2\theta_2 Q > 0$. This gives positive eigenvalue of the matrix Thus, E_1 is unstable in F-direction whenever E_2 or E_3 exists.

Now, we study the local stability behavior of the equilibriums E_2 and E_3 by using Routh-Hurwitz criterion. Then the matrix M_2 at equilibrium point E_2 as follows.

$$M_2 = \begin{bmatrix} -\frac{r}{L}F_2^* & -\beta F_2^* & 0 & -\kappa_1 F_2^* & -\kappa_2 F_2^* \\ \lambda & \beta_1 F_2^* - \pi_1 I_2^* - \varphi_1 & 0 & 0 & 0 \\ 0 & -\pi_2 I_2^* & -\varphi_2 & 0 & 0 \\ 0 & \theta_1 & 0 & -\delta_1 F_2^* - \lambda_W & 0 \\ -\delta_2 P_{I_2}^* & 0 & \theta_2 & 0 & -\delta_2 F_2^* - \lambda_I \end{bmatrix}.$$

Let $C_{22} = \varphi_1 - \beta_1 F_2^* + \pi_1 I_2^*$, $C_{44} = \delta_1 F_2^* + \lambda_W$ and $C_{55} = \delta_2 F_2^* + \lambda_I$. Now the corresponding characteristic equation of the Jacobian matrix M_2 is given as,

$$B^5 + q_1 B^4 + q_2 B^3 + q_3 B^2 + q_4 B + q_5 = 0, \tag{23}$$

where

$$
\begin{aligned}
q_1 =& \frac{r}{L}F_2^* + C_{22} + \varphi_2 + C_{44} + C_{55},\\
q_2 =& C_{55}\left(C_{22} + \frac{r}{L}F_2^* + \varphi_2 + C_{44}\right) + \left((\varphi_2 + C_{44})\left(C_{22} + \frac{r}{L}F_2^*\right) + C_{44}\varphi_2 + \frac{r}{L}F_2^* + \lambda\beta F_2^*\right)\\
&- \kappa_2\delta_2 P_{I_2}^* F_2^*,\\
q_3 =& C_{55}\left((\varphi_2 + C_{44})\left(C_{22} + \frac{r}{L}F_2^*\right) + C_{44}\varphi_2 + \frac{r}{L}F_2^* C_{22} + \lambda\beta F_2^*\right)\\
&+ \left(\kappa_1\theta_1\lambda F_2^* + (\varphi_2 + C_{44})\left(\frac{r}{L}F_2^* C_{22} + \lambda\beta F_2^*\right) + C_{44}\varphi_2\left(C_{22} + \frac{r}{L}F_2^*\right)\right)\\
&- \kappa_2\delta_2 P_I F_2^*(\varphi_2 + C_{44}) - \kappa_2 F_2^* C_{44},\\
q_4 =& C_{55}\left(\kappa_1\theta_1\lambda F_2^* + (\varphi_2 + C_{44})\left(\frac{r}{L}F_2^* C_{22} + \lambda\beta F_2^*\right) + C_{44}\varphi_2\left(C_{22} + \frac{r}{L}F_2^*\right)\right)\\
&+ \left(\kappa_1\theta_1\lambda\varphi_2 F_2^* + C_{44}\varphi_2\left(\frac{r}{L}F_2^* C_{22} + \lambda\beta F_2^*\right)\right) - \kappa_2 C_{44}(\varphi_2 + C_{44}) F_2^*\\
&- \kappa_2 C_{22}\varphi_2 F_2^* - \kappa_2\theta_2\pi_2\lambda F_2^* I_2^*,\\
q_5 =& C_{55}\left(\kappa_1\theta_1\lambda\varphi_2 F_2^* + C_{44}\varphi_2\left(\frac{r}{L}F_2^* C_{22} + \lambda\beta F_2^*\right)\right) - \kappa_2 C_{44} C_{22}\varphi_2 F_2^* - \kappa_2\theta_2\pi_2\lambda I_2^* C_{44} F_2^*.
\end{aligned}
$$

Here it is easy to see that $q_1 > 0$ if $\varphi_1 - \beta_1 F_2^* + \pi_1 I_2^* > 0$. Thus for the local stability of the equilibrium E_2, we have the following result.

