Modelling herd behavior of prey: analysis of a prey-predator model[*]

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Abstract. In this paper, a prey-predator model with a social activity of prey population has been analyzed. In some ecological situations, the prey-predator interaction occurs only at the outer surface of a herd formed by prey population. To model this phenomenon, the square root of prey density has been used in the functional response. The basic model is formulated with this modified functional response. A steady-state analysis has been performed. Mathematical analysis including effect of time-delay is presented. Numerical computations are carried out to validate the analytical findings. Biological implications of the analytical and numerical findings are discussed critically.

Keywords: prey-predator system, herd, stability, time-delay

1 Introduction

Interaction between food and its eater is the basic rule of nature. Understanding the dynamical relationship between prey and predator is the central goal of ecology. Ever since the pioneering work of Lotka[19] and Volterra[28], thousand of models have been considered and studied extensively to describe the prey-predator relationship. Mathematicians as well as biologists are too much interested on such modeling analysis. Ecologists serve the real-field data to form mathematical models, and mathematicians study the various characteristics of the models that return the biologists to gain insight into complex ecosystems.

To describe the dynamical mechanism between shark and fishes in Adriatic sea, the classical Lotka-Volterra model was introduced. This model has been consequently used as a machine to introduce numerous theoretical and mathematical concepts in population modelling. The classical Lotka-Volterra model has been modified incorporating Verhulst or Logistic growth to take into account the fact that the resource is limited, and also to avoid the structural unstable nature of the model.

The mass action predation term (in Lotka-Volterra model), though reasonable, is not the only possibility. Depending upon the behaviour of populations, more suitable 'functional response' has been developed as a quantification of the relative responsiveness of the predation rate to change in prey density at various populations of prey. In this connection, Holling family of functional responses[13–15] are the most focused, and in particular, the Holling type-II functional response has served as basis for a very large literature in prey-predator theory[1, 3, 4, 22, 24, 25]. The type-II functional response includes the fact that a single individual can feed only until the stomach is not full, and so a saturation function would be better to describe the intake of food. This is similar to the concept of the 'law of diminishing returns' borrowed from operations research, via

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a hyperbola rising up to an asymptotic value. In other words, the functional response would be of the following form

$$F(X) = \frac{\alpha X}{1 + T_h \alpha X},$$

where $X(T)$ is the prey density at time T, α is the search efficiency of the predator for prey, T_h is the average handling time for each prey.

In natural ecosystems, most of the living beings live forming groups and all members of a group do not interact at a time. The literature abounds with ecological systems where the prey population exhibits herd behaviour. In marine ecology, the herd or schooling behaviour of the sardine is reported by many researchers[8, 31]. Recently, some prey-predator systems with herd behavior have been analyzed by some researchers[5, 27, 29, 30]. As a consequence, suitable form of functional response was searched for to describe the social behaviour of such populations. Some researchers have discussed the characteristics of group defense in this regard[5, 11, 26, 29]. To explore the consequence of forming spatial group of fixed shape by predators, Cosner et al.[9] made the assumption that the number of predators in a shape is proportional to the area of the group in two dimension, and to its volume in three dimension. As a result, the encounter rate $E(X, Y) = \alpha XY$ should change its form to $E(X, Y) = \alpha Y \sqrt{X}$ in two dimension, and in three dimension it should be $E(X, Y) = \alpha Y X^{\frac{2}{3}}$. Recently Ajraldi et al.[2] argued that when prey population exhibit herd behaviour, the functional response should be in terms of square root of prey population. They also claimed that on a large savanna, the herbivorous and their large predators entirely fit their suggested model. Their work, in which the prey population assumed to form a herd in defense or searching for food purpose, lead to some unexpected behaviours, particularly near origin.

Current trend in mathematical ecology modelling includes time-delay as the unavoidable instrument. In reality, time-delay occurs in almost every situation, and so to omit it is to avoid reality. Prey-predator systems may counter time-delay at several stages but mainly as the time-lag between predation and the conversion of prey biomass to predator biomass. To study the effect of time-delay, the delay differential equations have become more prevalent in theoretical ecology. In the classical books of Gopalsamy[12], Kuang[17], Macdonald[20], detailed arguments on importance of time-delay in realistic models have been made.

In this paper, we have considered a prey-predator model where the functional response is of Holling type-II but the prey density term is replaced by its square root. The paper is structured as follows. In the next section, we have discussed the details of the assumptions in the model and the significance of the parameters used in it. Section 3 deals with positivity and boundedness of the model, and these ensure the biological relevance of the model. In Section 4, all the possible equilibrium points and their existence criteria are discussed. Stability and bifurcation analysis are presented in Section 5. The analysis of the model in presence of time-delay is carried out in Section 6. Section 7 contains the numerical verification of our analytical findings. Biological significance of the analytical and numerical findings is discussed in Section 8.

