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A dynamical model for the spread of cholera near an aquatic reservoir

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Abstract. Cholera is a disease of small intestine caused by a bacterium named Vibrio Cholerae. It has been a threat to our society and challenge to medical science. In the present work we have formulated a dynamical model for the spread of cholera in a region under the consideration that this disease is transmitted there by contaminated water only. The present model is expected to capture long time-scale dynamics of the spread of cholera in an efficient manner as in addition to the issue of deaths occurring due to cholera and medical recovery from this disease in the group of infected individuals and falling of the recovered individuals again in the susceptible group it also assumes a variable population understanding birth and death (not by cholera) in the susceptible. We have calculated the corresponding equilibrium points, discussed about their local asymptotic stability and we have analyzed the possibility of Hopf bifurcation. We have also presented a numerical example to show the time dependent profiles of the susceptible, infected and concentration of Vibrio Cholerae using our model.

Keywords: cholera, equilibrium points, local asymptotic stability, Hopf-bifurcation

1 Introduction

Since 1855 cholera has been a subject of thorough study. Primarily it is a waterborne disease transmitted by a bacterium viz. Vibrio Cholerae (V. Cholerae) which affects small intestine badly resulting profuse watery diarrhea and vomiting. These two symptoms often cause rapid dehydration and electrolytic imbalance. If not detected early and treated properly this may cause death too within hours of infection. Contaminated water and food are main transmitters of this disease. It is noticed that in 1st world countries mainly seafood is responsible for transmission of the disease whereas in under developed and developing countries water is the main transmitter. In most epidemic outbreak cases responsible agent is V. Cholerae O1 (also recently reported V. Cholerae O139)[3, 16]. Although cholera may be life threatening, if proper sanitation is practiced it may be prevented. As in developed countries advanced water treatment and sewage systems are available in those countries cholera is no more a major health issue. For example in United States last major outbreak in cholera occurred in 1910-11.

Primary treatment may be oral rehydration solution (ORS) effective for a few hours and if no improvement is observed intravenous fluid must be injected. Anti-bacterial drugs are also useful to reduce the duration and severity of the disease. Approximately 100 million bacteria are to be consumed by a normal healthy adult to get infected by cholera, although this amount is less for children, persons having blood type O, gastric problems, lower immunity, AIDS etc. However it is notable that any individual may experience a severe case of cholera and the person should be treated individually measuring loss of fluid from body with respect to time. A person infected by cholera may produce 10-20 liters of diarrhea a day. V. Cholerae bacteria produce hollow

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cylindrical protein flagellin to make flagella to propel themselves to the mucus layer of small intestine. After reaching that layer V. Cholerae starts producing toxic protein which results watery diarrhea in infected body. At the same time multiple generations of V. Cholerae are produced and they are released to nature through infected persons diarrhea or vomiting. If proper sanitation is not present it will be gradually consumed by another host. Generally a rapid dip-stick test is done to test presence of V. Cholerae, if test result is positive then further diagnosis is required to measure antibiotic resistance [1, 13, 25].

