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The modified Kudryashov method for solving some seventh order nonlinear PDEs in mathematical physics*

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Abstract. With the aid of computer algebraic system Maple, we apply in this paper the modified Kudryashov method to construct the exact traveling wave solutions of the nonlinear seventh-order Sawada-Kotera-Ito equation, the nonlinear seventh-order Kaup-Kupershmidt equation and the nonlinear seventh-order Lax equation. The new contribution of this paper is to show that the solutions of these nonlinear partial differential equations (PDEs) can be expressed in terms of the symmetrical Lucas sine and Lucas cosine functions. The obtained solutions are new and not found elsewhere. The graphs for some of these solutions have been presented by choosing suitable values of parameters to visualize the mechanism of the given PDEs.

Keywords: the modified Kudryashov method, nonlinear PDEs, exact solutions, symmetrical Lucas functions, symmetrical Fibonacci functions

1 Introduction

Exact soliton solutions for nonlinear evolution equations play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, optics, condensed matter physics, plasma physics, and so on. The achievements and the research direction in the research area are the construction of the exact soliton solutions of these nonlinear PDEs using many mathematical methods. In recent decades, many effective methods are established and well-known for us, such as the inverse scattering transform[2], the Hirota method[8], the truncated Painlevé expansion method[12], the Bäcklund transform method[2, 8, 12, 22], the exp-function method[4, 10, 33], the simplest equation method[14, 54], the Weierstrass elliptic function method[11], the Jacobi elliptic function method[15, 16, 36, 51, 55], the tanh-function method[25, 29, 41], the $(\frac{G'}{G})$ -expansion method[1, 3, 5, 23, 32, 39], the modified simple equation method[9, 38, 46, 49, 50, 53], the Kudryashov method[13, 26, 27, 45, 52], the multiple exp-function algorithm method[18, 21], the transformed rational function method[19], the Frobenius decomposition technique[20], the local fractional variation iteration method[37], the local fractional series expansion method[17], the $(\frac{G'}{G}, \frac{1}{G})$-expansion method[40, 42–44, 47, 48] and so on.

The objective of this paper is to employ the modified Kudryashov method for finding the exact soliton solutions of the nonlinear seventh-order Sawada-Kotera-Ito equation, the nonlinear seventh-order Kaup-Kupershmidt equation and the nonlinear seventh-order Lax equation. These equations have been discussed before by many authors (see for example [6, 7, 34]) using other methods such as the Hirota direct method, the tanh-coth method, the sech method, the rational exp-function method and the exp-function method, but they are not investigated before using the modified Kudryashov method. Therefore, the new contribution of this paper is to use the later method to find new solutions of these equations in terms of symmetrical Lucas sine and Lucas cosine functions. This paper is organized as follows: In section 2, we give the description of the modified Kudryashov method. In Section 3, we apply this method with the aid of Maple to solve three seventh-order

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nonlinear PDEs indicated above. In Section 4, we present the physical explanations of the obtained soliton solutions. In Section 5, some conclusions are given.

2 Description of the modified kudryashov method

Suppose we have a nonlinear evolution equation in the form

$$F(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method[24]:

Step 1. Using the wave transformation

$$u(x,t) = u(\xi), \qquad \xi = kx + \omega t, \tag{2}$$

to reduce Eq. (1) to the following ODE:

$$P(u, u', u'', \cdots) = 0, \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while k, ω are constants and $' = d/d\xi$.

Step 2. We suppose that Eq. (3) has the formal solution

$$u(\xi) = \sum_{n=0}^{N} a_n Q(\xi)^n, \tag{4}$$

where a_n $(n = 0, 1, \cdots, N)$ are constants to be determined, such that $a_N \neq 0$, and $Q(\xi)$ is the solution of the equation

$$Q'(\xi) = \left[Q^2(\xi) - Q(\xi)\right] \ln(a), \tag{5}$$

Eq. (5) has the solutions

$$Q(\xi) = \frac{1}{1 \pm a^{\xi}}, \tag{6}$$

where $a > 0$, $a \neq 1$ is a number. If $a = e$, then we have the Kudryashov method which has been applied by many authors, see for example [13, 26, 27, 45, 52].

Step 3. We determine the positive integer N in Eq. (4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (3).

Step 4. Substitute Eq. (4) along with Eq. (5) into Eq. (3), we calculate all the necessary derivatives $u', u'', \cdots$ of the function $u(\xi)$. As a result of this substitution, we get a polynomial of $Q^i(\xi)$, $(i = 0, 1, 2, \cdots)$. In this polynomial we gather all terms of same powers of $Q^i(\xi)$ and equating them to zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica to get the unknown parameters a_n $(n = 0, 1, \cdots, N)$, k and ω. Consequently, we obtain the exact solutions of Eq. (1).