Theorem 1. *The equilibrium E_2 is locally asymptotically stable if the following conditions are satisfied: $q_5 >$, $q_1q_2 - q_3 > 0$, $q_3(q_1q_2 - q_3) - q_1(q_1q_4 - q_5) > 0$ and $q_4(q_3(q_1q_2 - q_3) - q_1(q_1q_4 - q_5)) - q_5(q_2(q_1q_2 - q_3) - (q_1q_4 - q_5)) > 0$, where q_1, q_2, q_3, q_4 and q_5 are defined as above.*

Now the matrix at the equilibrium E_3 point is given as,

$$
M_3 = \begin{bmatrix}
-\frac{r}{L}F^* & -\beta F^* & 0 & -\kappa_1\delta_1 F^* & -\kappa_2\delta_2 F^*\\
\lambda + \beta_1 W^* & \beta_1 F^* - \pi_1 I^* - \varphi_1 & -\pi_1 W^* & 0 & 0\\
0 & -\pi_2 I^* & -\pi_2 W^* - \varphi_2 & 0 & 0\\
-\delta_1 P_W^* & \theta_1\varphi_1 & 0 & -\delta_1 F^* - \lambda_W & 0\\
-\delta_2 F_I^* & 0 & \theta_2\varphi_2 & 0 & -\delta_2 F^* - \lambda_I
\end{bmatrix}.
$$

Let $a_{21}^* = \lambda + \beta_1 W^*$, $a_{22}^* = \varphi_1 - \beta_1 F^* + \pi_1 I^*$, $a_{33}^* = \pi_2 W^* + \varphi_2$, $a_{44}^* = \delta_1 F^* + \lambda_W$ and $a_{55}^* = \delta_2 F^* + \lambda_I$. Now the corresponding characteristic equation of the Jacobian matrix M_3 as follows,

$$
A^5 + p_1 A^4 + p_2 A^3 + p_3 A^2 + p_4 A + p_5 = 0, \tag{24}
$$

where

$$
\begin{aligned}
p_1 =& \frac{r}{L}F^* + a_{22}^* + a_{33}^* + a_{44}^* + a_{55}^*,\\
p_2 =& a_{55}^*\left(a_{22}^* + a_{33}^* + \frac{r}{L}F^* + a_{44}^*\right)\\
&+ \left(a_{44}^*\left(a_{22}^* + a_{33}^* + \frac{r}{L}F^*\right) + \left(a_{22}^* a_{33}^* + \frac{r}{L}F^*(a_{22}^* + a_{33}^*) - \pi_1\pi_2 W^* I^* + \beta F^* a_{21}^*\right) - \kappa_1\delta_1 F^* P_W^*\right)\\
&- \kappa_2\delta_1 F^* P_I^*,\\
p_3 =& a_{44}^*\left(a_{22}^* a_{33}^* + \frac{r}{L}F^*(a_{22}^* + a_{33}^*) - \pi_1\pi_2 W^* I^* + \beta F^* a_{21}^*\right)\\
&+ \left(\frac{r}{L}F^* a_{22}^* a_{33}^* + \beta F^* a_{21}^* a_{33}^* - \frac{r}{L}\pi_1\pi_2 W^* I^* F^*\right)\\
&- \kappa_1\delta_1 F^* P_W^*(a_{22}^* + a_{33}^*) - \kappa_1\theta_1 F^* a_{21}^*\\
&+ a_{55}^*\left(a_{44}^*\left(a_{22}^* + a_{33}^* + \frac{r}{L}F^*\right) + \left(a_{22}^* a_{33}^* + \frac{r}{L}F^*(a_{22}^* + a_{33}^*) - \pi_1\pi_2 W^* I^* + \beta F^* a_{21}^*\right) - \kappa_1\delta_1 F^* P_W^*\right)\\
&- \kappa_2 F^*\left(\delta_2 P_I^*(a_{22}^* + a_{33}^*) + a_{44}^*\delta_2 P_I^*\right),
\end{aligned}
$$