Up to 19^{th} century cholera was an epidemic in Indian sub-continent only with two seasonal peaks one during March-April and another is just after monsoon period (September-December)[10, 22]. From this confined region seven outbreaks of cholera occurred throughout the world[7]. In Indonesia the last pandemic started in 1961 which hit Asian continent during 60's and the next target was Africa in 1970[4, 9, 18]. The number of susceptible; exposure to untreated water and sanitation and the presence of an aquatic reservoir with the concentration of V. Cholerae are some deterministic parameters of the persistence of the disease in a particular region[10, 18, 22]. Although we can find a number of literatures on the clinical perspectives and bacteriological distribution in cholera disease[2, 5, 8, 11, 17, 19, 20, 23, 24, 27] we have a very few literatures to the best of our knowledge on the mathematical modeling of spreading of cholera[4, 14, 15, 26, 28]. In the present work, as a whole, we have made an effort to construct a mathematical model of the spread of cholera in a geographical domain under the assumption that water is the only transmitter of this disease. The point is to mention in this regard that Codeco's model[4], which is a modified version of the model proposed by Capasso and Paveri-Fontana[28], does not consider the birth and death (not by cholera) in the susceptible population and in that model the total population was supposed to be fixed. Therefore it is not possible to understand the long-scale dynamics of the spread of cholera in view of this model. Moreover, Codecos model[4] did not capture the death occurring by this disease in the infected group. To overcome these limitations we have proposed our model introducing birth and death (not by cholera) in susceptible as well as the death of a portion of the infected group of people by cholera. We can find one more inconsistency in that model[4] from the medical point of view. Codeco[4] considered a term indicating clinical recovery from cholera while modelling rate of change in numbers in infected group ($\frac{dI}{dt}$) but that was not incorporated in the expression of the rate of change in numbers in susceptible ($\frac{dS}{dt}$). This somehow looks inconsistent from medical perspective as it indicates that the person once recovered from cholera will not have any further chance to get infected by the same in future which is not true in general. The same inconsistency is evident in [14] and [26] although in [26] population was supposed to vary with time. In view of this, in our present model, we have framed $\frac{dS}{dt}$ and $\frac{dI}{dt}$ accordingly.

In the Theory section we have first demonstrated the formulation of the model. Next we have identified the equilibrium point (s) at different conditions and have discussed about their local asymptotic stability. At the very next, we have discussed on the possibility of generation of the Hopf bifurcation in the system. In the Result section we have considered a numerical example taking the magnitudes of some of the parameters involved from [15] to show a time dependent profile of the spread of this disease and the matter has been also demonstrated graphically. In the Discussion section we have summarized the theme and outcome of the present work and commented on the future scope of this work.

2 Theory

2.1 Formulation of the model

To construct the model we assume the following:

S = number of people susceptible for cholera at time t;
I = number of people infected by the disease at time t;
B = concentration of V. Cholerae in water reservoir nearby the population region at time t;
b = birth rate of the population d=death rate of the population;
d = death rate of the population;

γ = recovery rate of the population from the disease;
α = mortality rate of the population due to cholera;
e = contribution of each infected person in spreading cholera;
a = rate of infection per susceptible per unit concentration of V. Cholerae;
m = growth rate of V. Cholerae;
n = loss rate of V. Cholerae.

We can construct the following model which is an extended and modified version of the model proposed by Codeco[4] taking in to account the issue of deaths occurring due to cholera and medical recovery from this disease in the group of infected and falling of the recovered individuals again in the susceptible group as well as a variable population understanding birth and death (not by cholera) in the susceptible:

$$\begin{cases} \frac{dS}{dt} &= (b-d)S + \gamma I - \alpha BS, \\ \frac{dI}{dt} &= \alpha BS - \gamma I - \alpha I, \\ \frac{dB}{dt} &= (m-n)B + eI. \end{cases} \tag{1}$$

We put $b - d = r$ and $m - n = r'$ and then we have from (1)

$$\begin{cases} \frac{dS}{dt} &= rS + \gamma I - \alpha BS, \\ \frac{dI}{dt} &= \alpha BS - \gamma I - \alpha I, \\ \frac{dB}{dt} &= r'B + eI. \end{cases} \tag{2}$$

2.2 Search for the equilibrium points

For equilibrium points we must have

$$\frac{dS}{dt} = \frac{dI}{dt} = \frac{dB}{dt} = 0,$$

which gives

$$\begin{cases} rS + \gamma I - aBS = 0, \\ aBS - \gamma I - \alpha I = 0, \\ r'B + eI = 0. \end{cases} \tag{3}$$

Solving (3) we get the equilibrium points as $(0, 0, 0)$ and $\left\{\frac{-(\alpha+\gamma)r'}{ae}, \frac{-rr'(\alpha+\gamma)}{ae\alpha}, \frac{(\alpha+\gamma)r}{a\alpha}\right\}$.

