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Positive solutions for a class of chemically reacting systems with sign-changing weights

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Abstract. We study the existence of positive solution for the system

$$\begin{cases} -\Delta u = \lambda a(x)[f(v) - \frac{1}{u^\alpha}], & x \in \Omega \\ -\Delta v = \lambda b(x)[g(v) - \frac{1}{v^\beta}], & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter, Ω is a bounded domain with smooth boundary, α, $\beta \in (0, 1)$. Here $a(x)$ and $b(x)$ are C^1 sing-changing functions that maybe negative near the boundary and f, g are C^1 nondecreasing functions, such that $f, g : (0, \infty) \to (0, \infty)$; $f(s) > 0, g(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} \frac{f(Mg(s)}{s} = 0$. We discuss the existence of positive solution when f, g, $a(x)$ and $b(x)$ satisfy certain additional conditions. We use the method of subsupersolution to establish our results.

Keywords: positive solutions, chemically reacting systems, sub-supersolutions

1 Introduction

We study the existence of positive solution for the system

$$\begin{cases} -\Delta u = \lambda a(x)[f(v) - \frac{1}{u^\alpha}], & x \in \Omega \\ -\Delta v = \lambda b(x)[g(v) - \frac{1}{v^\beta}], & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where λ is a positive parameter, Ω is a bounded domain with smooth boundary, α, $\beta \in (0, 1)$. Here $a(x)$ and $b(x)$ are C^1 sing-changing functions that maybe negative near the boundary and f, g are C^1 nondecreasing functions, such that $f, g : (0, \infty) \to (0, \infty)$; $f(s) > 0, g(s) > 0$ for $s > 0$.

Systems of singular equations like (1) are the stationary counterpart of general evolu- tionary problems of the form

$$\begin{cases} u_t = \eta\Delta u + \lambda\Big(f(v) - \frac{1}{u^\alpha}\Big), & x \in \Omega \\ v_t = \delta\Delta v + \lambda\Big(g(u) - \frac{1}{v^\beta}\Big), & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where η and δ are positive parameters. This system is motivated by an interesting applications in chemically reacting systems, where u represents the density of an activator chemical substance and v is an inhibitor. The slow diffusion of u and the fast diffusion of v is translated into the fact that η is small and δ is large (see [1]).

Recently, such problems have been studied in [3, 5, 6]. Also in [6], the authors have studied the existence results for the system (1) in the case $a \equiv 1$, $b \equiv 1$. Here we focus on further extending the study in [5]

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to the system (1). In fact, we study the existence of positive solution to the system (1) with sign-changing weight functions $a(x), b(x)$. Due to these weight functions, the extensions are challenging and nontrivial. Our approach is based on the method of sub-super solutions (see [4, 7]).

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda\phi, & x \in \Omega \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue λ_1 of (2) such that $\phi(x) > 0$ in Ω and $||\phi||_\infty = 1$. Let $m, \mu, \delta > 0$ be such that

$$\mu \le \phi \le 1, \ \ x \in \Omega - \overline{\Omega_\delta}, \tag{3}$$

$$\frac{2}{1+s}(1 - \frac{2s}{1+s})|\nabla\phi|^2 \ge m, \ \ x \in \overline{\Omega_\delta}, \tag{4}$$

for $s = \alpha, \beta$, where $\overline{\Omega_\delta} := \{x \in \Omega \,|\, d(x, \partial\Omega) \le \delta\}$. This is possible since $|\nabla\phi| \neq 0$ on $\partial\Omega$ while $\phi = 0$ on $\partial\Omega$. We will also consider the unique solution $e \in W_0^{1,2}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta e = 1, & x \in \Omega \\ e = 0 & x \in \partial\Omega, \end{cases}$$

to discuss our existence result, it is known that $e > 0$ in Ω and $\frac{\partial e}{\partial n} < 0$ on $\partial\Omega$.