Remark 1. The obtained solutions can be depended on the symmetrical hyperbolic Lucas functions and Fibonacci functions proposed by Stakhov and Rozin[30, 31]. The symmetrical Lucas sine, cosine, tangent and cotangent functions are respectively, defined as

$$\begin{gathered} sLs\,(\xi) = a^{\xi} - a^{-\xi}, \quad cLs\,(\xi) = a^{\xi} + a^{-\xi}. \\ tLs\,(\xi) = \frac{a^{\xi} - a^{-\xi}}{a^{\xi} + a^{-\xi}} = \frac{sLs\,(\xi)}{cLs\,(\xi)}, \quad ctLs\,(\xi) = \frac{a^{\xi} + a^{-\xi}}{a^{\xi} - a^{-\xi}} = \frac{cLs\,(\xi)}{sLs\,(\xi)}, \end{gathered} \tag{7}$$

while the symmetrical Fibonacci sine, cosine tangent, and cotangent functions are respectively, defined as

$$sFs(\xi)=\frac{a^{\xi}-a^{-\xi}}{\sqrt{5}},\quad cFs(\xi)=\frac{a^{\xi}+a^{-\xi}}{\sqrt{5}}.$$
$$tFs(\xi)=\frac{a^{\xi}-a^{-\xi}}{a^{\xi}+a^{-\xi}}=\frac{sFs(\xi)}{cFs(\xi)},\quad ctFs(\xi)=\frac{a^{\xi}+a^{-\xi}}{a^{\xi}-a^{-\xi}}=\frac{cFs(\xi)}{sFs(\xi)}. \tag{8}$$

Also, these functions satisfy the following formulas:

$$[cLs(\xi)]^2-[sLs(\xi)]^2=4, \tag{9}$$
$$[cFs(\xi)]^2-[sFs(\xi)]^2=\frac{4}{5}. \tag{10}$$

The obtained solutions in this paper can be obtained in terms of the symmetrical hyperbolic Lucas functions.

3 Applications

In this section, we apply the modified Kudryashov method to find the exact solutions of the following nonlinear partial differential equations:

3.1 example 1. the nonlinear seventh-order Sawada-Kotera-Ito equation

This equation is well known[7, 28, 34, 35] and has the form

$$u_t+252u^3u_x+63u_x^3+378uu_xu_{xx}+126u^2u_{3x}+63u_{xx}u_{3x}+42u_xu_{4x}+21uu_{5x}+u_{7x}=0. \tag{11}$$

Let us now solve Eq. (11) by using the modified Kudryashov method. To this end, we use the wave transformation (2) to reduce Eq. (11) to the following ODE:

$$\omega u'+252ku^3u'+63k^3u'^3+378k^3uu'u''+126k^3u^2u^{(3)}+63k^5u''u^{(3)} \tag{12}$$
$$+42k^5u'u^{(4)}+21k^5uu^{(5)}+k^7u^{(7)}=0. \tag{13}$$

Balancing $u^{(7)}$ with u^3u' yields $N=2$. Consequently, Eq. (13) has the formal solution

$$u(\xi)=a_0+a_1Q(\xi)+a_2Q(\xi)^2, \tag{14}$$

where a_0, a_1 and a_2 are constants to be determined such that $a_2\neq 0$. From (14), we can obtain

$$u'=(\ln a)(a_1+2Qa_2)Q(Q-1), \tag{15}$$
$$u''=(\ln a)^2Q(Q-1)[(-1+2Q)a_1+2Q(3Q-2)a_2], \tag{16}$$
$$u^{(3)}=(\ln a)^3Q(Q-1)[(1-6Q+6Q^2)a_1+2Q(4-15Q+12Q^2)a_2], \tag{17}$$
$$u^{(4)}=(\ln a)^4Q(Q-1)[(-1+14Q-36Q^2+24Q^3)a_1+2Q(-8+57Q-108Q^2+60Q^3)a_2] \tag{18}$$
$$u^{(5)}=(\ln a)^5Q(Q-1)[(1-30Q+150Q^2-240Q^3+120Q^4)a_1+2Q(16-195Q+660Q^2-840Q^3+360Q^4)a_2], \tag{19}$$
$$u^{(6)}=(\ln a)^6Q(Q-1)[(-1+62Q-540Q^2+1560Q^3-1800Q^4+720Q^5)a_1+2Q(-32+633Q-3420Q^2+7500Q^3-7200Q^4+2520Q^5)a_2], \tag{20}$$
$$u^{(7)}=(\ln a)^7Q(Q-1)[(1-126Q+1806Q^2-8400Q^3+16800Q^4-15120Q^5+5040Q^6)a_1+2Q(64-1995Q+16212Q^2-54600Q^3+88200Q^4-68040Q^5+20160Q^6)a_2], \tag{21}$$
$$Q^9: 40\,320k^7a_2(\ln a)^7+34\,272k^5a_2^2(\ln a)^5+8064k^3a_2^3(\ln a)^3+504ka_2^4(\ln a)=0,$$