Here the parameters a, e, α, γ and r are all positive. Hence the above the non-trivial equilibrium point as above will arise only if $r' \leq 0$.Otherwise we will only have the trivial equilibrium point.

2.3 Local asymptotic stability analysis of the equilibrium points

The characteristic equation for the equilibrium point $(0, 0, 0)$ is given by

$$\begin{vmatrix} r-\lambda & \gamma & 0 \\ 0 & -\gamma-\alpha-\lambda & 0 \\ 0 & e & r'-\lambda \end{vmatrix} = 0,$$

i.e.

$$(\lambda - r)(\lambda - r')(\lambda + \gamma + \alpha) = 0, \tag{4}$$

which gives $\lambda = r, r', -(\alpha + \gamma)$.

In real world situations, $r = b - d > 0$. Again as $\alpha > 0$ and $\gamma > 0$ we have $-(\alpha + \gamma) < 0$. Hence the equilibrium point $(0, 0, 0)$ cannot be a stable node irrespective of the sign of r'.

Again the characteristic equation for the non-trivial equilibrium point, which exists only for $r' \leq 0$, is given by

$$\lambda^3 + (\alpha+\gamma) - r' + \frac{rr'\gamma}{\alpha}\lambda - rr'(\alpha+\gamma) = 0. \tag{5}$$

According to Routh-Hurwitz criterion if $\lambda^3 + A\lambda^2 + B\lambda + C = 0$ be the characteristic equation corresponding to a non-trivial equilibrium point then the equilibrium point is stable if $B > 0$, $C > 0$ and $AB - C > 0$.

Here for $r' < 0$ as α, γ and r are all positive we have $B = -\frac{rr'\gamma}{\alpha} > 0$. For $r' = 0$ we have $B = 0$.

Again for $r' < 0$ as α, γ and r are all positive we have $C = -rr'(\alpha+\gamma) > 0$. For $r' = 0$ We have $C = 0$.

Now to have $AB - C > 0$ for $r' < 0$, we must have $-\{(\alpha+\gamma) - r' + \frac{r\gamma}{\alpha}\}\frac{rr'\gamma}{\alpha} + rr'(\alpha+\gamma) > 0$ from which we get $(\alpha^2 - \gamma^2)\alpha - \gamma(r\gamma - r'\alpha) < 0$ (as $r > 0$ and $r' < 0$ gives $rr' < 0$).

Now as α, γ and r are all positive hence if $\gamma \geq \alpha$, the above condition holds trivially for $r' < 0$. So for $r' < 0$ one condition for corresponding stability is $\gamma \geq \alpha$. Otherwise if $\gamma < \alpha$, for $r' < 0$ the condition for local asymptotic stability for the non-trivial equilibrium point is given by $(\alpha^2 - \gamma^2)\alpha - \gamma(r\gamma - r'\alpha) < 0$.

For $r' = 0$ as we have $B = 0$, $C = 0$ and $A = \{(\alpha+\gamma) + \frac{r\gamma}{\alpha}\} > 0$ (as α, γ and r all are positive) we have a pseudo stability for the non-trivial equilibrium point.

2.4 Condition for Hopf bifurcation

If the characteristic equation corresponding to any equilibrium point is given by

$$\rho_0 + \rho_1\lambda + \rho_2\lambda^2 + \rho_3\lambda^3 = 0, \tag{6}$$

the conditions for the Hopf-Bifurcation are

$$\rho_0 > 0, \rho_1 > 0 \text{ and } \rho_1\rho_2 - \rho_0\rho_3 = 0. \tag{7}$$

Now for the trivial equilibrium point $(0,0,0)$ the characteristic equation is $(\lambda - r)(\lambda - r')(\lambda + \gamma + \alpha) = 0$ which indicates that all the three roots here are real and due to that reason Hopf bifurcation is not possible in this case.