2   The mathematical model

The model we consider here is originally proposed by Braza[7]. However, in his paper, Braza has presented the analysis of a simpler form of the model, assuming the average handling time to be zero. In the following, we discuss how the model is constructed.

Let $X(T)$ denote the prey population density at time T. In the absence of predators, the prey population is assumed to have a logistic growth with intrinsic growth rate r and environmental carrying capacity K. Let $Y(T)$ denote the density of the predator that has the only food source X. Our main consideration is that the prey population live in herd, and only the prey of the border-line interact with predators. So it is more reasonable to take the predation functional response as $F(X) = \frac{\alpha \sqrt{X}}{1 + T_h \alpha \sqrt{X}}$ in spite of mass action or Holling type-II functional response. Thus we consider the model under the framework of the following pair of non-linear differential equations:

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right) - \frac{\alpha\sqrt{X}Y}{1 + T_h\alpha\sqrt{X}},$$
$$\frac{dY}{dT} = -\delta Y + \frac{c\alpha\sqrt{X}Y}{1 + T_h\alpha\sqrt{X}}. \tag{1}$$

Here α is the search efficiency of the predator for prey, T_h is the average handling time for each prey, c is the biomass conversion rate of prey population to predator population, and δ is the natural death rate for predator population. All parameters are assumed to be positive.

To reduce the number of parameters and to make it easier, we non-dimensionalize the system (1) with the following scaling $x = \frac{X}{K}, y = \frac{\alpha Y}{r\sqrt{K}}, t = rT$. Then the system (1) takes the form (after some simplification)

$$\frac{dx}{dt} = x(1 - x) - \frac{\sqrt{x}y}{1 + a\sqrt{x}},$$
$$\frac{dy}{dt} = -dy + \frac{b\sqrt{x}y}{1 + a\sqrt{x}}, \tag{2}$$

where $a = T_h\alpha\sqrt{K}, b = \frac{c\alpha\sqrt{K}}{r}, d = \frac{\delta}{r}$.

3 Boundedness and positivity

The following theorem ensures the boundedness of the system (2).

Theorem 1. *All solutions of the system (2) that start in $\mathbb{R}_+^2$ are uniformly bounded.*

Proof. Let $(x(t), y(t))$ be any solution of the system (2). Since

$$\frac{dx}{dt} \leq x(1 - x),$$

we have

$$\limsup_{t\to\infty} x(t) \leq 1.$$

Let

$$W = bx + y.$$

Therefore

$$\begin{aligned}
\frac{dW}{dt} &\leq bx(1 - x) - dy, \\
&\leq b(1 + d)x - dW, \\
&\leq b(1 + d) - dW.
\end{aligned}$$

Hence

$$\frac{dW}{dt} + dW \leq b(1 + d) = \gamma \text{ (say)}.$$

Applying a theorem on differential inequalities [6], we obtain

$$0 \leq W(x, y) \leq \frac{\gamma}{d} + \frac{(x(0), y(0))}{e^{dt}},$$

and for $t \to \infty$,

$$0 \leq W \leq \frac{\gamma}{d}.$$

Thus, all the solutions of (2) enter into the region

$$B = \left\{(x, y) : 0 \leq W \leq \frac{\gamma}{d} + \epsilon, \text{ for any } \epsilon > 0\right\}.$$

Hence the theorem.   □

Theorem 2. *All solutions of the system (2) that start in $\mathbb{R}_+^2$ remain positive forever.*

The proof is simple and therefore it is omitted.

4 Equilibrium points

The following theorem gives all possible equilibrium points of system (2) and the criteria for their existence. For the sake of brevity, its proof is omitted.

Theorem 3. *The system (2) always has the trivial equilibrium $E_0(0,0)$ and predator-free equilibrium $E_1(1,0)$. If $d < b/(1+a)$ the interior equilibrium point $E^*(x^*, y^*)$ exists and is given by:*

$$
\begin{aligned}
x^* &= \frac{d^2}{(b-ad)^2}, \\
y^* &= \frac{bd\left\{(b-ad)^2 - d^2\right\}}{(b-ad)^4}.
\end{aligned}
$$

5 Stability and bifurcation analysis

5.1 The extinction of predator

Before we delve into the stability analysis of the equilibrium points, we have the following theorem on the extinction of the predator.