Substituting (14) - (21) into (13) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$Q^8: -176\,400k^7a_2(\ln a)^7+5040a_1k^7(\ln a)^7-124\,236k^5a_2^2(\ln a)^5+29\,988a_1k^5a_2(\ln a)^5$$
$$-20\,412k^3a_2^3(\ln a)^3+15\,876a_1k^3a_2^2(\ln a)^3-504ka_2^4(\ln a)+1764a_1ka_2^3(\ln a)=0,$$
$$Q^7: -20\,160k^7a_1(\ln a)^7+312\,480k^7a_2(\ln a)^7+4284k^5a_1^2(\ln a)^5-103\,824k^5a_1a_2(\ln a)^5$$
$$+173\,376k^5a_2^2(\ln a)^5+15\,120a_0k^5a_2(\ln a)^5+9450k^3a_1^2a_2(\ln a)^3-39\,312k^3a_1a_2^2(\ln a)^3$$
$$+16\,884k^3a_2^3(\ln a)^3+10\,584a_0k^3a_2^2(\ln a)^3+2268ka_1^2a_2^2(\ln a)-1764ka_1a_2^3(\ln a)$$
$$+1512a_0ka_2^3(\ln a)=0,$$
$$Q^6: 31\,920k^7a_1(\ln a)^7-285\,600k^7a_2(\ln a)^7-13\,734k^5a_1^2(\ln a)^5+136\,626k^5a_1a_2(\ln a)^5$$
$$+2520a_0k^5a_1(\ln a)^5-115\,122k^5a_2^2(\ln a)^5-50\,400a_0k^5a_2(\ln a)^5+1575k^3a_1^3(\ln a)^3$$
$$-22\,680k^3a_1^2a_2(\ln a)^3+31\,626k^3a_1a_2^2(\ln a)^3+11\,340a_0k^3a_1a_2(\ln a)^3-4536k^3a_2^3(\ln a)^3$$
$$-25\,704a_0k^3a_2^2(\ln a)^3+1260ka_1^3a_2(\ln a)-2268ka_1^2a_2^2(\ln a)+3780a_0ka_1a_2^2(\ln a)$$
$$-1512a_0ka_2^3(\ln a)=0,$$
$$Q^5: -25\,200k^7a_1(\ln a)^7+141\,624k^7a_2(\ln a)^7-7560k^5a_0a_1(\ln a)^5+63\,000k^5a_0a_2(\ln a)^5$$
$$+16\,338k^5a_1^2(\ln a)^5-83\,874k^5a_1a_2(\ln a)^5+35\,742k^5a_2^2(\ln a)^5+3024k^3a_0^2a_2(\ln a)^3$$
$$+2268k^3a_0a_1^2(\ln a)^3-26\,460k^3a_0a_1a_2(\ln a)^3+20\,160k^3a_0a_2^2(\ln a)^3-3591k^3a_1^3(\ln a)^3$$
$$+17\,514k^3a_1^2a_2(\ln a)^3-8190k^3a_1a_2^2(\ln a)^3+1512ka_0^2a_2^2(\ln a)+3024ka_0a_1^2a_2(\ln a)$$
$$-3780ka_0a_1a_2^2(\ln a)+252ka_1^4(\ln a)-1260ka_1^3a_2(\ln a)=0,$$
$$Q^4: 10\,206k^7a_1(\ln a)^7-36\,414k^7a_2(\ln a)^7+8190k^5a_0a_1(\ln a)^5-35\,910k^5a_0a_2(\ln a)^5$$
$$-8715k^5a_1^2(\ln a)^5+23\,289k^5a_1a_2(\ln a)^5-4032k^5a_2^2(\ln a)^5+756k^3a_0^2a_1(\ln a)^3$$
$$-6804k^3a_0^2a_2(\ln a)^3-4914k^3a_0a_1^2(\ln a)^3+19\,656k^3a_0a_1a_2(\ln a)^3-5040k^3a_0a_2^2(\ln a)^3$$
$$+2583k^3a_1^3(\ln a)^3-4284k^3a_1^2a_2(\ln a)^3+2268ka_0^2a_1a_2(\ln a)-1512ka_0^2a_2^2(\ln a)$$
$$+756ka_0a_1^3(\ln a)-3024ka_0a_1^2a_2(\ln a)-252ka_1^4(\ln a)=0,$$
$$Q^3: -1932k^7a_1(\ln a)^7+4118a_2k^7(\ln a)^7-3780k^5a_0a_1(\ln a)^5+8862a_2k^5a_0(\ln a)^5$$
$$+1953k^5a_1^2(\ln a)^5-2205a_2k^5a_1(\ln a)^5-1512k^3a_0^2a_1(\ln a)^3+4788a_2k^3a_0^2(\ln a)^3$$
$$+3276k^3a_0a_1^2(\ln a)^3-4536a_2k^3a_0a_1(\ln a)^3-567k^3a_1^3(\ln a)^3+504a_2ka_0^3(\ln a)$$
$$+756ka_0^2a_1^2(\ln a)-2268a_2ka_0^2a_1(\ln a)-756ka_0a_1^3(\ln a)+2\omega a_2(\ln a)=0,$$
$$Q^2: 127k^7a_1(\ln a)^7-128a_2k^7(\ln a)^7+651k^5a_0a_1(\ln a)^5-672a_2k^5a_0(\ln a)^5-126k^5a_1^2(\ln a)^5$$
$$+882k^3a_0^2a_1(\ln a)^3-1008a_2k^3a_0^2(\ln a)^3-630k^3a_0a_1^2(\ln a)^3+252ka_0^3a_1(\ln a)$$
$$-504a_2ka_0^3(\ln a)-756ka_0^2a_1^2(\ln a)+\omega a_1(\ln a)-2\omega a_2(\ln a)=0,$$
$$Q^1: -a_1k^7(\ln a)^7-21a_1k^5a_0(\ln a)^5-126a_1k^3a_0^2(\ln a)^3-252a_1ka_0^3(\ln a)-\omega a_1(\ln a)=0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Case 1.