Now for the non-trivial equilibrium point the characteristic equation is given by $\lambda^3 + \{(\alpha+\gamma) - r + \frac{r\gamma}{\alpha}\}\lambda^2 - \frac{rr'\gamma}{\alpha}\lambda - rr'(\alpha+\gamma) = 0$.

Comparing the above with (6) we get,

$$\begin{cases} \rho_0 = -rr'(\alpha+\gamma), \\ \rho_1 = -\frac{rr'\gamma}{\alpha}, \\ \rho_2 = \alpha + \gamma - r' + \frac{r\gamma}{\alpha}, \\ \rho_3 = 1. \end{cases} \tag{8}$$

As $r > 0$, $\gamma > 0$ and $\alpha > 0$, for $r' < 0$, we have $\rho_0 = -rr'(\alpha+\gamma) > 0$ and $\rho_1 = -\frac{rr'\gamma}{\alpha} > 0$. Under the same assumption, to have $\rho_1\rho_2 - \rho_0\rho_3 = 0$ we must have $(\alpha^2 - \gamma^2)\alpha - \gamma(r\gamma - r'\alpha) = 0$. This is the only condition to have Hopf bifurcation in this case.

For $r' = 0$ as $\rho_0 = 0$ and $\rho_1 = 0$ we cannot find Hopf bifurcation in this case.

3 Result

We frame an example by choosing the parameters as $r = 0.000036$, $r' = -0.15$, $\alpha = 0.0001$, $\gamma = 0.0002$, $a = 0.0007$, and $e = 0.001$. We have chosen some of these values on the basis of the example provided in [15]. We start the simulation by taking $S = 5000$, $I = 10$ and $B = 0.001$ at time $t = 0$ and then we take in to account a discrete version of (2). We take an increment of 1 day and count the change in S, I and