$$p_4 = a_{55}^* \begin{pmatrix} a_{44}^*\left(a_{22}^* a_{33}^* + \frac{r}{L}F^*\left(a_{22}^* + a_{33}^*\right) - \pi_1\pi_2 W^* I^* + \beta F^* a_{21}^*\right) \\ +\left(\frac{r}{L}F^* a_{22}^* a_{33}^* F^* + \beta F^* a_{21}^* a_{33}^* - \frac{r}{L}\pi_1\pi_2 W^* I^* F^*\right) - \kappa_1\delta_1 F^* P_W^*\left(a_{22}^* + a_{33}^*\right) - \kappa_1\theta_1 F^* a_{22}^* \end{pmatrix}$$
$$+ a_{44}^*\left(\frac{r}{L}F^* a_{22}^* a_{33}^* + \beta F^* a_{21}^* a_{33}^* - \frac{r}{L}\pi_1\pi_2 W^* I^* F^*\right) - \kappa_1 F^*\left(a_{22}^* a_{33}^* \delta_1 P_W^* - \pi_1\pi_2\delta_1 W^* P_W^*\right)$$
$$- \kappa_1\theta_1 F^* a_{21}^* a_{33}^* - \kappa_2 F^*\left(\theta_2\pi_2 I^* a_{21}^* + a_{22}^* a_{33}^* \delta_2 P_I^* - \pi_1\pi_2\delta_2 W^* P_I^* + \delta_2 P_I^* a_{44}^*\left(a_{22}^* + a_{33}^*\right)\right),$$
$$p_5 = a_{55}^* \begin{pmatrix} a_{44}^*\left(\frac{r}{L}F^* a_{22}^* a_{33}^* + \beta F^* a_{21}^* a_{33}^* - \frac{r}{L}\pi_1\pi_2 W^* I^* F^*\right) - \kappa_1 F^*\left(a_{22}^* a_{33}^* \delta_1 P_W^* - \pi_1\pi_2\delta_1 W^* P_W^*\right) \\ -\kappa_1\theta_1 F^* a_{21}^* a_{33}^* \end{pmatrix}$$
$$- a_{44}^*\left(\pi_2\theta_2 I^* a_{21}^* + a_{22}^* a_{33}^* \delta_2 P_I^* - \pi_1\pi_2\delta_2 W^* P_I^*\right).$$

Here it is easy to see that $p_1 > 0$ if $\varphi_1 - \beta_1 F^* + \pi_1 I^* > 0$. Thus for the local stability of the equilibrium E_3, we have the following result.

Theorem 2. *The equilibrium is locally asymptotically stable if the following conditions are satisfied:* $p_5 > 0$, $p_1p_2 - p_3 > 0$, $p_3(p_1p_2 - p_3) - p_1(p_1p_4 - p_5) > 0$ *and* $p_4\left(p_3\left(p_1p_2 - p_3\right) - p_1\left(p_1p_4 - p_5\right)\right) - p_5\left(p_2\left(p_1p_2 - p_3\right) - \left(p_1p_4 - p_5\right)\right) > 0$, *where* p_1, p_2, p_3, p_4 *and* p_5 *are defined as above. The result on global stability is given in the following theorem.*

Theorem 3. *If the following inequalities hold in* Ω,

(1) $4\kappa_1\delta_1 P_W^* < \frac{r}{L}\lambda_W$,
(2) $4\kappa_2\delta_2 P_I^* < \frac{r}{L}\lambda_I$,
(3) $4\pi_1\pi_2 W^* I^* < \varphi_2\left(\varphi_1 - \beta_1 F^*\right)$,
(4) $\theta_1^2 < \left(\frac{m_1}{m_3}\right)\left(\varphi_1 - \beta_1 F^*\right)\lambda_W$,
(5) $\theta_2^2 < \left(\frac{m_2}{m_4}\right)\varphi_2\lambda_I$.

Then equilibrium E_3 *is globally asymptotically stable in the region* Ω.

Proof. To prove this, we take a positive definite function and under some conditions, we show that its derivative is negative. Proof of this theorem can be seen in Appendix A.

4 Numerical simulation

In order to check the feasibility of the results obtained in the above section for the existence of E_3 we conduct some numerical computations of the model (1) using MATLAB. The following parameter values have been chosen:

$$L = 100, Q = 50, \lambda = 0.8, r = 4, \beta = 0.04, \beta_1 = 0.003, \lambda_I = 1, \lambda_W = 1, \varphi_1 = 3, \varphi_2 = 5,$$
$$\pi_1 = 0.5, \pi_2 = 0.3, \delta_1 = 0.02, \delta_2 = 0.01, \kappa_1 = 0.5, \kappa_2 = 0.5, \theta_1 = 0.1, \theta_2 = 0.7$$