Here we assume that the weight functions $a(x)$ and $b(x)$ take negative values in $\overline{\Omega_\delta}$, but require $a(x)$ and $b(x)$ be strictly positive in $\Omega - \overline{\Omega_\delta}$. To be precise we assume that there exist positive constants a_0 , a_1 , b_0 and b_1 such that $a(x) \ge -a_0$, $b(x) \ge -b_0$ on $\overline{\Omega_\delta}$ and $a(x) \ge a_1$, $b(x) \ge b_1$ on $\Omega - \overline{\Omega_\delta}$.

2 Existence result

In this section, we shall establish our existence result via the method of sub- supersolution. A pair of nonnegative functions $(\psi_1, \psi_2), (z_1, z_2)$ are called a subsolution and supersulution of (1) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$\begin{aligned} -\Delta\psi_1 &\le \lambda a(x)[f(\psi_2) - \frac{1}{\psi_1^\alpha}], && x \in \Omega, \\ -\Delta\psi_2 &\le \lambda b(x)[g(\psi_1) - \frac{1}{\psi_2^\beta}], && x \in \Omega, \\ -\Delta z_1 &\ge \lambda a(x)[f(z_2) - \frac{1}{z_1^\alpha}], && x \in \Omega, \\ -\Delta z_2 &\ge \lambda b(x)[f(z_1) - \frac{1}{z_2^\beta}], && x \in \Omega. \end{aligned}$$

Then the following result holds :

Lemma 1. *(See [2]) Suppose there exist sub and supersolutions (ψ_1, ψ_2) and (z_1, z_2) respectively of (1) such that $(\psi_1, \psi_2) \le (z_1, z_2)$. Then (1) has solution (u, v) such that $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.*

We make the following assumptions :

(H1) $f, g : [0, \infty) \to [0, \infty)$ are C^1 nondecreasing functions such that $f(s), g(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} g(s) = \infty$.

(H2) $\lim_{s\to\infty} \frac{f(Mg(s)^{\frac{1}{q-1}})}{s^{p-1}} = 0$, for all $M > 0$.

(H3) Suppose that there exists $\epsilon > 0$ such that :

(i)$f(\frac{\mu\epsilon}{2}) > (\frac{2}{\mu\epsilon})^{\alpha}$,

(ii)$g(\frac{\mu\epsilon}{2}) > (\frac{2}{\mu\epsilon})^{\beta}$,

(iii)$\frac{\lambda_1 f(\epsilon)}{m} \leq \min\left\{\frac{2^{\alpha-1}(1+\alpha)}{\epsilon^{\alpha}}, \frac{Na_1(1+\alpha)}{2a_0}, \frac{(1+\beta)b_0 2^{\beta-1}}{a_0\epsilon^{\beta}}, \frac{(1+\beta)Mb_1}{2a_0}\right\}$,

(iv)$\frac{\lambda_1 g(\epsilon)}{m} \leq \min\left\{\frac{2^{\beta-1}(1+\beta)}{\epsilon^{\beta}}, \frac{Na_1(1+\alpha)}{2b_0}, \frac{(1+\alpha)a_0 2^{\beta-1}}{b_0\epsilon^{\alpha}}, \frac{(1+\beta)Mb_1}{2b_0}\right\}$,

where

$$N = f(\frac{\mu\epsilon}{2}) - (\frac{2}{\mu\epsilon})^{\alpha},$$

and

$$M = g(\frac{\mu\epsilon}{2}) - (\frac{2}{\mu\epsilon})^{\beta}.$$

We are now ready to give our existence result.

Theorem 1. *Let(H1)-(H3) hold. Then there exists a positive solution of (1) for every* $\lambda \in [\lambda_*(\epsilon), \lambda^*(\epsilon)]$, *where*

$$\lambda^* = \min\left\{\frac{m\epsilon}{2a_0 f(\epsilon)}, \frac{m\epsilon}{2b_0 g(\epsilon)}\right\},$$

and

$$\lambda_* = \max\left\{\frac{\epsilon^{\alpha+1}\lambda_1}{2^{\alpha}a_0(1+\alpha)}, \frac{\epsilon^{\beta+1}\lambda_1}{2^{\beta}b_0(1+\beta)}, \frac{\lambda_1\epsilon}{(1+\alpha)Na_1}, \frac{\lambda_1\epsilon}{(1+\beta)Mb_1}\right\}.$$

Remark 1. Note that $(H3)$ implies $\lambda_* < \lambda^*$.