Theorem 4. *If $d > (b/a)$ then $\lim\limits_{t\to\infty} y(t) = 0$.*

Proof. From the 2nd equation of system (2)

$$
\begin{aligned}
\frac{dy}{dt} &\leq -dy + \frac{b\sqrt{x}y}{1+a\sqrt{x}}, \\
&\leq -dy + \frac{by}{a}\left(1 - \frac{1}{1+a\sqrt{x}}\right), \\
&\leq \left(\frac{b-ad}{a}\right)y.
\end{aligned}
$$

That implies $y(t) \leq y_0 e^{-\left(\frac{ad-b}{a}\right)t}$.

Now for $d > b/a$ we have $\frac{ad-b}{a} > 0$ and hence $\lim\limits_{t\to\infty} y(t) = 0$. $\qquad\square$

5.2 The trivial equilibrium

At the origin $E_0(0,0)$, the variational matrix becomes indeterminate. The system (2) being not linearizable, due to the square root term, the stability of the origin can not be evaluated. Rescaling the variable $x = p^2$ the singularity may be overcome[2]. But such rescaling may hide the true dynamics in case of prey-predator system.

We think the study of Braza[2] is more realistic to highlight the effect of the square root term by a local nonlinear analysis of the system (2) to uncover the singular dynamics near origin. As the population densities approaches origin, it is reasonable to assume $x(t)$ sufficiently small with the initial value $x_0 = x(0)$ near to origin so that (i) x^2 or higher order terms vanishes, (ii) $1 + a\sqrt{x} \approx 1$ and (iii) $b\sqrt{x}y$ is negligible to y. Under this approach, the system (2) becomes

$$
\begin{aligned}
\frac{dx}{dt} &= x - y\sqrt{x}, \\
\frac{dy}{dt} &= -dy.
\end{aligned}
$$

If the prey population is considerably smaller than predator population, i.e. $x = O(y^\alpha)$ with $\alpha \geq 2$, then the prey population first extincts and the predator population follows suit. For $\alpha < 2$, origin becomes a saddle causing the system (2) to be unstable near origin.

5.3   The predator-free (or prey-only) equilibrium

At $E_1(1,0)$ the variational matrix $V(E_1)$ is given by

$$V(E_1) = \begin{bmatrix} -1 & -\frac{1}{(1+a)} \\ 0 & \frac{b}{1+a} - d \end{bmatrix}.$$

The corresponding eigenvalues are $-1, -(d - \frac{b}{1+a})$ and hence the system (2) is stable at $E_1(1,0)$ if $d > \frac{b}{1+a}$. Thus we may find a threshold value of the predator mortality rate over which the predator population goes to extinction.

5.4   The interior equilibrium

At the interior equilibrium $E^*(x^*, y^*)$, the variational matrix $V(E^*)$ is given by

$$V(E^*) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix},$$

where a_{11}, a_{12} and a_{21} are given by

$$
\begin{aligned}
a_{11} &= \frac{(b + ad)(b - ad)^2 - (ad + 3b)d^2}{2b(b - ad)^2}, \\
a_{12} &= -\frac{d}{b}, \\
a_{21} &= \frac{(b - ad)^2 - d^2}{2(b - ad)}.
\end{aligned}
$$

Then we have the following theorem on stability of E^*.

Theorem 5. *If* $(ad + 3b)d^2 - (b + ad)(b - ad)^2 > 0$ *then* $E^*(x^*, y^*)$ *is locally asymptotically stable.*

Proof. The characteristic equation of $V(E^*)$ is given by

$$\lambda^2 + A\lambda + B = 0, \tag{3}$$

where

$$
\begin{aligned}
A &= \frac{(ad + 3b)d^2 - (b + ad)(b - ad)^2}{2b(b - ad)^2}, \\
B &= \frac{d((b - ad)^2 - d^2)}{2b(b - ad)}.
\end{aligned}
$$

For existence of E^* we have $d < \frac{b}{1+a}$, that ensures $B > 0$. And for the proposed condition it is obvious that $A > 0$. Thus the roots of (3) given by,

$$\lambda_{1,2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2},$$

are negative or with negative real part. Hence the theorem.   □

5.5   Hopf bifurcation near E*

The parameter b, i.e. the biomass conversion rate, takes an important role in stability of the system (2) at E^*. The following theorem ensures the Hopf bifurcation near E^* treating b as the bifurcation parameter.