It is found that under the above mentioned numerical values, the equilibrium values of $E_3(F^*,\ W^*,\ I^*,\ P_W^*,\ P_I^*)$, are obtained as:

$$F^* = 45.8055,\ W^* = 5.5399,\ I^* = 7.506,\ P_W^* = 0.2891,\ P_I^* = 3.6033.$$

The eigenvalues of the Jacobian matrix M_3 corresponding to the equilibrium $E_3\left(F^*,\ W^*,\ I^*,\ P_W^*,\ P_I^*\right)$ for the model system (1) are:

$$-9.1568, -3.4231 - 1.1845i, -3.4231 + 1.1845i, -1.9972, -0.4837.$$

Here, it may be noted that three eigenvalues of matrix are negative and the other two eigenvalues are with negative real parts. Therefore, it implies that the interior equilibrium $E_3\left(F^*,\ W^*,\ I^*,\ P_W^*,\ P_I^*\right)$ is locally asymptotically stable.

For the above set of parameters, the computer generated graphs are plotted in Figs. 1-7 to predict the sensitivity of the model system under the change of different parameter values. Fig. 1 shows the change in the density of forest resources (F) with respect to time (t) for different values of β. Here, it may be noted that F decreases as β increases and finally attains its equilibrium. Figs. 2 and 3 respectively depict the effect of pollutants emitted from wood and non-wood industries on the density of forest resources by taking different values of δ_1 and δ_2. These figures also show that density of forest resources decreases as δ_1 and δ_2 decreases. The same effects (as in Fig. 1) can also be seen (Fig. 4) on the density of forest resources with change in the growth rate of wood industries due to forest resources β_1. But, β_1 as increases beyond certain limit, the model system has a destabilizing effect in the direction of density of forest resources. This is because of equilibrium condition $(\varphi_1 - \beta_1 F^*) > 0$, which may be violated for larger values of β_1. Thus, β_1 is an important parameter and it should remain under control in order to have the stable system i.e. the use of forest resources for wood-based industries should be held in check.

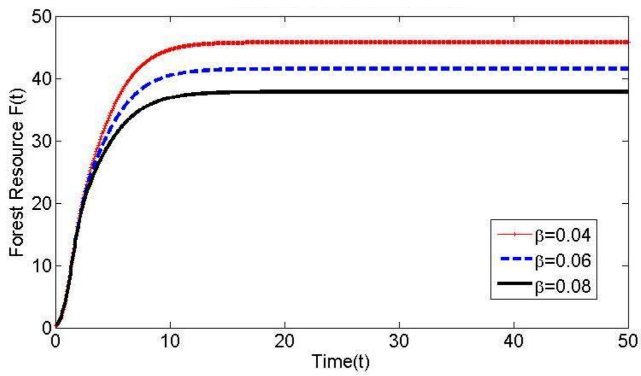

Fig. 1: Effect of β on forest resource

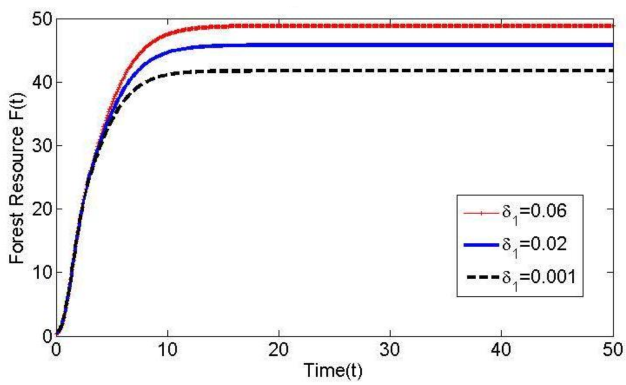

Fig. 2: Effect of δ_1 on forest resource

The variation of density of forest resources (F) and wood-based industries (W) with time for different values of Q are plotted in Figs. 5 and 6 respectively. The graphs show that both F and W decrease as the size of Q increases. This implies that though F and W based industries are not directly dependent on the growth of non-wood industries (Q), but indirectly it affects both. Fig. 5 also reveals that as Q increases beyond certain level (i.e. for $Q = 60$), the density of forest resources (F) leads to extinction. Similar effect can also be visualized on density of wood-based industries W for the same value of Q. This also describes that there is a competition effect between wood and non-wood industries, so if we increase the size of non-wood industries, the size of wood-based industries decreases. From this discussion, it can be said that Q has an important role in the above model system and it should be controlled, otherwise forest resources can become extinct.

