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The $(\frac{G'}{G}, \frac{1}{G})$-expansion method and its applications for constructing the exact solutions of the nonlinear Zoomeron equation

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Abstract. In this paper, we apply the $(\frac{G'}{G}, \frac{1}{G})$-expansion method to find the explicit traveling wave solutions with parameters of an incognito evolution equation, that called Zoomeron equation. This method can be thought of as the generalization of the well-known $(\frac{G'}{G})$-expansion method given recently by M. Wang et al. When these parameters are taken special values, the solitary wave solutions can be found from the traveling waves. On using this method, we obtain new solutions of the nonlinear Zoomeron equation which play an important role in mathematical physics.

Keywords: the $(\frac{G'}{G}, \frac{1}{G})$-expansion method, the Zoomeron equation, exact solutions, traveling wave solutions, solitary wave solutions, nonlinear evolution equations

1 Introduction

In mathematical physics, it is well known that many nonlinear evolution equations are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics solid-state physics, plasma physics, plasma waves, biology and so on. Investigation of explicit traveling wave solutions to these equations play an important role in the study of nonlinear physical phenomenon. Searching for exact solutions of the nonlinear evolution equations using various methods has become extremely valuable, and many powerful and sufficient methods such as the inverse scattering transform method[3], the Hirota method[13], the painleve expansion method[16–18, 28], the Backlund transform method[24, 25], the exp-function method[7, 8, 12, 15, 29, 40], the tanh function method[2, 11, 30, 42], the Jacobi elliptic function expansion method[22, 23, 27], the $(\frac{G'}{G})$-expansion method[4–7, 14, 19, 20, 27, 33, 41], the first integral method[26], the $(\frac{G'}{G}, \frac{1}{G})$-expansion method[21, 31, 34–39] and so on.

The objective of this paper is to apply the $(\frac{G'}{G}, \frac{1}{G})$-expansion method to construct the exact traveling wave solutions of the following nonlinear evolution equations:

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2\left(u^2\right)_{xt} = 0\,, \tag{1}$$

where $u(x, y, t)$ is the amplitude of the relevant wave mode. This equation is an incognito evolution equation, that called Zoomeron equation. Recently, Reza Abazari[1] has discussed Eq. (1) by using the $(\frac{G'}{G})$-expansion method. According to our recent search, there are very few articles about this equation. We know that this equation was introduced by Calogero and Degasperis[9]. The $(\frac{G'}{G}, \frac{1}{G})$-expansion method gives new solutions of Eq. (1) which can be considered as a generalization of the results of [1].

The rest of this paper is organized as follows : In Sec. 2, we give the description of the $(\frac{G'}{G}, \frac{1}{G})$-expansion method. In Sec. 3, we apply this method to solve Eq. (1). In Sec. 4, some conclusions are given.

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2 Description of the two variable $(\frac{G'}{G}, \frac{1}{G})$-expansion method

Before, we describe the main steps of this method, we need the following remarks (see [21, 31, 34–39]):

Remark 1. If we consider the second order linear ODE:

$$G^{''}(\xi) + \lambda G(\xi) = \mu \,, \tag{2}$$

and set $\phi = G'/G \, \psi = 1/G$, then we get

$$\phi' = -\phi^2 + \mu\psi - \lambda \,, \qquad \psi^{'} = -\phi\psi \,. \tag{3}$$

Remark 2. If $\lambda < 0$, then the general solutions of Eq. (2) has the form:

$$G(\xi) = A_1 \sinh(\xi \sqrt{-\lambda}) + A_2 \cosh(\xi \sqrt{-\lambda}) + \frac{\mu}{\lambda} \,, \tag{4}$$

where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma^2 + \mu^2}(\phi^2 - 2\mu \, \psi + \lambda) \,, \tag{5}$$

where $\sigma = A_1^2 - A_2^2$.

Remark 3. If $\lambda > 0$, then the general solutions of Eq. (2) has the form:

$$G(\xi) = A_1 \sin(\xi \sqrt{\lambda}) + A_2 \cos(\xi \sqrt{\lambda}) + \frac{\mu}{\lambda} \,, \tag{6}$$

and hence

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma^2 - \mu^2}(\phi^2 - 2\mu \, \psi + \lambda) \,, \tag{7}$$

where $\sigma = A_1^2 + A_2^2$.