Theorem 6. *Assume that* E^* *exists with* $d < \frac{b}{a+1}$. *Then if* $a < 1$, *a Hopf bifurcation occurs at the unique value* b^* *of* b, *where* b^* *is the only positive real root of* $b^3 - adb^2 - d^2(a^2 + 3)b - (a - a^3)d^3 = 0$, *provided* $3b^{*2} - 2adb^* - (3 + a^2)d^2 \neq 0$.

Proof. If $a < 1$ the equation $b^3 - adb^2 - d^2(a^2 + 3)b - (a - a^3)d^3 = 0$ has only one positive root b^* and obviously $b^{*3} - adb^{*2} - d^2(a^2 + 3)b^* - (a - a^3)d^3 = 0$. Then from the Jacobian matrix $V(E^*)$ it is clear that

(i) $[A]_{b=b^*} = 0$,

(ii) $[B]_{b=b^*} > 0$,

(iii) Since for $b = b^*$, (i) holds true; the characteristic Eq. (3) becomes

$$\lambda^2 + [\det V(E^*)]_{b=b^*} = 0,$$

whose roots are purely imaginary (using(ii)),

(iv) $\frac{d}{db}[A]_{b=b^*} = 3b^{*2} - 2adb^* - (3 + a^2)d^2 \neq 0$.

Therefore all conditions of Hopf bifurcation theorem are satisfied, and hence the theorem is established. □

6 Analysis with discrete delay

Time-delay is an important and obvious factor in biological systems. Here we have introduced a time-delay τ regarding it as the time-lag from predation of a prey to convert it into predator biomass. Thus generalization of the model (2) involving discrete time-delay becomes:

$$\begin{aligned}
\frac{dx}{dt} &= x(1 - x) - \frac{\sqrt{x}y}{1 + a\sqrt{x}}, \\
\frac{dy}{dt} &= -dy + \frac{b\sqrt{x(t - \tau)}y}{1 + a\sqrt{x(t - \tau)}}.
\end{aligned} \tag{4}$$

The system (4) has the same equilibria as system (2). In this section we want to study the stability behaviour of $E^*(x^*, y^*)$ in the presence of discrete delay ($\tau \neq 0$).

To study the stability behaviour of $E^*(x^*, y^*)$ for the system (4), we linearize the system (4) using the following transformations :

$$x = x^* + \bar{x}, \ \ y = y^* + \bar{y}.$$

Then the linear system is given by

$$\frac{du}{dt} = Au(t) + Bu(t - \tau), \tag{5}$$

where

$$u(t) = [\bar{x}, \bar{y}]^T, \ A = (a_{ij})_{2\times 2}, \ B = (b_{ij})_{2\times 2},$$

$$a_{11} = 1 - 2x^* - \frac{y^*}{2\sqrt{x^*}(1 + a\sqrt{x^*})^2}, \ \ a_{12} = -\frac{\sqrt{x^*}}{1 + a\sqrt{x^*}},$$

$$a_{21} = a_{22} = 0, \ \ b_{11} = b_{12} = b_{22} = 0, \ \ b_{21} = \frac{by^*}{2\sqrt{x^*}(1 + a\sqrt{x^*})^2}.$$

We look for solution of the model (5) in the form of $u(t) = \rho e^{\lambda t}, 0 \neq \rho \in \mathbb{R}^2$. This gives the following characteristic equation:

$$\lambda^2 + a_1\lambda + b_1 e^{-\lambda\tau} = 0, \tag{6}$$

where

$$a_1 = -a_{11}, \ \ \ b_1 = -a_{12}b_{21}.$$

The stability behavior of E^* depends on the the roots of the characteristic Eq. (6). Substituting $\lambda = \xi + i\eta$ in (6), and equating real and imaginary parts separately we have

$$\xi^2 - \eta^2 + a_1\xi + b_1 \cos(\eta\tau)e^{-\xi\tau} = 0, \tag{7}$$

$$2\xi\eta + a\eta - b_1 \sin(\eta\tau)e^{-\xi\tau} = 0. \tag{8}$$

The characteristic Eq. (6) should have purely imaginary roots for stability change of E^*. Hence, to obtain the stability criterion, we set $\xi = 0$ in (7) and (8) and get

$$\eta^2 - b_1 \cos(\eta\tau) = 0, \tag{9}$$

and

$$a_1\eta - b_1 \sin(\eta\tau) = 0. \tag{10}$$

Squaring and adding these two, we get the τ-eliminated equation

$$\eta^4 + a_1^2\eta^2 - b_1^2 = 0. \tag{11}$$

Substituting $\eta^2 = \sigma$ we get a quadratic equation given by

$$F(\sigma) \equiv \sigma^2 + a_1^2\sigma - b_1^2 = 0. \tag{12}$$

It is obvious that the quadratic (12) always has one and only one positive real root. Let σ_0 be the unique positive root of (12) Then there exist $\eta_0^2 = \sigma_0$ that satisfy equation (11). Now we have the following theorem regarding stability switch at E^*.