Example 1. Let $f(s) = e^{\frac{s}{s+1}}$, $g(s) = e^s$. Here $f(s)$, $g(s) > 0$ for $s > 0$, f , g are non-decreasing functions and

$$\lim_{s\to\infty} \frac{f(Mg(s))}{s} = 0,$$

for all $M > 0$, and $\lim_{s\to\infty} g(s) = \infty$. We can choose $\epsilon > 0$ so small that f, g satisfy (H3).

Proof. We shall verify that

$$(\psi_1, \psi_2) = \big(\frac{1}{2}\epsilon\phi^{\frac{2}{1+\alpha}}, \frac{1}{2}\epsilon\phi^{\frac{2}{1+\beta}}\big),$$

is a sub-solution of (1). Then a calculation shows that

$$\nabla\psi_1 = \frac{1}{2}\epsilon\big(\frac{2}{1+\alpha}\big)\phi^{\frac{1-\alpha}{1+\alpha}}\nabla\phi,$$

and

$$\begin{aligned}
-\Delta\psi_1 &= -\nabla(\nabla\psi_1)\\
&= -\nabla\Big(\frac{1}{2}\epsilon(\frac{2}{1+\alpha})\phi^{\frac{1-\alpha}{1+\alpha}}|\nabla\phi|\Big)\\
&= -\frac{\epsilon}{1+\alpha}\big\{(\frac{1-\alpha}{1+\alpha})\,\phi^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi|^2 + \phi^{\frac{1-\alpha}{1+\alpha}}\,\Delta\phi\big\}\\
&= \frac{\epsilon}{1+\alpha}\big\{\phi^{\frac{1-\alpha}{1+\alpha}}\,(-\Delta\phi) - (\frac{1-\alpha}{1+\alpha})\,\phi^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi|^2\big\}\\
&= \frac{\epsilon}{1+\alpha}\big\{\phi^{\frac{1-\alpha}{1+\alpha}}\,(\lambda_1\phi) - (\frac{1-\alpha}{1+\alpha})\,\phi^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi|^2\big\}\\
&= \frac{\epsilon}{\alpha+1}\,\phi^{\frac{-2\alpha}{1+\alpha}}\big\{\lambda_1\phi^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2\big\}.
\end{aligned}$$

First we consider the case when $x \in \overline{\Omega_\delta}$. We have

$$-\frac{2}{1+\alpha}\big(1-\frac{2\alpha}{1+\alpha}\big)|\nabla\phi|^2 \le -m.$$

Hence

$$-\frac{\epsilon}{1+\alpha}\phi^{-\frac{2\alpha}{1+\alpha}}\big(\frac{1-\alpha}{1+\alpha}\big)|\nabla\phi|^2 \le -\frac{m\epsilon}{2},$$

and since $\lambda \le \lambda^*$, then

$$-\frac{m\epsilon}{2} \le -\lambda a_0 f(\epsilon) \le -\lambda a_0 f\big(\frac{\epsilon}{2}\phi^{\frac{2}{1+\alpha}}\big).$$

Hence

$$-\frac{\epsilon}{1+\alpha}\phi^{-\frac{-2\alpha}{1+\alpha}}\big(\frac{1-\alpha}{1+\alpha}\big)|\nabla\phi|^2 \le -\lambda a_0 f\big(\frac{\epsilon}{2}\phi^{\frac{2}{1+\beta}}\big), \tag{5}$$

and since $\lambda \ge \lambda_*$ we have

$$\frac{\epsilon}{1+\alpha}\phi^{\frac{2}{1+\alpha}}\lambda_1 \le \frac{\lambda_1\epsilon}{1+\alpha} \le \frac{\lambda a_0}{(\frac{2\epsilon}{2})^\alpha} \le \frac{\lambda a_0}{(\frac{\epsilon}{2}\phi^{\frac{2}{1+\alpha}})^\alpha}. \tag{6}$$