$$a_0=-\frac{(\ln a)^2k^2}{3},\ \ a_1=4(\ln a)^2k^2,\ \ a_2=-4(\ln a)^2k^2,\ \ \omega=\frac{4(\ln a)^6k^7}{3},\ \ a=a. \tag{22}$$

From (6), (7), (14), (22), we obtain the following exact solutions of Eq. (11)

$$u_1(x,t)=-\frac{(\ln a)^2k^2}{3}-\left(\frac{2(\ln a)k}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{23}$$

$$u_2(x,t)=-\frac{(\ln a)^2k^2}{3}+\left(\frac{2(\ln a)k}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{24}$$

where $\xi = kx + \frac{4(\ln a)^6 k^7}{3} t$.

Case 2.

$$\begin{aligned} a_0 &= a_0, \;\; a_1 = 2(\ln a)^2 k^2, \;\; a_2 = -2(\ln a)^2 k^2, \\ \omega &= -252 k a_0^3 - 126 k^3 a_0^2 (\ln a)^2 - 21 k^5 a_0 (\ln a)^4 - k^7 (\ln a)^6, \;\; a = a. \end{aligned} \tag{25}$$

In this case, we deduce the following exact solutions of Eq. (11)

$$u_3(x,t) = a_0 - 2\left(\frac{(\ln a)k}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{26}$$

$$u_4(x,t) = a_0 + 2\left(\frac{(\ln a)k}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{27}$$

where $\xi = kx - \left(252 k a_0^3 + 126 k^3 a_0^2 (\ln a)^2 + 21 k^5 a_0 (\ln a)^4 + k^7 (\ln a)^6\right) t$.

3.2 example 2. the nonlinear seventh-order kaup-kupershmidt equation

This equation is well known[7, 28, 35] and has the form

$$u_t + 2016 u^3 u_x + 630 u_x^3 + 2268 u u_x u_{xx} + 504 u^2 u_{3x} + 252 u_{xx} u_{3x} + 147 u_x u_{4x} + 42 u u_{5x} + u_{7x} = 0. \tag{28}$$

Let us now solve Eq. (28) by using the modified Kudryashov method. To this end, we use the wave transformation (2) to reduce Eq. (28) to the following ODE:

$$\begin{aligned} &\omega u' + 2016 k u^3 u' + 630 k^3 u'^3 + 2268 k^3 u u' u'' + 504 k^3 u^2 u^{(3)} + 252 k^5 u'' u^{(3)} \\ &+ 147 k^5 u' u^{(4)} + 42 k^5 u u^{(5)} + k^7 u^{(7)} = 0. \end{aligned} \tag{29}$$

Balancing $u^{(7)}$ with $u^3 u'$ yields $N = 2$. Consequently, Eq. (29) has the formal solution (14). Substituting (14) - (21) into (29) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$\begin{aligned} Q^9 :& 40\,320 k^7 a_2 (\ln a)^7 + 101\,808 k^5 a_2^2 (\ln a)^5 + 44\,352 k^3 a_2^3 (\ln a)^3 + 4032 k a_2^4 (\ln a) = 0, \\ Q^8 :& -176\,400 k^7 a_2 (\ln a)^7 + 5040 a_1 k^7 (\ln a)^7 - 376\,992 k^5 a_2^2 (\ln a)^5 + 81\,144 a_1 k^5 a_2 (\ln a)^5 \\ & - 114\,912 k^3 a_2^3 (\ln a)^3 + 84\,672 a_1 k^3 a_2^2 (\ln a)^3 - 4032 k a_2^4 (\ln a) + 14\,112 a_1 k a_2^3 (\ln a) = 0, \\ Q^7 :& -20\,160 k^7 a_1 (\ln a)^7 + 312\,480 k^7 a_2 (\ln a)^7 + 11\,592 k^5 a_1^2 (\ln a)^5 - 286\,272 k^5 a_1 a_2 (\ln a)^5 \\ & + 539\,532 k^5 a_2^2 (\ln a)^5 + 30\,240 a_0 k^5 a_2 (\ln a)^5 + 49\,140 k^3 a_1^2 a_2 (\ln a)^3 - 214\,704 k^3 a_1 a_2^2 (\ln a)^3 \\ & + 97\,776 k^3 a_2^3 (\ln a)^3 + 51\,408 a_0 k^3 a_2^2 (\ln a)^3 + 18\,144 k a_1^2 a_2^2 (\ln a) - 14\,112 k a_1 a_2^3 (\ln a) \\ & + 12\,096 a_0 k a_2^3 (\ln a) = 0, \end{aligned}$$