Theorem 7. *Suppose that E^* exists with $d < \frac{b}{1+a}$. If $(ad + 3b)d^2 - (b + ad)(b - ad)^2 > 0$, E^* is locally asymptotically stable for all $\tau \in [0, \tau^*]$ and exhibits a supercritical Hopf bifurcation near E^* for $\tau = \tau^*$ where τ^* is given by $\tau^* = \frac{1}{\eta_0} \arccos\left[\frac{\eta_0^2}{b_1}\right]$.*

Proof. From equation (9) we find the expression for τ^* as given in Theorem 7. For $\tau = \tau^*$ the characteristic equation (6) has purely imaginary roots, $\pm i\eta_0$. Again, if $\eta = \pm i\eta_0$ are solutions of (9) and (10), then η_0^2 is the unique root of (11). Now it is sufficient to proof that the $\left[\dfrac{d\xi}{dt}\right]_{\tau=\tau^*} > 0$.

Differentiating equation (7) and (8) with respect to τ and setting $\xi = 0$ we obtain

$$
\begin{aligned}
a_1\frac{d\xi}{d\tau} - 2\eta\frac{d\eta}{d\tau} &= b_1\eta\sin(\eta\tau), \\
2\eta\frac{d\xi}{d\tau} + a_1\frac{d\eta}{d\tau} &= b_1\eta\cos(\eta\tau).
\end{aligned}
$$

Solving these two equations with $\tau = \tau^*$ and $\eta = \eta_0$ we have

$$\left[\frac{d\xi}{dt}\right]_{\tau=\tau^*} = \frac{\eta_0^2(a_1^2 + 2\eta_0^2)}{a_1^2 + 4\eta_0^2} > 0.$$

And hence the theorem. $\qquad\square$

7 Numerical simulation

Beside analytical findings, numerical simulations are also important; because simulations can be used to validate the analytical findings. For various choices of the parameters of the model, we have performed the simulations using MATLAB. It is observed that they are in good agreement with our analytical findings.

From the discussion of subsection 5.3, it is seen that if d is greater than $b/(1 + a)$; the equilibrium $E_1(1, 0)$ is stable. For $a = 0.2$, $b = 0.85$, we see that $d = 0.8[> 0.7083 = b/(1 + a)]$. The corresponding phase portrait of the system (2) is presented in Fig. 1, which shows that the prey population approaches the equilibrium value 1 and the predator population goes to extinction with time.

If we take the parameters as $a = 0.2$, $b = 0.85$, $d = 0.5$; then by Theorem 4, the coexistence equilibrium $E^*(x^* = 0.4444, y^* = 0.4198)$ exists as d is less than $b/(a + 1)[= 0.7083]$. It is also noticed that $(ad +$
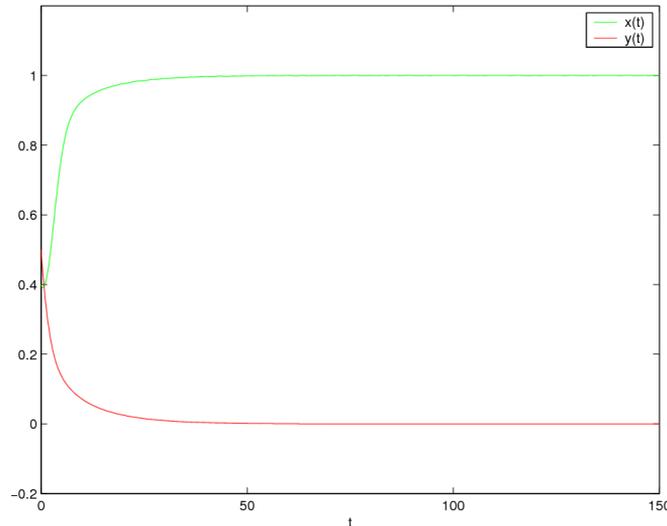
Fig. 1. For $x(0) = 0.4$, $y(0) = 0.5$, $a = 0.2$, $b = 0.85$, $d = 0.8$; the phase portrait of system (2) showing that $E_1(1, 0)$ is locally asymptotically stable. Clearly x approaches 1 and y approaches 0 in finite time
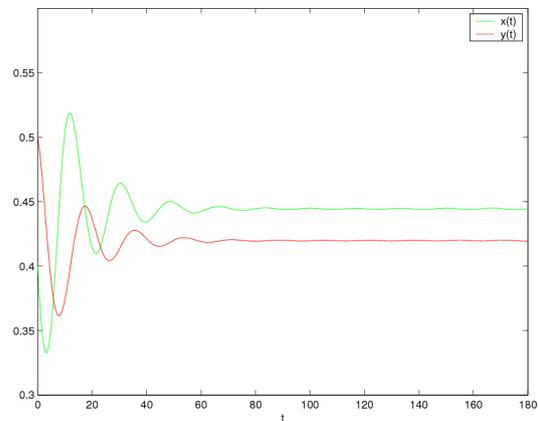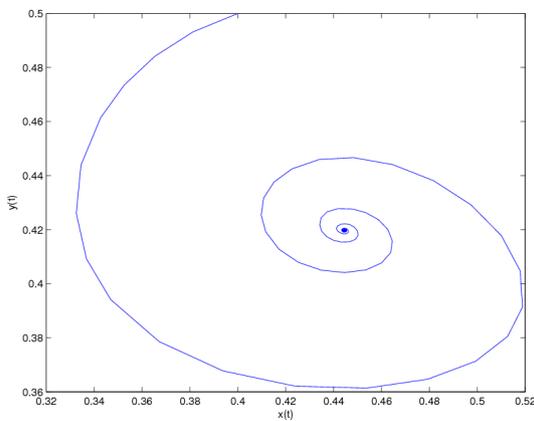