Combining (5) and (6) we see that

$$\begin{aligned}
-\Delta\psi_1 &\le -\lambda a_0[f(\frac{\epsilon}{2}\phi^{\frac{2}{1+\beta}}) - \frac{1}{(\frac{\epsilon}{2}\phi^{\frac{2}{1+\alpha}})^\alpha}] \\
&= -\lambda a_0[f(\psi_2) - \frac{1}{\psi_1^\alpha}] \\
&\le \lambda a(x)[f(\psi_2) - \frac{1}{\psi_1^\alpha}].
\end{aligned}$$

On the other hand on $\Omega - \overline{\Omega_\delta}$ we have $1 \ge \phi \ge \mu$. Also $a(x) \ge a_1, b(x) \ge b_1$ and since $\lambda \ge \lambda_*$, we have

$$\frac{\lambda_1\epsilon}{(1+\alpha)Na_1} \le \lambda.$$

Hence

$$\begin{aligned}
-\Delta\psi_1 &= \frac{\epsilon}{\alpha+1}\,\phi^{\frac{-2\alpha}{1+\alpha}}\big\{\lambda_1\phi^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2\big\} \\
&\le \frac{\lambda_1\epsilon\phi^{\frac{2}{1+\alpha}}}{1+\alpha} \\
&\le \lambda a_1 N \\
&= \lambda\, a_1\big[f(\frac{\mu\epsilon}{2}) - (\frac{2}{\mu\epsilon})^\alpha\big] \\
&\le \lambda a_1\big[f(\frac{\epsilon}{2}\phi^{\frac{2}{1+\beta}}) - \frac{1}{(\frac{\epsilon}{2}\phi^{\frac{2}{1+\alpha}})^\alpha}\big] \\
&= \lambda a_1\big[f(\psi_2) - \frac{1}{\psi_1^\alpha}\big] \\
&\le \lambda a(x)\big[f(\psi_2) - \frac{1}{\psi_1^\alpha}\big].
\end{aligned}$$

A similar argument shows that

$$-\Delta\psi_2 \leq \lambda b(x)\big[g(\psi_1) - \frac{1}{\psi_2^{\beta}}\big],$$

i.e., (ψ_1, ψ_2) is a sub-solution of (1).

Now we will prove there exist a c large enough so that $(z_1, z_2) = \Big(c\,e(x)\,,\ \lambda\|b\|_\infty g(c\|e(x)\|_\infty e(x)\Big)$, is a supersolution of (1). By (H2) we can choose c large enough so that

$$\frac{1}{\lambda\|a\|_\infty} \geq \frac{f\Big(\lambda\|b\|_\infty g(c\|e\|_\infty)\|e\|_\infty\Big)}{c}.$$

Hence

$$\begin{aligned}
-\Delta z_1 = c &\geq \lambda||a||_\infty f\Big(\lambda\|b\|_\infty g(c\|e\|_\infty)\|e\|_\infty\Big)\\
&\geq \lambda a(x) f(z_1)\\
&\geq \lambda a(x)[f(z_1) - \frac{1}{(ce)^{\alpha}}]\\
&= \lambda a(x)[f(z_1) - \frac{1}{z_2^{\alpha}}].
\end{aligned}$$

Also

$$\begin{aligned}
-\Delta z_2 &= \lambda\|b\|_\infty g(c\|e\|_\infty)\\
&\geq \lambda b(x) g(c\,e(x))\\
&= \lambda b(x) g(z_1)\\
&\geq \lambda b(x)[g(z_1) - \frac{1}{z_2^{\beta}}],
\end{aligned}$$

i.e., (z_1, z_2) is a supersolution of (1) with $z_i \geq \psi_i$ for c large and $i = 1, 2$ (note $|\nabla e| \neq 0; \partial\Omega$). Thus, there exist a positive solution (u, v) of (1) such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$. This completes the proof of Theorem 1.

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