

# Extension of the analysis of the MISR model by applying the laplace-adomian decomposition method

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(Received December 1 2013, Accepted May 14 2014)

**Abstract.** In this paper, model analysis is presented using analytic and numerical methods and the analytic series solution of the epidemic model is approximated. Laplace-Adomian decomposition method (LADM) is employed to approximate the series solution of the epidemic model.

**Keywords:** Series solution, Epidemic model, Laplace-Adomian decomposition method, Approximate solution, System of non-linear differential equations

## 1 Introduction

Models of biological systems are represented by non-linear ordinary differential equations. In recent biological studies, mathematical modeling and simulations of these models are playing very important roles. In scientific literature, the epidemic model has taken attention in recent years.

MSIR models<sup>[4, 5]</sup>, are epidemic models that consist of four classes, that is, a proportion of infants born with passive immunity, susceptible individuals, infected individuals and individuals who have recovered with immunity. These models describes infectious diseases where an infant is born with passive immunity from its mother. Susceptible individuals are defined as those individuals in the population who are not yet infected but have the possibility of being infected. In contrast, infected individuals are defined as those individuals who have acquired the disease or infection and have revealed signs and symptoms of the disease or infection.

In this manuscript, we present an MSIR model in general by computing equilibrium points and the reproduction number as well as their analysis. Furthermore, we extend the analysis of the model by using the Laplace Adomian decomposition method<sup>[6, 7]</sup> to find a series solution of the system of equations.

## 2 Model description and analysis of equilibrium points

Consider the following MSIR model describe by the compartmental model Fig. 1:

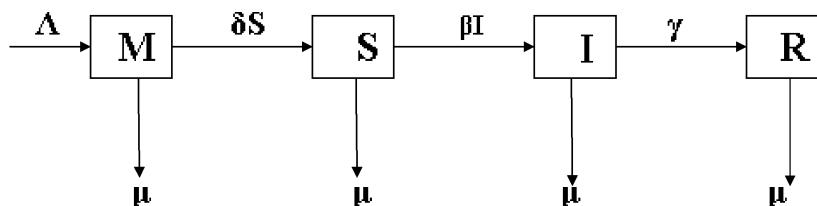


Fig. 1. Structure of the model

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At any time  $t$ ,

$M(t)$  represents a proportion of people born with passive immunity;

$S(t)$  represents a proportion of susceptible individuals;

$I(t)$  represents a proportion of infected individuals;

$R(t)$  represents a proportion of recovered individuals with immunity.

$\Lambda$  is the recruitment rate

$\frac{1}{\gamma}$  is an average infectious period

$\delta$  is an average temporary immunity period

$\beta$  is a contact rate and

$\mu$  is the natural death rate<sup>[4, 5]</sup>.

The mathematical system of equations that describes MSIR model is given by:

$$\begin{aligned}\frac{dM}{dt} &= \Lambda - \delta MS - \mu M, \\ \frac{dS}{dt} &= \delta MS - \beta SI - \mu S, \\ \frac{dI}{dt} &= \beta SI - (\gamma + \mu)I, \\ \frac{dR}{dt} &= \gamma I - \mu R,\end{aligned}\tag{1}$$

with the initial conditions  $M(0) = M_0$ ,  $S(0) = S_0$ ,  $I(0) = I_0$  and  $R(0) = R_0$ .

Let  $Q_0$  be the disease free equilibrium point of the system (1). Then

$$Q_0 = \left( \frac{\mu}{\delta}, \frac{\Lambda - \mu^2}{\mu\delta}, 0, 0 \right).$$

Following van den Driessche and Watmough<sup>[3]</sup>, the reproduction number of the system (1) is given by:

$$R = \frac{\beta(\Lambda - \mu^2)}{\mu\delta(\mu + \gamma)}$$

and deduce following theorem.

**Theorem 1.** *The disease free equilibrium point  $Q_0$  is both globally and locally asymptotically stable whenever  $R < 1$ .*

Let  $Q^*$  be the endemic equilibrium point. It follows that:

$$Q^* = \left( \frac{\mu + \gamma}{\beta}, \frac{\Lambda\beta}{\delta(\mu + \gamma) - \mu\beta}, \frac{\delta\Lambda\beta - \mu(\delta\mu + \delta\gamma - \mu\beta)}{\beta(\delta\mu + \delta\gamma - \mu\beta)}, \frac{\gamma\delta\Lambda\beta - \mu(\delta\mu + \delta\gamma - \mu\beta)}{\mu\beta(\delta\mu + \delta\gamma - \mu\beta)} \right).$$