Remark 4. If $\lambda = 0$, then the general solutions of Eq. (2) has the form:

$$G(\xi) = \frac{\mu}{2} \, \xi^2 + A_1\xi + A_2 \tag{8}$$

and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu \, A_2}(\phi^2 - 2\mu \, \psi). \tag{9}$$

Suppose we have the following nonlinear evolution align

$$F(u \,, u_x \,, u_y \,, u_t \,, u_{xx} \,, u_{xy} \,, u_{xt} \,, u_{yy}, \cdots) = 0 \,, \tag{10}$$

where F is a polynomial in $u(x, y, t)$ and its partial derivatives. In the following, we give the main steps of the $(\frac{G'}{G}, \frac{1}{G})$-expansion method:

Step 1. The traveling wave transformation

$$u(x \,, y \,, t) = u(\xi), \qquad \xi = x \, - c \, y \, - w \, t, \tag{11}$$

where c and w are constant, reduces Eq. (10) to an ODE in the form:

$$P\left(u \,, u^{'} \,, u^{''} \,, \cdots\right) = 0 \,, \tag{12}$$

where P is a polynomial of $u(\xi)$ and its total derivatives with respect to ξ.
Step 2. Assuming that the solution of Eq. (12) can be expressed by a polynomial in the two variables ϕ and ψ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i\, \phi^i + \sum_{i=1}^{N} b_i\, \phi^{i-1}\, \psi\,, \tag{13}$$

where a_i $(i = 0, 1, \cdots, N)$ and b_i $(i = 1, \cdots, N)$ are constants to be determined later.
Step 3. Determine the positive integer N in Eq. (13) by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in Eq.(11).
Step 4. Substitute Eq. (13) into Eq. (12) along with Eq. (3) and Eq. (5), the left- hand side Eq.(11) can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not longer than 1. Equating each coefficients of this polynomial to zero, yields a system of algebraic aligns which can be solved by using the Maple or Mathematica to get the values of a_i , b_i , w, μ, A_1, A_2, and λ where $\lambda < 0$.
Step 5. Similar to step 4, substitute Eq. (13) into Eq. (12) along with Eq. (3) and Eq. (7) for $\lambda > 0$, (or Eq. (3) and Eq. (9) for $\lambda = 0$), we obtain the exact solutions of Eq. (12) expressed by trigonometric functions (or by rational functions) respectively.

3 An application

In this section, we will apply the $(\frac{G'}{G}, \frac{1}{G})$- expansion method described in Sec. 2 to find the explicit traveling wave solutions of the nonlinear Zoomeron align (1). To this end, we use the traveling wave transformation (11) to reduce Eq. (1) into the following ODE:

$$c(1-w^2)u'' - 2\, wu^3 - Ru = 0, \tag{14}$$

where R is a constant of integration and $w \neq \{0, 1\}$.
By balancing between u'' with u^3 in Eq. (14) we get $N = 1$. Consequently, we get

$$u(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi), \tag{15}$$

where a_0, a_1 and b_1 are constants to be determined later.
There are three cases to be discussed as follows:

Case 1. Hyperbolic function solutions $(\lambda < 0)$
If $\lambda < 0$, substituting Eq. (15) into Eq. (14) and using Eq. (3) and Eq. (5), the left-hand side of Eq. (14) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero, yields a system of algebraic aligns in a_0 , a_1 , b_1 , μ , λ, and w as follows:

$$\begin{aligned}
\phi^3 \quad &: \quad 2c(1-w^2)a_1 - 2w\left(a_1^3 - \frac{3\,\lambda\;\, a_1\, b_1^2}{\lambda^2\,\sigma + \mu^2}\right) = 0,\\
\phi^2\psi \quad &: \quad 2\,c(1-w^2)b_1 - 2w\left(3\;\, a_1^2\, b_1 - \frac{\lambda\, b_1^3}{\lambda^2\,\sigma + \mu^2}\right) = 0,\\
\phi^2 \quad &: \quad c(1-w^2)\frac{\lambda\,\mu\, b_1}{\lambda^2\,\sigma + \mu^2} - 2w\left(3\;\, a_1^2\, a_0 - \frac{3\,\lambda\, a_0\, b_1^2}{\lambda^2\,\sigma + \mu^2} - \frac{2\,\lambda^2\,\mu\, b_1^3}{(\lambda^2\,\sigma + \mu^2)^2}\right) = 0,\\
\phi\psi \quad &: \quad -3c(1-w^2)\,\mu\, a_1 - 2w\left(6\, a_0\, a_1\, b_1 + \frac{6\,\lambda\,\mu\, a_1\, b_1^2}{\lambda^2\,\sigma + \mu^2}\right) = 0\,,\\
\psi \quad &: \quad c(1-w^2)\left(\lambda\, b_1 - \frac{2\,\lambda\,\mu^2\, b_1}{\lambda^2\,\sigma + \mu^2}\right) - 2w\left(3\, a_0^2\, b_1 + \frac{6\,\lambda\,\mu\, a_0\, b_1^2}{\lambda^2\,\sigma + \mu^2} + \frac{4\,\lambda^2\,\mu^2\, b_1^3}{(\lambda^2\,\sigma + \mu^2)^2} - \frac{\lambda^2\, b_1^3}{\lambda^2\,\sigma + \mu^2}\right) - R\, b_1 = 0,\\
\phi \quad &: \quad 2c(1-w^2)\,\lambda\, a_1 - 2w\left(3\, a_0^2\, a_1 - \frac{3\,\lambda^2\, a_1\, b_1^2}{\lambda^2\,\sigma + \mu^2}\right) - R\, a_1 = 0\,,\\
\phi^0 \quad &: \quad c(1-w^2)\frac{\lambda^2\,\mu\, b_1}{\lambda^2\,\sigma + \mu^2} - 2w\left(a_0^3 - \frac{3\,\lambda^2\, a_0\, b_1^2}{\lambda^3\,\sigma + \mu^2} - \frac{2\,\lambda\,\mu\, b_1^3}{(\lambda\,\sigma + \mu^2)^2}\right) - R\, a_0 = 0,
\end{aligned}$$

On solving the above algebraic aligns using the Maple or Mathematica, we get the following results:

$$a_0 = 0\,, \qquad a_1 = \pm\sqrt{\frac{c(1-w^2)}{4\,w}}\,, \qquad b_1 = \pm\sqrt{\frac{c(1-w^2)(\lambda^2\,\sigma + \mu^2)}{-4\,\lambda\,w}}\,, \qquad R = \frac{1}{2}\lambda\, c(1-w^2).$$

In this case, the exact solution of Eq. (14) has the form :

$$u(\xi)=\pm\sqrt{\frac{c\,\lambda\,(1-w^2)}{-4\,w}}\left[\frac{A_1\cosh(\xi\sqrt{-\lambda})+A_2\sinh(\xi\sqrt{-\lambda})}{A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\frac{\mu}{\lambda}}\right]$$
$$\pm\sqrt{\frac{c(1-w^2)(\lambda^2\,\sigma\;+\mu^2)}{-4\,\lambda\,w}}\left[\frac{1}{A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\frac{\mu}{\lambda}}\right], \tag{16}$$

where $\sigma=A_1^2-A_2^2$.
If $A_1=0$, $A_2\neq0$ and $\mu=0$, then we have the solitary wave solution

$$u(\xi)=\pm\sqrt{\frac{c\,\lambda\,(1-w^2)}{-4\,w}}\,[\tanh(\xi\sqrt{-\lambda})\;+i\;sech(\xi\sqrt{-\lambda})]\;,\;i=\sqrt{-1}, \tag{17}$$

If $A_1\neq0$, $A_2=0$ and $\mu=0$, then we have the solitary wave solution

$$u(\xi)=\pm\sqrt{\frac{c\,\lambda\,(1-w^2)}{-4\,w}}\,[\coth(\xi\sqrt{-\lambda})\;+\;cosech(\xi\sqrt{-\lambda})]. \tag{18}$$

Case 2. Trigonometric function solutions $(\lambda>0)$
If $\lambda>0$, substituting (15) into (14) and using Eq. (3) and Eq. (7), the left-hand side of Eq. (14) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero, yields a system of algebraic aligns in a_0, a_1, b_1, μ, λ and w as follows:

$$\begin{aligned}
\phi^3 &: \; 2\,c(1-w^2)a_1\;-2w\left(a_1^3+\tfrac{3\,\lambda\;a_1\,b_1^2}{\lambda^2\,\sigma\,-\mu^2}\right)=0\,,\\
\phi^2\psi &: \; 2\,c(1-w^2)b_1\;-2w\left(3\;a_1^2\,b_1+\tfrac{\lambda\,b_1^3}{\lambda^2\,\sigma\,-\mu^2}\right)=0\,,\\
\phi^2 &: \; -c(1-w^2)\tfrac{\lambda\,\mu\,b_1}{\lambda^2\,\sigma\,-\mu^2}-2w\left(3\;a_1^2\,a_0+\tfrac{3\,\lambda\,a_0\,b_1^2}{\lambda^2\,\sigma\,-\mu^2}-\tfrac{2\,\lambda^2\,\mu\,b_1^3}{(\lambda^2\,\sigma\,-\mu^2)^2}\right)=0\,,\\
\phi\psi &: \; -3\,c(1-w^2)\,\mu\,a_1\;-2w\left(6\,a_0\,a_1\,b_1-\tfrac{6\,\lambda\,\mu\,a_1\,b_1^2}{\lambda^2\,\sigma\,-\mu^2}\right)=0\,,\\
\psi &: \; c(1-w^2)\left(\lambda\,b_1+\tfrac{2\,\lambda\,\mu^2\,b_1}{\lambda^2\,\sigma\,-\mu^2}\right)-2w\left(3\,a_0^2\,b_1-\tfrac{6\,\lambda\,\mu\,a_0\,b_1^2}{\lambda^2\,\sigma\,-\mu^2}+\tfrac{4\,\lambda^2\,\mu^2\,b_1^3}{(\lambda^2\,\sigma\,-\mu^2)^2}+\tfrac{\lambda^2\,b_1^3}{\lambda^2\,\sigma\,-\mu^2}\right)-\;R\,b_1=0\,,\\
\phi &: \; 2\,c(1-w^2)\,\lambda\,a_1\;-2w\left(3\,a_0^2\,a_1+\tfrac{3\,\lambda^2\,a_1\,b_1^2}{\lambda^2\,\sigma\,-\mu^2}\right)-\;R\,a_1=0\,,\\
\phi^0 &: \; -\,c(1-w^2)\tfrac{\lambda^2\,\mu\,b_1}{\lambda^2\,\sigma\,-\mu^2}-2w\left(a_0^3\;+\tfrac{3\,\lambda^2\,a_0\,b_1^2}{\lambda^3\,\sigma\,-\mu^2}-\tfrac{2\,\lambda^2\,\mu\,b_1^3}{(\lambda^2\,\sigma\,-\mu^2)^2}\right)-\;R\,a_0=0\,,
\end{aligned}$$

On solving the above algebraic aligns using the Maple or Mathematica, we get the following results:

$$a_0=0\,,\qquad a_1=\pm\sqrt{\frac{c(1-w^2)}{4\,w}},\qquad b_1=\pm\sqrt{\frac{c(1-w^2)(\lambda^2\,\sigma\;-\mu^2)}{4\,\lambda\,w}},\qquad R=\frac{1}{2}\lambda\,c(1-w^2).$$

In this case, the exact solution of Eq. (14) has the form :

$$u(\xi)=\pm\sqrt{\frac{c\,\lambda(1-w^2)}{4\,w}}\left[\frac{A_1\cos(\xi\sqrt{\lambda})-A_2\sin(\xi\sqrt{\lambda})}{A_1\sin(\xi\sqrt{\lambda})+A_2\cos(\xi\sqrt{\lambda})+\frac{\mu}{\lambda}}\right]$$
$$\pm\sqrt{\frac{c(1-w^2)(\lambda^2\,\sigma\;-\mu^2)}{4\,\lambda\,w}}\left[\frac{1}{A_1\sin(\xi\sqrt{\lambda})+A_2\cos(\xi\sqrt{\lambda})+\frac{\mu}{\lambda}}\right], \tag{19}$$

where $\sigma=A_1^2+A_2^2$.
If $A_1=0$, $A_2\neq0$ and $\mu=0$, then we have the periodic solutions

$$u(\xi)=\pm\sqrt{\frac{c\,\lambda(1-w^2)}{4\,w}}\,[\tan(\xi\sqrt{\lambda})\;+\;sec(\xi\sqrt{\lambda})]. \tag{20}$$

If $A_1\neq0$, $A_2=0$ and $\mu=0$, then we have the periodic solutions

$$u(\xi)=\pm\sqrt{\frac{c\,\lambda(1-w^2)}{4\,w}}\,[\cot(\xi\sqrt{\lambda})\;+\;cosec(\xi\sqrt{\lambda})]. \tag{21}$$