$$\begin{aligned}
Q^6 :& 31\,920k^7a_1(\ln a)^7 - 285\,600k^7a_2(\ln a)^7 - 38\,052k^5a_1^2(\ln a)^5 + 385\,518k^5a_1a_2(\ln a)^5 \\
&+ 5040a_0k^5a_1(\ln a)^5 - 369\,348k^5a_2^2(\ln a)^5 - 100\,800a_0k^5a_2(\ln a)^5 + 8190k^3a_1^3(\ln a)^3 \\
&- 120\,960k^3a_1^2a_2(\ln a)^3 + 177\,912k^3a_1a_2^2(\ln a)^3 + 52\,920a_0k^3a_1a_2(\ln a)^3 - 27\,216k^3a_2^3(\ln a)^3 \\
&- 127\,008a_0k^3a_2^2(\ln a)^3 + 10\,080ka_1^3a_2(\ln a) - 18\,144ka_1^2a_2^2(\ln a) + 30\,240a_0ka_1a_2^2(\ln a) \\
&- 12\,096a_0ka_2^3(\ln a) = 0, \\
Q^5 :& - 25\,200k^7a_1(\ln a)^7 + 141\,624k^7a_2(\ln a)^7 - 15\,120k^5a_0a_1(\ln a)^5 + 126\,000k^5a_0a_2(\ln a)^5 \\
&+ 46\,662k^5a_1^2(\ln a)^5 - 243\,726k^5a_1a_2(\ln a)^5 + 119\,112k^5a_2^2(\ln a)^5 + 12\,096k^3a_0^2a_2(\ln a)^3 \\
&+ 10\,584k^3a_0a_1^2(\ln a)^3 - 125\,496k^3a_0a_1a_2(\ln a)^3 + 101\,808k^3a_0a_2^2(\ln a)^3 - 19\,278k^3a_1^3(\ln a)^3 \\
&+ 96\,516k^3a_1^2a_2(\ln a)^3 - 47\,880k^3a_1a_2^2(\ln a)^3 + 12\,096ka_0^2a_2^2(\ln a) + 24\,192ka_0a_1^2a_2(\ln a) \\
&- 30\,240ka_0a_1a_2^2(\ln a) + 2016ka_1^4(\ln a) - 10\,080ka_1^3a_2(\ln a) = 0, \\
Q^4 :& 10\,206k^7a_1(\ln a)^7 - 36\,414k^7a_2(\ln a)^7 + 16\,380k^5a_0a_1(\ln a)^5 - 71\,820k^5a_0a_2(\ln a)^5 \\
&- 25\,935k^5a_1^2(\ln a)^5 + 70\,392k^5a_1a_2(\ln a)^5 - 14\,112k^5a_2^2(\ln a)^5 + 3024k^3a_0^2a_1(\ln a)^3 \\
&- 27\,216k^3a_0^2a_2(\ln a)^3 - 23\,436k^3a_0a_1^2(\ln a)^3 + 95\,256k^3a_0a_1a_2(\ln a)^3 - 26\,208k^3a_0a_2^2(\ln a)^3 \\
&+ 14\,490k^3a_1^3(\ln a)^3 - 24\,696k^3a_1^2a_2(\ln a)^3 + 18\,144ka_0^2a_1a_2(\ln a) - 12\,096ka_0^2a_2^2(\ln a) \\
&+ 6048ka_0a_1^3(\ln a) - 24\,192ka_0a_1^2a_2(\ln a) - 2016ka_1^4(\ln a) = 0, \\
Q^3 :& - 1932k^7a_1(\ln a)^7 + 4118a_2k^7(\ln a)^7 - 7560k^5a_0a_1(\ln a)^5 + 17\,724a_2k^5a_0(\ln a)^5 \\
&+ 6174k^5a_1^2(\ln a)^5 - 7056a_2k^5a_1(\ln a)^5 - 6048k^3a_0^2a_1(\ln a)^3 + 19\,152a_2k^3a_0^2(\ln a)^3 \\
&+ 16\,128k^3a_0a_1^2(\ln a)^3 - 22\,680a_2k^3a_0a_1(\ln a)^3 - 3402k^3a_1^3(\ln a)^3 + 4032a_2ka_0^3(\ln a) \\
&+ 6048ka_0^2a_1^2(\ln a) - 18\,144a_2ka_0^2a_1(\ln a) - 6048ka_0a_1^3(\ln a) + 2\omega a_2(\ln a) = 0, \\
Q^2 :& 127k^7a_1(\ln a)^7 - 128a_2k^7(\ln a)^7 + 1302k^5a_0a_1(\ln a)^5 - 1344a_2k^5a_0(\ln a)^5 \\
&- 441k^5a_1^2(\ln a)^5 + 3528k^3a_0^2a_1(\ln a)^3 - 4032a_2k^3a_0^2(\ln a)^3 - 3276k^3a_0a_1^2(\ln a)^3 \\
&+ 2016ka_0^3a_1(\ln a) - 4032a_2ka_0^3(\ln a) - 6048ka_0^2a_1^2(\ln a) + \omega a_1(\ln a) - 2\omega a_2(\ln a) = 0, \\
Q^1 :& - a_1k^7(\ln a)^7 - 42a_1k^5a_0(\ln a)^5 - 504a_1k^3a_0^2(\ln a)^3 - 2016a_1ka_0^3(\ln a) - \omega a_1(\ln a) = 0.
\end{aligned}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following result:

$$a_0 = -\frac{(\ln a)^2k^2}{24},\ \ a_1 = \frac{(\ln a)^2\,k^2}{2},\ \ a_2 = -\frac{(\ln a)^2k^2}{2},\ \ \omega = \frac{(\ln a)^6k^7}{48},\ \ a = a. \tag{30}$$

From (6), (7), (14), (30), we obtain the following exact solutions of Eq. (28).

$$u_1(x,t) = -\frac{(\ln a)^2k^2}{24} - \frac{1}{2}\left(\frac{(\ln a)k}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{31}$$

$$u_2(x,t) = -\frac{(\ln a)^2k^2}{24} + \frac{1}{2}\left(\frac{(\ln a)k}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{32}$$

where $\xi = kx + \frac{(\ln a)^6k^7}{48}t$.

3.3 example 3. the nonlinear seventh-order lax equation

This equation is well known [7, 28, 35] and has the form

$$u_t + 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u^2u_{3x} + 70u_{xx}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0. \tag{33}$$

Let us now solve Eq. (33) by using the modified Kudryashov method. To this end, we use the wave transformation (2) to reduce Eq. (33) to the following ODE:

$$\omega u' + 140ku^3u' + 70k^3u'^3 + 280k^3uu'u'' + 70k^3u^2u^{(3)} + 70k^5u''u^{(3)} + 42k^5u'u^{(4)} + 14k^5uu^{(5)} + k^7u^{(7)} = 0. \tag{34}$$