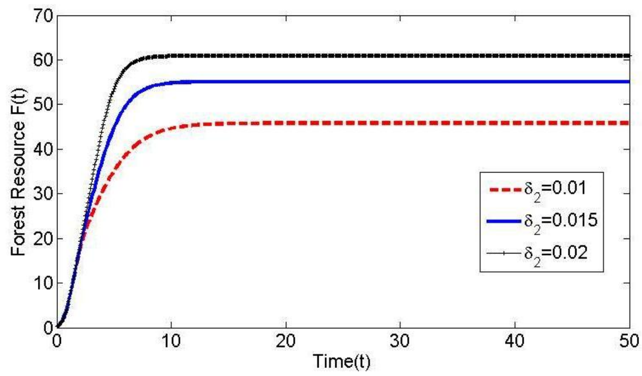

Fig. 3: Effect of δ_2 on forest resource

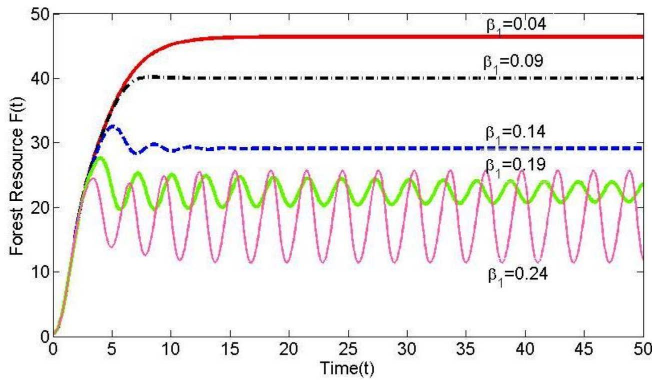

Fig. 4: Effect of β_1 on forest resource

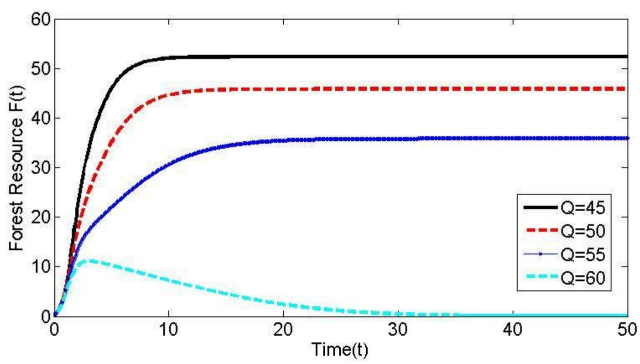

Fig. 5: Effect of Q on wood industry

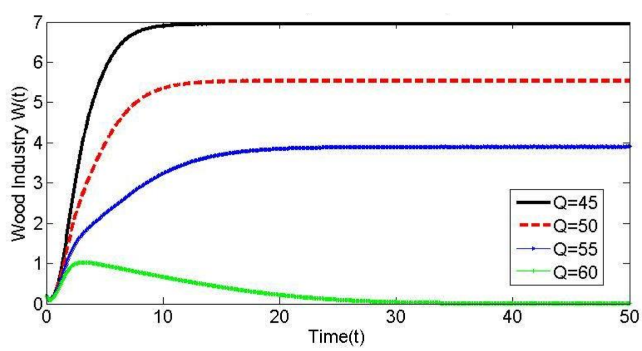

Fig. 6: Effect of Q on wood industry

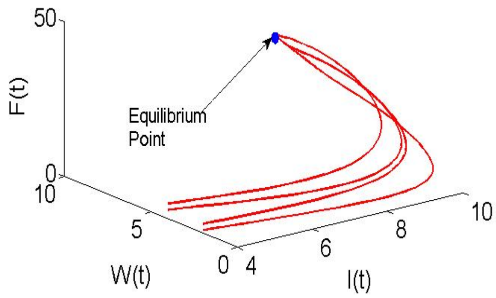

Fig. 7:

In order to illustrate the global stability of E_3 in $F-W-I$ plane, numerical simulation is performed for different initial starts and displayed in Fig. 7. From this figure, we can see that all trajectories initiating inside the region of attraction approach towards the equilibrium value.