**Theorem 2.** *The endemic equilibrium point  $Q^*$  exist for  $R > 1$ .*

*Proof.* The endemic equilibrium point exist when  $\frac{dI(t)}{dt} > 0$  and  $I(t) > 0$ . This implies that,  $\beta SI - (\gamma + \mu)I > 0$  for which  $\beta S - (\gamma + \mu) > 0$ , since  $I > 0$  and  $\beta \frac{\Lambda - \mu^2}{\mu\delta} - (\gamma + \mu) > 0$ , since  $S = \frac{\Lambda - \mu^2}{\mu\delta}$ . Therefore  $R > 1$  as desired.

**Theorem 3.** *The endemic equilibrium point  $Q^*$  is both globally and locally asymptotically stable whenever it exist.*

*Proof.* Since, global asymptotic stability implies local asymptotic stability, it suffices to prove that the endemic equilibrium point is globally asymptotically stable. We now set

$$M(t) = x(t), S(t) = y(t), I(t) = z(t) \text{ and } R(t) = w(t)$$

and consider a function  $V(x, y, z, w)$  given by

$$V = x - x^* \ln x + y - y^* \ln y + z - z^* \ln z + w - w^* \ln w. \quad (2)$$

where  $(x^*, y^*, z^*, w^*)$  is the endemic equilibrium point  $Q^*$ .

**Claim:** The function  $V$  is a Lyapunov function<sup>[1, 2]</sup> to which the endemic equilibrium point is the only stationary point and is a global minimum point.

It follows that,

$$\frac{\partial V}{\partial x} = 1 - \frac{x^*}{x}, \frac{\partial V}{\partial y} = 1 - \frac{y^*}{y}, \frac{\partial V}{\partial z} = 1 - \frac{z^*}{z}, \frac{\partial V}{\partial w} = 1 - \frac{w^*}{w}.$$

By equating the partial derivatives to zero and solve for  $x, y, z, w$  we get:

$$x = x^*, y = y^*, z = z^*, w = w^*.$$

The endemic equilibrium point is the only stationary point of the function  $V$ . Now

$$\frac{\partial^2 V}{\partial x^2} = \frac{x^*}{x^2}, \frac{\partial^2 V}{\partial y^2} = \frac{y^*}{y^2}, \frac{\partial^2 V}{\partial z^2} = \frac{z^*}{z^2}, \frac{\partial^2 V}{\partial w^2} = \frac{w^*}{w^2}.$$

it follows that the second partial derivatives of the function  $V$  with respect to  $x, y, z, w$  are positive, hence the endemic equilibrium point is the global minimum point of the function  $V$  and  $V' = 0$  only at the endemic equilibrium point.

Further, the function  $V$  is continuous everywhere by properties of continuous functions and has continuous first order partial derivatives. Therefore to conclude that the function  $V$  is a Lyapunov function, it requires us to prove that, the rate of change of the function  $V$  with respect to time is less than or equal to zero as  $t$  approaches infinity. Now

$$V'(t) = (x - x^*) \frac{x'(t)}{x(t)} + (y - y^*) \frac{y'(t)}{y(t)} + (z - z^*) \frac{z'(t)}{z(t)} + (w - w^*) \frac{w'(t)}{w(t)}.$$

Since the set of solutions for the system of equations (1) is bounded, there exist a positive real number  $k$  such that:  $k = \min x, y, z, w$  hence

$$V'(t) < \frac{1}{k} ((x - x^*)x'(t) + (y - y^*)y'(t) + (z - z^*)z'(t) + (w - w^*)w'(t)).$$

Also, since  $N'(t) = \Lambda - \mu N$  where  $0 < N \leq \frac{\Lambda}{\mu}$ , we have  $0 \leq N'(t) < \Lambda$  and hence, the derivatives are also bounded, that is  $x'(t), y'(t), z'(t), w'(t) \in L^\infty$ . Thus

$$V'(t) < \frac{\Lambda}{k} (x - x^* + y - y^* + z - z^* + w - w^*).$$

Using the result derived from Barbalat lemma<sup>[1]</sup>,

$$x - x^* + y - y^* + z - z^* + w - w^* \rightarrow 0, \text{ as } t \rightarrow \infty.$$

hence  $V'(t) < 0$ , this implies that the function  $V$  is a strict Lyapunov function with the endemic equilibrium point only as its stationary point, which is a global minimum point. Thus the endemic equilibrium point  $Q^*$  is globally asymptotically stable for  $R > 1$ .