Balancing $u^{(7)}$ with u^3u' yields $N = 2$. Consequently, Eq. (34) has the formal solution (14). Substituting (14) - (21) into (34) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$\begin{aligned}
Q^9 :& 40\,320k^7a_2(\ln a)^7 + 30\,240k^5a_2^2(\ln a)^5 + 5600k^3a_2^3(\ln a)^3 + 280ka_2^4(\ln a) = 0,\\
Q^8 :& 5040a_1k^7(\ln a)^7 - 176\,400k^7a_2(\ln a)^7 - 111\,384k^5a_2^2(\ln a)^5 + 24\,696a_1k^5a_2(\ln a)^5\\
&- 14\,420k^3a_2^3(\ln a)^3 + 10\,780a_1k^3a_2^2(\ln a)^3 - 280ka_2^4(\ln a) + 980a_1ka_2^3(\ln a) = 0,\\
Q^7 :& 312\,480k^7a_2(\ln a)^7 - 20\,160k^7a_1(\ln a)^7 + 3528k^5a_1^2(\ln a)^5 - 86\,688k^5a_1a_2(\ln a)^5\\
&+ 158\,424k^5a_2^2(\ln a)^5 + 10\,080a_0k^5a_2(\ln a)^5 + 6300k^3a_1^2a_2(\ln a)^3 - 27\,160k^3a_1a_2^2(\ln a)^3\\
&+ 12\,180k^3a_2^3(\ln a)^3 + 6720a_0k^3a_2^2(\ln a)^3 + 1260ka_1^2a_2^2(\ln a) - 980ka_1a_2^3(\ln a)\\
&+ 840a_0ka_2^3(\ln a) = 0,\\
Q^6 :& 31\,920k^7a_1(\ln a)^7 - 285\,600k^7a_2(\ln a)^7 - 11\,508k^5a_1^2(\ln a)^5 + 116\,032k^5a_1a_2(\ln a)^5\\
&+ 1680a_0k^5a_1(\ln a)^5 - 107\,660k^5a_2^2(\ln a)^5 - 33\,600a_0k^5a_2(\ln a)^5 + 1050k^3a_1^3(\ln a)^3\\
&- 15\,400k^3a_1^2a_2(\ln a)^3 + 22\,330k^3a_1a_2^2(\ln a)^3 + 7000a_0k^3a_1a_2(\ln a)^3 - 3360k^3a_2^3(\ln a)^3\\
&- 16\,520a_0k^3a_2^2(\ln a)^3) + 700ka_1^3a_2(\ln a) - 1260ka_1^2a_2^2(\ln a + 2100a_0ka_1a_2^2(\ln a)\\
&- 840a_0ka_2^3(\ln a) = 0,\\
Q^5 :& 141\,624k^7a_2(\ln a)^7 - 25\,200k^7a_1(\ln a)^7 - 5040k^5a_0a_1(\ln a)^5 + 42\,000k^5a_0a_2(\ln a)^5\\
&+ 14\,000k^5a_1^2(\ln a)^5 - 72\,800k^5a_1a_2(\ln a)^5 + 34\,412k^5a_2^2(\ln a)^5 + 1680k^3a_0^2a_2(\ln a)^3\\
&+ 1400k^3a_0a_1^2(\ln a)^3 - 16\,520k^3a_0a_1a_2(\ln a)^3 + 13\,160k^3a_0a_2^2(\ln a)^3 - 2450k^3a_1^3(\ln a)^3\\
&+ 12\,180k^3a_1^2a_2(\ln a)^3 - 5950k^3a_1a_2^2(\ln a)^3 + 840ka_0^2a_2^2(\ln a) + 1680ka_0a_1^2a_2(\ln a)\\
&- 2100ka_0a_1a_2^2(\ln a) + 140ka_1^4(\ln a) - 700ka_1^3a_2(\ln a) = 0,\\
Q^4 :& 10\,206k^7a_1(\ln a)^7 - 36\,414k^7a_2(\ln a)^7 + 5460k^5a_0a_1(\ln a)^5 - 23\,940k^5a_0a_2(\ln a)^5\\
&- 7700k^5a_1^2(\ln a)^5 + 20\,818k^5a_1a_2(\ln a)^5 - 4032k^5a_2^2(\ln a)^5 + 420k^3a_0^2a_1(\ln a)^3\\
&- 3780k^3a_0^2a_2(\ln a)^3 - 3080k^3a_0a_1^2(\ln a)^3 + 12\,460k^3a_0a_1a_2(\ln a)^3 - 3360k^3a_0a_2^2(\ln a)^3\\
&+ 1820k^3a_1^3(\ln a)^3 - 3080k^3a_1^2a_2(\ln a)^3 + 1260ka_0^2a_1a_2(\ln a) - 840ka_0^2a_2^2(\ln a)\\
&+ 420ka_0a_1^3(\ln a) - 1680ka_0a_1^2a_2(\ln a) - 140ka_1^4(\ln a) = 0,\\
Q^3 :& 4118a_2k^7(\ln a)^7 - 1932k^7a_1(\ln a)^7 - 2520k^5a_0a_1(\ln a)^5 + 5908a_2k^5a_0(\ln a)^5\\
&+ 1806k^5a_1^2(\ln a)^5 - 2058a_2k^5a_1(\ln a)^5 - 840k^3a_0^2a_1(\ln a)^3 + 2660a_2k^3a_0^2(\ln a)^3\\
&+ 2100k^3a_0a_1^2(\ln a)^3 - 2940a_2k^3a_0a_1(\ln a)^3 - 420k^3a_1^3(\ln a)^3 + 280a_2ka_0^3(\ln a)\\
&+ 420ka_0^2a_1^2(\ln a) - 1260a_2ka_0^2a_1(\ln a) - 420ka_0a_1^3(\ln a) + 2\omega a_2(\ln a) = 0,\\
Q^2 :& 127k^7a_1(\ln a)^7 - 128a_2k^7(\ln a)^7 + 434k^5a_0a_1(\ln a)^5 - 448a_2k^5a_0(\ln a)^5 - 126k^5a_1^2(\ln a)^5\\
&+ 490k^3a_0^2a_1(\ln a)^3 - 560a_2k^3a_0^2(\ln a)^3 - 420k^3a_0a_1^2(\ln a)^3 + 140ka_0^3a_1(\ln a)\\
&- 280a_2ka_0^3(\ln a) - 420ka_0^2a_1^2(\ln a) + \omega a_1(\ln a) - 2\omega a_2(\ln a) = 0,\\
Q^1 :& - a_1k^7(\ln a)^7 - 14a_1k^5a_0(\ln a)^5 - 70a_1k^3a_0^2(\ln a)^3 - 140a_1ka_0^3(\ln a) - \omega a_1(\ln a) = 0.
\end{aligned}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following result:

$$\begin{aligned}
a_0 &= a_0, \; a_1 = 2(\ln a)^2k^2, \; a_2 = -2(\ln a)^2k^2,\\
\omega &= -140ka_0^3 - 70k^3a_0^2(\ln a)^2 - 14k^5a_0(\ln a)^4 - k^7(\ln a)^6, \; a = a.
\end{aligned} \tag{35}$$

From (6), (7), (14), (35), we obtain the following exact solutions of Eq. (33).

$$u_1(x,t) = a_0 - 2\left(\frac{(\ln a)k}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{36}$$

$$u_2(x,t) = a_0 + 2\left(\frac{(\ln a)k}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{37}$$

where $\xi = kx - \left(140ka_0^3 + 70k^3a_0^2(\ln a)^2 + 14k^5a_0(\ln a)^4 + k^7(\ln a)^6\right)t$.