5 Conclusions

In this paper, we have developed a mathematical model to study the effects of wood, non-wood based industries and the pollution emitted from them on the depletion of forest resources. In the modeling process, logistic growth of forest resources and migration of wood based industries to the forest region have been considered. Using the stability theory of differential equations, model has been analyzed qualitatively to establish the following:

(1) boundedness and nonnegativeness of solutions
(2) identification of equilibrium points along with their existence and uniqueness
(3) the stability of each equilibrium point.

The qualitative analysis shows that that the equilibrium point E_1 is always unstable, whenever the other equilibrium points E_2 or E_3 exists, while the equilibrium points E_2 and E_3 are stable under certain conditions. Numerical simulation has also been performed to illustrate the feasibility of the obtained results.

The results of the model show that wood-based industries deplete the forest resources both directly (through harvesting) and indirectly (through pollutants), but that non-wood industries only deplete the forest resources indirectly (through pollution). On the other side, it has also been derived that the pollutants of non-wood industries also get reduced with increase in the density of forest resources, because forest resources also serve as a sink for these pollutants. It has also been observed that in order to have a stable model system, the utilization of forest resources for wood- based industries should be limited. From this, it can be said that non-wood industries can be the alternative for wood-based industries to fulfill with the increasing demands of population. But, because pollution generated from non-wood industries also affects the forest resources indirectly, non-wood industries can only be enhanced up to certain extent.

Therefore, it can be concluded that with more and more industrialization either in the form of wood or non-wood, the forest resources are affected and may become extinct. So in order to avoid this, efforts should be made to limit the wood and non-wood industries in a forest habitat. However, it appears difficult to achieve because of increasing population and their demands. In view of this, the better way out is to regulate the pollution emitted from these industries. Polluting industries should be held in check, while non-polluting industries should be motivated in a forest habitat. This aspect of regulating pollution through different control strategies through mathematical modeling in a forest habitat has been left for future research.

Appendix. proof of theorem 3

To prove this theorem we consider a positive definite function as follows,

$$U = \left(F - F^* - F^* \ln \frac{F}{F^*} \right) + \frac{1}{2} m_1 (W - W^*)^2 + \frac{1}{2} m_2 (I - I^*)^2 + \frac{1}{2} m_3 (P_W - P_W^*)^2 + \frac{1}{2} m_4 (P_I - P_I^*)^2, \quad (25)$$

where m_1, m_2, m_3 and m_4 are positive constants. On differentiating Eq. (25) with respect to t' we get,

$$\frac{dU}{dt} = (F - F^*) \frac{\dot{F}}{F} + m_1 (W - W^*) \dot{W} + m_2 (I - I^*) \dot{I} + m_3 (P_W - P_W^*) \dot{P}_W + m_4 (P_I - P_I^*) \dot{P}_I.$$

Substituting the values of the derivative from the model (1), we have

$$\begin{aligned} \frac{dU}{dt} = & - \frac{r}{L} (F - F^*)^2 - m_1 \pi_1 I (W - W^*)^2 - m_3 \delta_1 F (P_W - P_W^*)^2 - m_4 \delta_2 F (P_I - P_I^*)^2 \\ & - m_1 (\varphi_1 - \beta_1 F^*) (W - W^*)^2 - m_2 \varphi_2 (I - I^*)^2 - m_2 \pi_2 W (I - I^*)^2 - m_3 \lambda_W (P_W - P_W^*)^2 \\ & - m_4 \lambda_I (P_I - P_I^*)^2 + (m_1 \beta_1 W^* + m_1 \lambda - \beta) (W - W^*) (F - F^*) \\ & - (m_1 \pi_1 W^* + m_2 \pi_2 I^*) (W - W^*) (I - I^*) - (\kappa_2 + m_4 \delta_2 P_I^*) (P_I - P_I^*) (F - F^*) \\ & - (\kappa_1 + m_2 \delta_1 P_W^*) (P_W - P_W^*) (F - F^*) + m_3 \theta_1 (F - F^*) (P_W - P_W^*) + m_4 \theta_2 (I - I^*) (P_I - P_I^*). \end{aligned}$$