By using limits properties, we can also conclude that, as the recovery rate  $\gamma$  increases or as the average infectious period decreases, the reproduction number  $R$  decreases. This means to say, as the recovery rate increase the number of infected individuals decreases. Similarly, a decrease in the contact rate  $\beta$  decrease the rate at which the infection will spread.

### 3 Numerical simulations

In this section, we carry out detailed numerical simulations using Matlab programming language to assess the effect that certain parameters have on the population size.

#### 3.1 Sensitivity analysis

Here we represent graphically the sensitivity of the reproduction number  $R$  to parameters that defines it using Monte Carlo simulations, hyper-cube sampling graphs and partial rank correlation coefficients (PRCC).

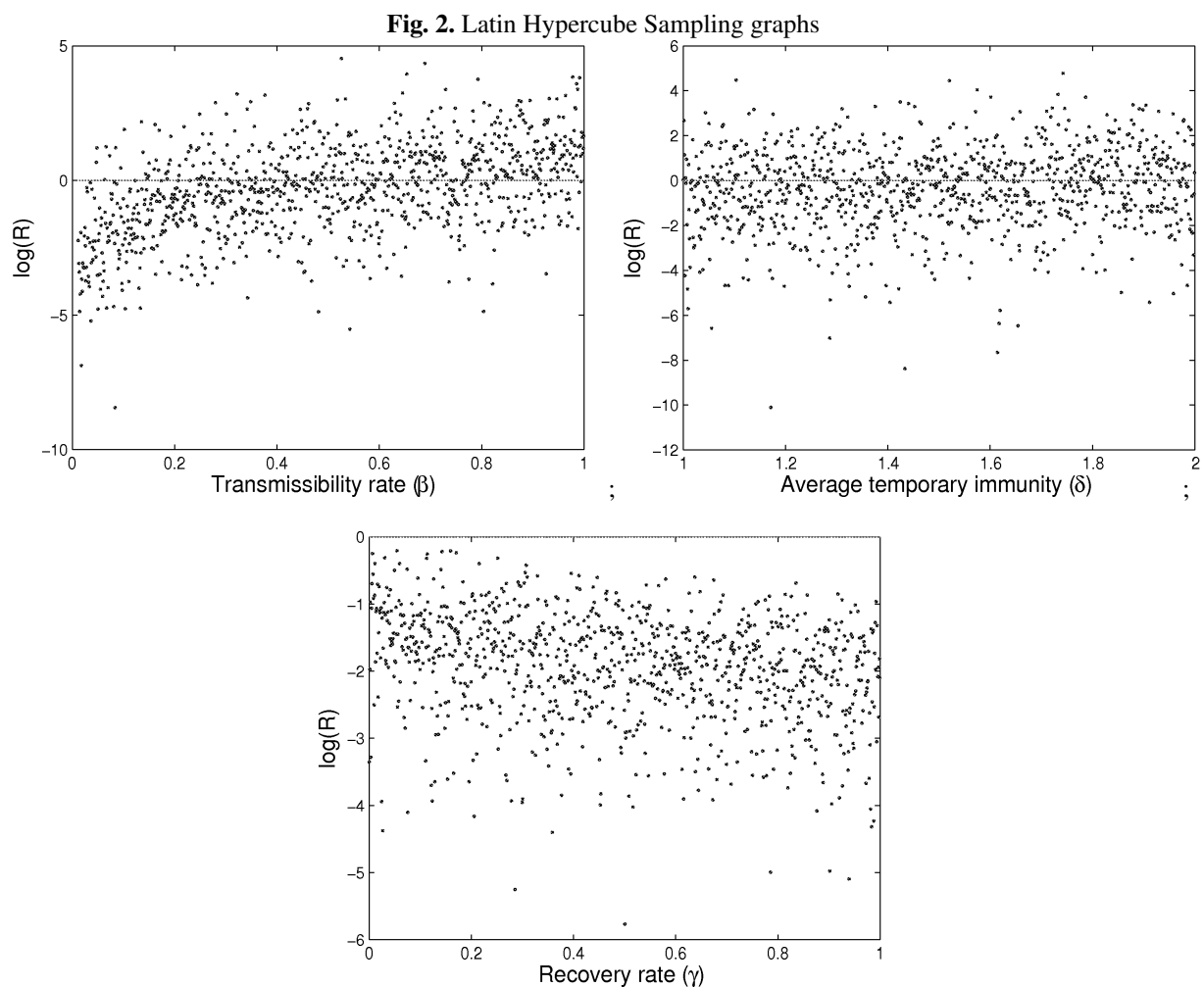


Fig. 2 are scatter graphs that relates the reproduction number to its parameters. It shows a strong positive correlation between the contact number and the reproduction number. Also, it reflects that the average temporary immunity has a negligible effect on the reproduction number, since the line of best fit is parallel to the horizontal axis. In addition, Fig. 2(c) indicates a strong negative relationship between the recovery rate and the reproduction number, that is, an increase in the value of the recovery rate yields a decrease in the value the reproductive number thereby controlling the rate at which the infection spreads.