Fig. 2. Here $x(0) = 0.4$, $y(0) = 0.5$, $a = 0.2$, $b = 0.85$, $d = 0.5$. The phase portrait of system (2) showing that E^* is locally asymptotically stable

Fig. 3. Here $x(0) = 0.4$, $y(0) = 0.5$, $a = 0.2$, $b = 0.85$, $d = 0.5$. x, y approach their equilibrium values $x^* = 0.4444, y^* = 0.4198$ in finite time

$3b)d^2 - (b + ad)(b - ad)^2 = 0.1281$, a positive quantity; and so the condition of Theorem 5 is also satisfied. Therefore, for this choice of parameters, E^* is locally asymptotically stable. The corresponding phase portrait is presented in Fig. 2, which is clearly a stable spiral converging to E^*. Fig. 3 shows that x and y approach their equilibrium values in finite time.

As stated in Theorem 6, we have taken b as the bifurcation parameter. For $a = 0.2$ and $d = 0.5$, we have computed the value of $b^* = 0.9377$. Then by Theorem 6, the system undergoes a Hopf bifurcation about E^*. We verify that for $b < b^*$, E^* is locally asymptotically stable. In particular, for $b = 0.85 < b^*$, the phase portrait and the behaviour with time are already shown in Fig. 2 and 3, respectively. For $b > b^*$, the equilibrium E^* becomes unstable. We see that for $b = 0.94 > b^*$, E^* is unstable and there is a periodic orbit near E^* (see the phase portrait in Fig. 4. Fig. 5 shows the sustained oscillations of x and y in time. A bifurcation diagram is also drawn taking b as the bifurcation parameter (see Fig. 6).