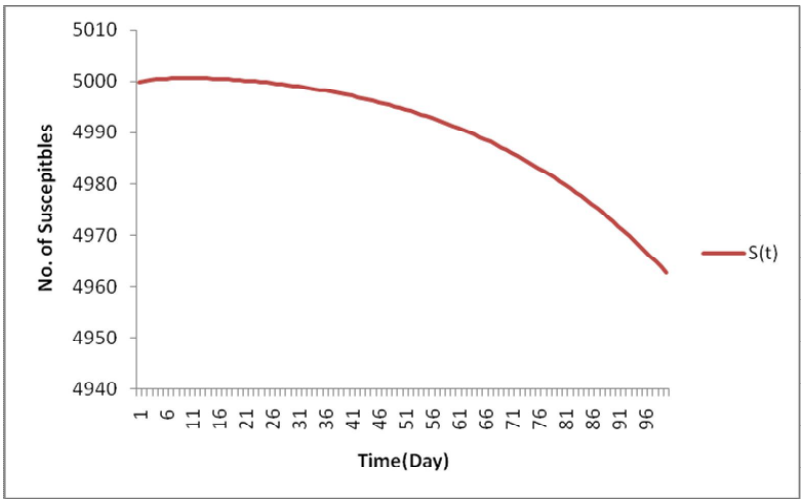

Fig. 1. Graph for S versus t

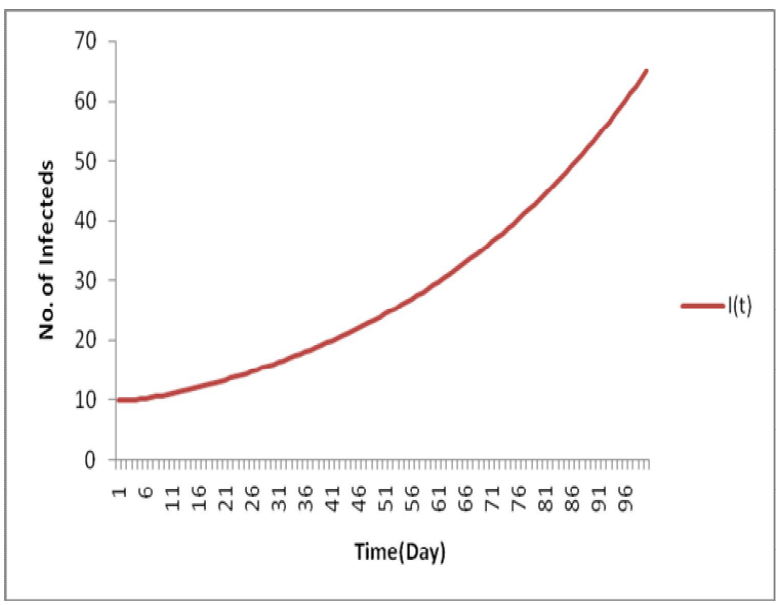

Fig. 2. Graph for I versus t

B at each stage by using that discrete version of (2) and add these to the corresponding values at the previous stage to have a fresh set of values of S, I and B for the next stage. The process is continued for 100 days. We present the graphs of S versus t, I versus t and B versus t as the outcomes of this computation.

For the above profile we can observe an alarming situation as in this case S shows a significant decreasing trend from the initial value 5000 after a slight increase at the very beginning and I shows a sharp increase from its initial value 10 with the increase in days and also the concentration of V. Cholerae increases with time.

4 Discussion

In the present work a mathematical model is prescribed under the consideration that contaminated water is the only transmitter of the disease. To frame this model we have taken in to account the number of susceptible, number of infected and concentration of V. Cholerae in water reservoir nearby the population region. We expect an efficient performance of this model as it has the ability to capture long time-scale dynamics of the spread of cholera. The reason behind this expectation is that the present model assumes a variable population understanding birth and death (not by cholera) in the susceptible as well as it also covers the issue of

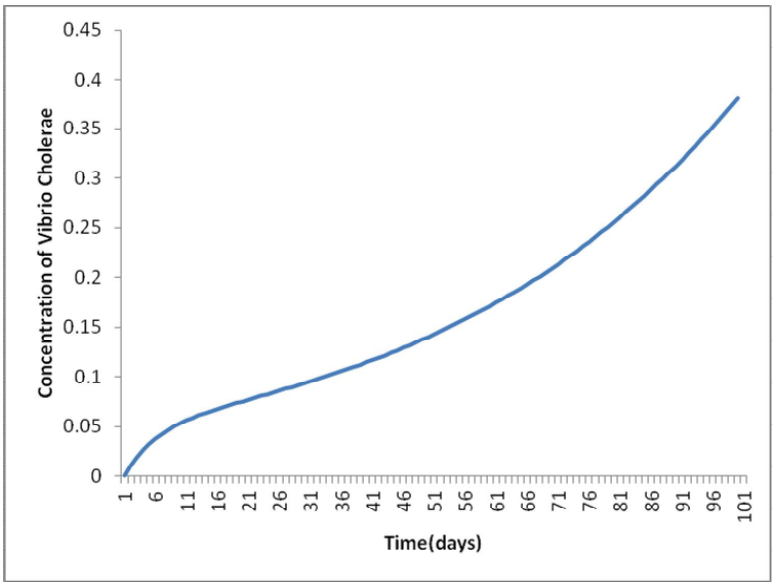

Fig. 3. Graph for I versus t

deaths occurring due to cholera and medical recovery from this disease in the infected group and falling of the recovered individuals again in the susceptible group which were not considered as a whole in the previous models[4, 14, 15, 26, 28]. We have obtained the corresponding equilibrium points and have gone for the analysis on existence of these equilibrium points. Moreover, we have discussed on the local asymptotic stability of the equilibrium points. In addition to this, we have identified the conditions for the exhibition of Hopf bifurcation in the system. We have considered one example corresponding to our model taking suitable values of the presently used parameters from [15] in order to portray a time dependent profile of the number of susceptible, number of infected and the concentration of V. Cholerae. Mass vaccination in cholera disease is under investigative study now days[6, 12, 21] and in future, we will try to incorporate the effect of intervention in our present model.

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