In Fig. 3, partial rank correlation coefficients (PRCC) were calculated to estimate the correlation between the values of the reproduction number  $R$  and the parameters that defines it. In this Fig. 3, a large PRCC is an indicative of high sensitivity to parameter estimate ( $PRCC > 0$  will increase  $R$  when the parameters are increased). In contrast, small PRCC reflects low sensitivity ( $PRCC < 0$  will decrease the value of  $R$  when the parameters are increased). Also, Fig. 3 shows that the reproduction number  $R$  is more sensitive to recovery rate  $\gamma$  and the contact rate  $\beta$ . The graph shows that as the recovery rate increase the value of the reproduction number decreases thereby reducing the spread of the infection or disease. In contrast, an increase in the value of the contact rate increases the value of the reproduction number.

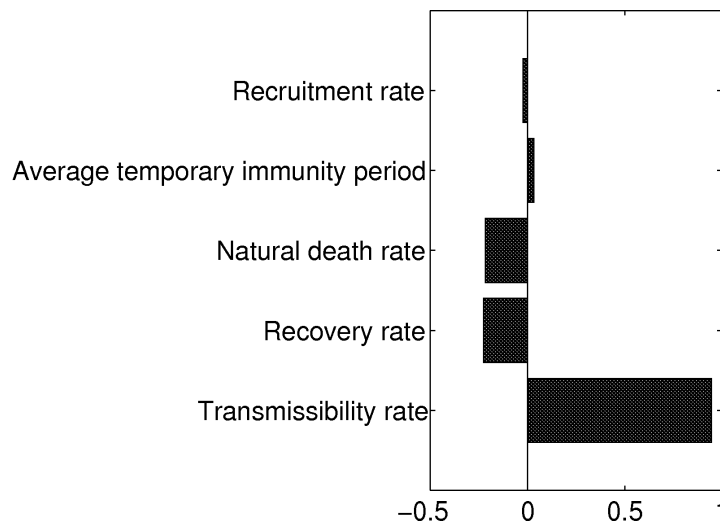


Fig. 3. Sensitivity of  $R$  to all parameters using partial rank correlation coefficients

#### 4 Application of adomian decomposition method on MSIR epidemic model

Due to the above change of variables the system of equations (1) becomes:

$$\frac{dx}{dt} = \Lambda - \delta xy - \mu x, \tag{3}$$

$$\frac{dy}{dt} = \delta xy - \beta yz - \mu y, \tag{4}$$

$$\frac{dz}{dt} = \beta yz - (\gamma + \mu)z, \tag{5}$$

$$\frac{dw}{dt} = \gamma z - \mu w. \tag{6}$$

With the initial conditions  $x(0) = x_0, y(0) = y_0, z(0) = z_0$  and  $w(0) = w_0$ , where at time  $t$ ,

$x(t)$  represents a proportion of people born with passive immunity;

$y(t)$  represents a proportion of susceptible individuals;

$z(t)$  represents a proportion of infected individuals;

$w(t)$  represents a proportion of recovered individuals with immunity.

Here, we seek to extend the application of LADM to obtain an approximate solution of the MSIR epidemic model (3).

##### 4.1 Solution procedure

We recall that the Laplace transform of  $x'_i(t)$  are defined by

$$\mathfrak{L}x' = s\mathfrak{L}x - x(0) \text{ for } i = 1, 2, 3, \dots, n.$$

Applying the Laplace transform both sides of (3) and simplifying yields

$$\begin{aligned}\mathfrak{L}x(t) &= \frac{x_0}{s} + \frac{A}{s} - \frac{\delta}{s}\mathfrak{L}x(t)y(t) - \frac{\mu}{s}\mathfrak{L}x(t), \\ \mathfrak{L}y(t) &= \frac{y_0}{s} + \frac{\delta}{s}\mathfrak{L}x(t)y(t) - \frac{\beta}{s}\mathfrak{L}x(t)z(t) - \frac{\mu}{s}\mathfrak{L}y(t), \\ \mathfrak{L}z(t) &= \frac{z_0}{s} + \frac{\beta}{s}\mathfrak{L}x(t)z(t) - \frac{(\mu + \gamma)}{s}\mathfrak{L}z(t), \\ \mathfrak{L}w(t) &= \frac{w_0}{s} + \frac{\gamma}{s}\mathfrak{L}z(t) - \frac{\mu}{s}\mathfrak{L}w(t).\end{aligned}\tag{7}$$