For similar choice of parameters (as in Fig. 2 and 3), the computed value of τ^*, as stated in Theorem 7, is 1.1198. Hence the system (4) should have a stable equilibrium for $\tau < \tau^*$ (see Fig. 7). When $\tau > \tau^*$, the system becomes unstable (see Fig. 8). Hence it is seen that τ capable of changing the stability of system (4).
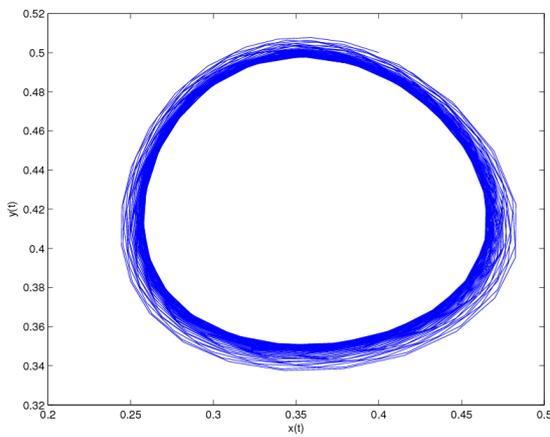
Fig. 4. The phase portrait of system (2) for $x(0) = 0.4$, $y(0) = 0.5$, $a = 0.2$, $d = 0.5$ and the bifurcation parameter $b = 0.94 > b^*(= 0.93)$.The periodic orbit near $E^*(0.4444, 0.4198)$
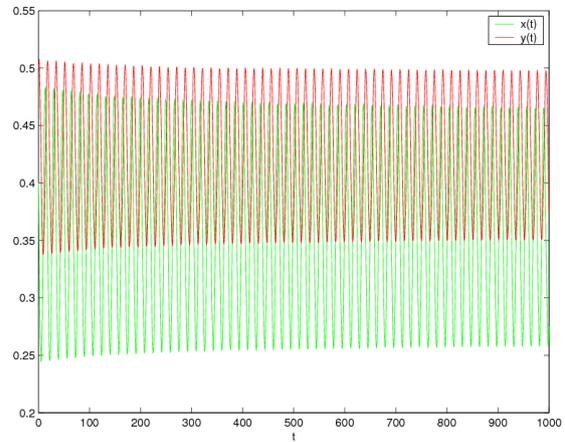
Fig. 5. The phase portrait of system (2) for $x(0) = 0.4$, $y(0) = 0.5$, $a = 0.2$, $d = 0.5$ and the bifurcation parameter $b = 0.94 > b^*(= 0.93)$.The periodic orbit near $E^*(0.4444, 0.4198)$
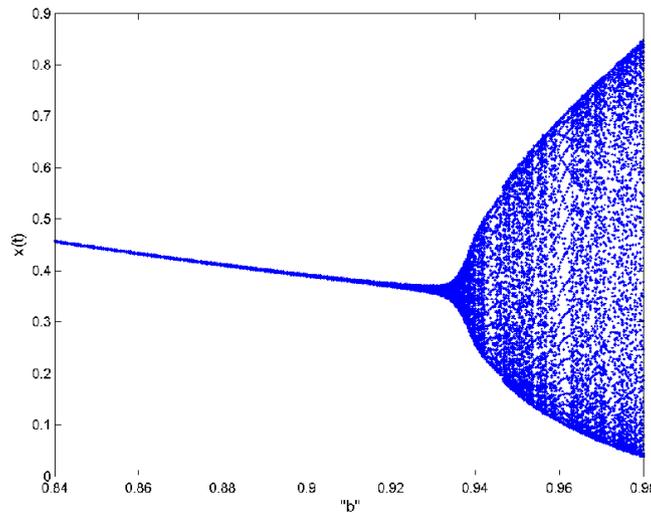

Fig. 6. Bifurcation diagram for $x(t)$ with b as the bifurcation parameter when $a = 0.2$, $d = 0.5$

8 Concluding remarks

It is a fact that prey-predator systems have been studied extensively in theoretical ecology. But only a little attention has been paid on one of the most realistic phenomenon that many prey populations live in groups, and therefore interaction with predators occur only with the outer members of the herds.

In this paper, we have studied a dynamical model with a modified functional response to describe the herd mechanism. The construction of the model has been discussed in details. The model we have considered is biologically well behaved, as it is bounded and remains in feasible range forever. In the absence of predators, prey population grows following logistic growth unto its carrying capacity. In our model, according to Theorem 4, the predator population dies off if $d > (b/a)$; i.e. in terms of original parameters, $\delta > (c/T_h)$. Hence the predator population is washed out from the system if its death rate is greater than the ratio of biomass conversion rate and the prey handling time. It is obvious that due to disease or any other natural circumstances, if predator death rate increases significantly; whole predator population may be wiped out. But the predator population also may extinct if its biomass conversion rate becomes low due to digestive problem or the predator takes much time in handling the prey.