4 physical explanations of the obtained solutions

We have shown in section 3, that the solutions of the three seventh order nonlinear PDEs are written in terms of the symmetrical Lucas sine and Lucas cosine functions. In this section, we will present some graphs of these solutions by choosing suitable values of the parameters a_0, k, a to visualize the mechanism of the original nonlinear PDEs. Using mathematical software Maple or Mathematica, we organize these graphs as follows: In Figs. 1 and 2, the plots of the solutions (26), (27) are drawn by choosing $a_0 = 0.0001$, $k = 0.005$ and $a = 1.5$. In Figs. 3 and 4, the plots of the solutions (31), (32) are drawn by choosing $k = 3$, $a = 2$. In Fig. 5, the plot of the solution (37) is drawn by choosing $a_0 = 1$, $k = 4$ and $a = 2$. All these figures are new and not found elsewhere, which include the graphs of the symmetrical Lucas sine and Lucas cosine functions.

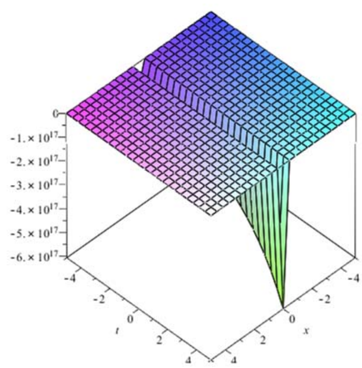

Fig. 1: The plots of the solutions (26) when $a_0 = 0.0001$, $k = 0.005$, $a = 1.5$

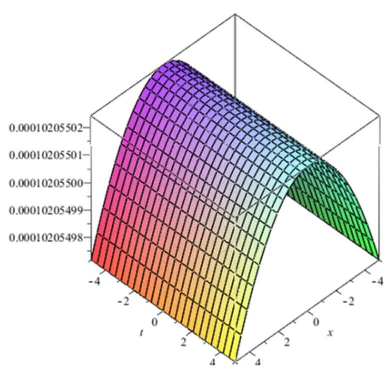

Fig. 2: The plots of the solutions (27) when $a_0 = 0.0001$, $k = 0.005$, $a = 1.5$

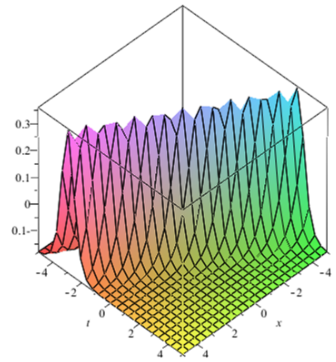

Fig. 3: The plots of the solutions (31) when $k = 3$, $a = 2$

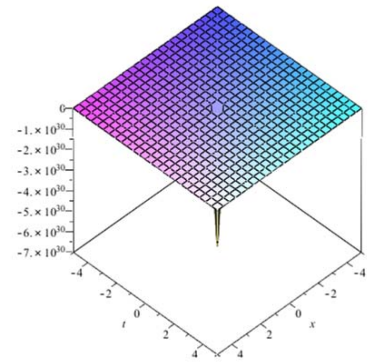

Fig. 4: The plots of the solutions (32) when $k = 3$, $a = 2$

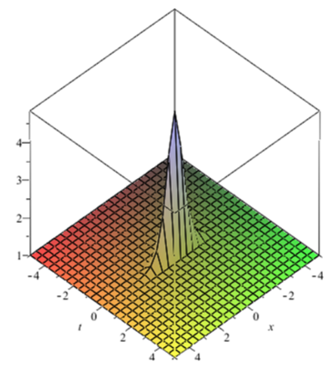

Fig. 5: The plot of the solution (37) when $a_0 = 1$, $k = 4$, $a = 2$

5 conclusions

In this paper, we have used the modified Kudryashov method to solve three nonlinear PDEs of seventh-order, namely, the nonlinear seventh-order Sawada-Kotera-Ito equation, the nonlinear seventh-order Kaup-Kupershmidt equation and the nonlinear seventh-order Lax equation. With help of Maple, we have obtained many solutions in terms of the symmetrical Lucas sine and Lucas cosine functions. On comparing our results (26) (27) of Eq. (11) and the results (31), (32) of Eq. (28) as well as the results (36), (37) of Eq. (33) with the well-known results obtained in [6, 7, 34] using different methods, we deduce that our results are new and not found elsewhere. All solutions obtained in this article have been checked with the Maple by putting them back into the original equations (11), (28) and (33). To our knowledge, these nonlinear equations and their soliton solutions can be applied to many fields such as the quantum mechanics and nonlinear optics. Finally, the modified Kudryashov method is direct, effective and can be applied to many other nonlinear PDEs in mathematical physics.

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