Choosing $m_1(\beta_1 W^* + \lambda) - \beta = 0$ which gives $m_1 = \frac{\beta}{\beta_1 W^* + \lambda}$ is positive constant. Now $\dot{U}$ can be written as,

$$\begin{aligned} \frac{dU}{dt} = & - m_1 \pi_1 I (W - W^*)^2 - m_2 \pi_2 W (I - I^*)^2 - m_3 \delta_1 F (P_W - P_W^*)^2 - m_4 \delta_2 F (P_I - P_I^*)^2 \\ & - \frac{1}{2} Z_{11} (F - F^*)^2 - Z_{12} (F - F^*) (P_W - P_W^*) - \frac{1}{2} Z_{22} (P_W - P_W^*)^2 \\ & - \frac{1}{2} Z_{11} (F - F^*)^2 - Z_{13} (F - F^*) (P_I - P_I^*) - \frac{1}{2} Z_{33} (P_I - P_I^*)^2 \\ & - \frac{1}{2} Z_{44} (W - W^*)^2 + Z_{42} (W - W^*) (P_W - P_W^*) - \frac{1}{2} Z_{22} (P_W - P_W^*) \\ & - \frac{1}{2} Z_{44} (W - W^*)^2 - Z_{45} (W - W^*) (I - I^*) - \frac{1}{2} Z_{55} (I - I^*)^2 \\ & - \frac{1}{2} Z_{55} (I - I^*)^2 + Z_{53} (I - I^*) (P_I - P_I^*) - \frac{1}{2} Z_{33} (P_I - P_I^*)^2, \end{aligned}$$

where $Z_{11} = \frac{r}{L}$, $Z_{12} = \kappa_1 + m_3 \delta_1 P_W^*$, $Z_{22} = m_3 \lambda_W$, $Z_{13} = \kappa_2 + m_4 \delta_2 P_I^*$, $Z_{33} = m_4 \lambda_I$, $Z_{44} = m_1 (\varphi_1 - \beta_1 F^*)$, $Z_{42} = m_3 \theta_1$, $Z_{45} = m_1 \pi_1 W^* + m_2 \pi_2 I^*$, $Z_{55} = m_2 \varphi_2$ and $Z_{53} = m_4 \theta_2$.

Now according to $Z_{ij}^2 < Z_{ii} Z_{jj}$ we have,

$$(\kappa_1 + m_3 \delta_1 P_W^*)^2 < \frac{r}{L} m_3 \lambda_W, \quad (26)$$

$$(\kappa_2 + m_4 \delta_2 P_I^*)^2 < \frac{r}{L} m_4 \lambda_I, \quad (27)$$

$$(m_1 \pi_1 W^* + m_2 \pi_2 I^*)^2 < m_1 m_2 (\varphi_1 - \beta_1 F^*) \varphi_2, \quad (28)$$

$$m_3^2 \theta_1^2 < m_1 m_3 (\varphi_1 - \beta_1 F^*) \lambda_W, \quad (29)$$

$$m_4^2 \theta_2^2 < m_2 m_4 \varphi_2 \lambda_I. \quad (30)$$

Eqs. (26), (27) and (28) can be written as,

$$(\kappa_1 - m_3 \delta_1 P_W^*)^2 + 4 \kappa_1 \delta_1 m_3 P_W^* < \frac{r}{L} m_3 \lambda_W.$$

Choosing $m_3 = \frac{\kappa_1}{\delta_1 P_W^*}$ then we have

$$4 \kappa_1 \delta_1 P_W^* < \frac{r}{L} \lambda_W. \quad (31)$$

Again

$$(\kappa_2 - m_4\delta_2 P_I^*)^2 + 4\kappa_2\delta_2 m_4 P_I^* \ < \ \frac{r}{L} m_4 \lambda_I.$$

Choosing $m_4 = \frac{\kappa_2}{\delta_2 P_I^*}$, then we have

$$4\kappa_2\delta_2 P_I^* < \frac{r}{L}\lambda_I. \tag{32}$$

Now from Eq. (28),

$$(m_1\pi_1 W^* - m_2\pi_2 I^*)^2 + 4m_1 m_2 \pi_1\pi_2 W^* I^* < m_1 m_2 (\varphi_1 - \beta_1 F^*) \varphi_2.$$

Choosing $m_2 = \frac{\pi_1 W^*}{\pi_2 I^*} m_1$, then we have

$$4\pi_1\pi_2 W^* I^* < \varphi_2 (\varphi_1 - \beta_1 F^*). \tag{33}$$

Hence all conditions of Theorem 3 are satisfied for $\frac{dU}{dt}$ to be negative in the region Ω. Therefore equilibrium E_3 is globally asymptotically stable in the region Ω.

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