Case 3. Rational function solutions $(\lambda = 0)$

If $\lambda = 0$, substituting (15) into (14) and using Eq. (3) and Eq. (9), the left-hand side of Eq. (14) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to the zero, yields a system of algebraic aligns in a_0, a_1, b_1, μ, λ and w as follows:

$$\begin{aligned}
\phi^3 &: \quad 2c(1-w^2)a_1 \; - 2w\left(a_1^3 + \tfrac{3\,a_1\,b_1^2}{A_1^2-2\mu\,A_2}\right) = 0,\\
\phi^2\psi &: \quad 2c(1-w^2)b_1 \; - 2w\left(3\;a_1^2\,b_1 + \tfrac{b_1^3}{A_1^2-2\mu\,A_2}\right) = 0,\\
\phi^2 &: \quad -c(1-w^2)\tfrac{\mu\,b_1}{A_1^2-2\mu\,A_2} - 2w\left(3\;a_1^2\,a_0 + \tfrac{3\,a_0\,b_1^2}{A_1^2-2\mu\,A_2} - \tfrac{2\,\mu\,b_1^3}{(\,A_1^2-2\mu\,A_2)^2}\right) = 0,\\
\phi\psi &: \quad -3c(1-w^2)\,\mu\,a_1 \; - 2w\left(6\,a_0\,a_1\,b_1 - \tfrac{6\,\mu\,a_1\,b_1^2}{A_1^2-2\mu\,A_2}\right) = 0\,,\\
\psi &: \quad 4c(1-w^2)\tfrac{\mu^2\,b_1}{A_1^2-2\mu\,A_2} - 2w\left(3\,a_0^2\,b_1 - \tfrac{6\,\mu\,a_0\,b_1^2}{A_1^2-2\mu\,A_2} + \tfrac{4\,\mu^2\,b_1^3}{(\,A_1^2-2\mu\,A_2)^2}\right) - \;R\,b_1 = 0,\\
\phi &: \quad -6\,w\,a_0^2\,a_1 - \;R\,a_1 = 0,\\
\phi^0 &: \quad -2\,w\;a_0^3 \;-\; R\,a_0 = 0.
\end{aligned}$$

On solving the above algebraic aligns using the Maple or Mathematica, we get the following results:

$$a_0 = 0, \qquad a_1 = \pm\sqrt{\tfrac{c(1-w^2)}{4\,w}}, \qquad b_1 = \pm\sqrt{\tfrac{c(1-w^2)(\,A_1^2-2\mu\,A_2)}{4\,w}}, \qquad R = \tfrac{2\,c\,\mu^2(1-w^2)}{A_1^2-2\mu\,A_2}.$$

In this case, the exact solution of Eq. (14) has the form :

$$u(\xi) = \pm\sqrt{\frac{c(1-w^2)}{4\,w}}\left[\frac{\mu\,\xi + A_1}{\frac{\mu}{2}\,\xi^2 + A_1\xi + A_2}\right] \pm \sqrt{\frac{c(1-w^2)(\,A_1^2-2\mu\,A_2)}{4\,\lambda\,w}}\left[\frac{1}{\frac{\mu}{2}\,\xi^2 + A_1\xi + A_2}\right]. \tag{22}$$

Remark 5. All solutions of this paper have been checked with Maple 14 by putting them back into the original align Eq. (1).

4 Conclusions

In this paper, the $(\frac{G'}{G},\frac{1}{G})$-expansion method was employed to obtain the explicit traveling wave solutions with parameters of an incognito evolution align, that called Zoomeron align (1). As the two parameters A_1 and A_2 take special values, we obtain the solitary wave solutions. When $\mu = 0$ and $b_i = 0$ in Eq. (2) and Eq. (13), the $(\frac{G'}{G},\frac{1}{G})$-expansion method reduces to the $(\frac{G'}{G})$-expansion method. So the $(\frac{G'}{G},\frac{1}{G})$-expansion method is an extension of the $(\frac{G'}{G})$-expansion method. The method in this paper is more effective and more general than the $(\frac{G'}{G})$-expansion method because it gives exact solutions in more general forms. In summary, the advantage of the $(\frac{G'}{G},\frac{1}{G})$-expansion method over the $(\frac{G'}{G})$-expansion method is that the solutions using the first method recover the solutions using the second one.

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