On stability analysis, it is observed that the Jacobian being indeterminate at origin due to square root term, stability analysis can not be carried out evaluating only its eigenvalues. Here we follow the pathway of
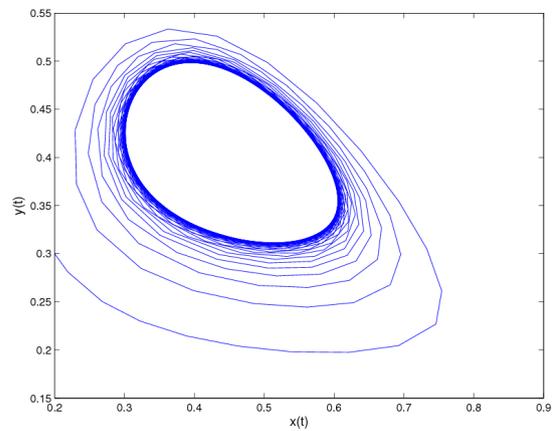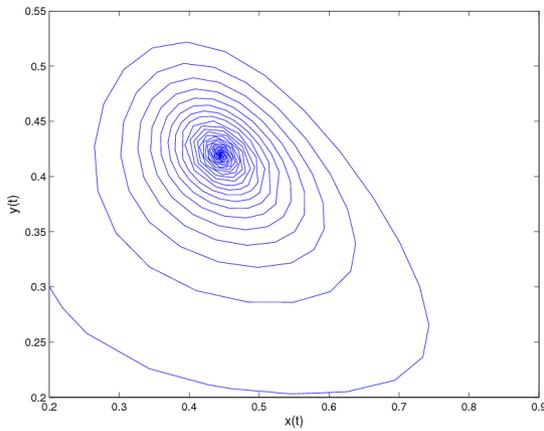
Fig. 7. Phase portrait of the system (4) for $x(0) = 0.2$, $y(0) = 0.3$, $a = 0.2$, $b = 0.85$, $d = 0.5$ and $\tau = 1.0 < \tau^*(= 1.1198)$ showing the local asymptotic stability at $E^*(0.4444, 0.4198)$

Fig. 8. Phase portrait of the system (4) for $x(0) = 0.2$, $y(0) = 0.3$, $a = 0.2$, $b = 0.85$, $d = 0.5$ and $\tau = 1.2 > \tau^*(= 1.1198)$ showing that the system (4) becomes unstable if τ exceeds its critical value τ^*
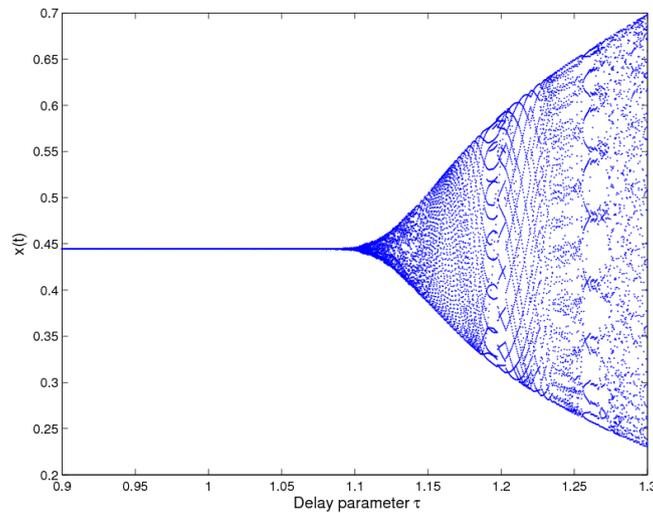

Fig. 9. Bifurcation diagram for $x(t)$ with the delay parameter τ as the bifurcation parameter for $x(0) = 0.2$, $y(0) = 0.3$, $a = 0.2$, $b = 0.85$, $d = 0.5$

approximation suggested by Braza[7]. We assume that near the origin, both the populations being small, their products or higher order terms may be neglected. And this gives an ecologically sound result that, if the prey density becomes smaller compared to the predator (i.e. $x = O(y^\alpha)$ with $\alpha \geq 2$), the system approaches its trivial equilibrium through stable path. This is a basic difference from the other models where origin being a saddle, no such possibility arises. Here the prey population first goes to extinction and then the predator population follows suit (see [7]).

We have already discussed that the system becomes predator-free if predator death rate is greater than a certain threshold value. Obviously, in sub-section 5.2, we observe that the predator-free equilibrium is stable when $d > (b/a)$ (because the stability condition $d > b/(1 + a)$ is automatically satisfied). The main aim of dynamical modelling is to study the coexistence stability that serves the expectation of ecosystems. In Theorem 4, we have discussed the condition for stable coexistence and survival of the species. The theorem has also been verified numerically, and graphical representation of the solution is depicted. It is well known that the stability of a system may be switched only by varying some sensitive parameter of the system. Here a bifurcation analysis has been carried out treating the biomass conversion rate as the bifurcation parameter. Change in digestive power may cause change in biomass conversion rate, and this in turn may lead from a stable system to an unstable one (see Fig. 4 and 5).

In mathematical modelling of bio-systems, the time-delays are unavoidable. In our system, we have considered the time-lag τ between predation and conversion to predator biomass. The resulting system is studied extensively in Section 6. Theorem 7 states that there is a critical value τ^* of the delay at which the stability switch occurs. In other words, the system exhibits a Hopf-bifurcation near the interior equilibrium for $\tau = \tau^*$ (see Fig. 7 to Fig. 9).

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