Let  $F(t) = x(t) \cdot y(t)$  and  $G(t) = x(t) \cdot z(t)$ . Now, we use the Adomian decomposition method [6] and the Adomian polynomials to handle system (7) and to address non-linear terms  $F(t) = x(t) \cdot y(t)$  and  $G(t) = x(t) \cdot z(t)$ . We now represent the solutions as infinite series, namely

$$x = \sum_{k=0}^{\infty} x_k, y = \sum_{k=0}^{\infty} y_k, z = \sum_{k=0}^{\infty} z_k, w = \sum_{k=0}^{\infty} w_k,\tag{8}$$

where the components  $x_k$  are to be recursively computed. Furthermore, the non-linear terms  $F(t) = x(t) \cdot y(t)$  and  $G(t) = x(t) \cdot z(t)$  will be represented by an infinite series of Adomian polynomials [7]

$$F(t, x, y) = \sum_{k=0}^{\infty} A_k, G(t, x, z) = \sum_{k=0}^{\infty} B_k,\tag{9}$$

where  $A_k, B_k, k \geq 0$  are defined by

$$\begin{aligned}A_k &= \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ F(t, \sum_{j=0}^k \lambda^j x_j, \sum_{j=0}^k \lambda^j y_j) \right], k = 0, 1, 2, \dots \\ B_k &= \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ G(t, \sum_{j=0}^k \lambda^j x_j, \sum_{j=0}^k \lambda^j z_j) \right], k = 0, 1, 2, \dots\end{aligned}\tag{10}$$

$A_k$  and  $B_k$  are called Adomian polynomials and can be evaluated for all forms of non-linear functions. Substituting (8) and (9) into (7) leads to

$$\begin{aligned}\mathfrak{L} \sum_{k=0}^{\infty} x_k &= \frac{x_0}{s} + \frac{A}{s} - \frac{\delta}{s} \mathfrak{L} \sum_{k=0}^{\infty} A_k - \frac{\mu}{s} \mathfrak{L} \sum_{k=0}^{\infty} x_k, \\ \mathfrak{L} \sum_{k=0}^{\infty} y_k &= \frac{y_0}{s} + \frac{\delta}{s} \mathfrak{L} \sum_{k=0}^{\infty} A_k - \frac{\beta}{s} \mathfrak{L} \sum_{k=0}^{\infty} B_k - \frac{\mu}{s} \mathfrak{L} \sum_{k=0}^{\infty} y_k, \\ \mathfrak{L} \sum_{k=0}^{\infty} z_k &= \frac{z_0}{s} + \frac{\beta}{s} \mathfrak{L} \sum_{k=0}^{\infty} B_k - \frac{(\mu + \gamma)}{s} \mathfrak{L} \sum_{k=0}^{\infty} z_k, \\ \mathfrak{L} \sum_{k=0}^{\infty} w_k &= \frac{w_0}{s} + \frac{\gamma}{s} \mathfrak{L} \sum_{k=0}^{\infty} z_k - \frac{\mu}{s} \mathfrak{L} \sum_{k=0}^{\infty} w_k.\end{aligned}\tag{11}$$

Matching both sides of (11) we obtain the following iterative algorithm:

$$\begin{aligned}
\mathfrak{L}\{x_0\} &= \frac{x_0}{s}, \quad \mathfrak{L}\{x_{k+1}\} = \frac{\Lambda}{s} - \frac{\delta}{s}\mathfrak{L}\{A_k\} - \frac{\mu}{s}\mathfrak{L}\{x_k\}, \\
\mathfrak{L}\{y_0\} &= \frac{y_0}{s}, \quad \mathfrak{L}\{y_{k+1}\} = \frac{\delta}{s}\mathfrak{L}\{A_k\} - \frac{\beta}{s}\mathfrak{L}\{B_k\} - \frac{\mu}{s}\mathfrak{L}\{y_k\}, \\
\mathfrak{L}\{z_0\} &= \frac{z_0}{s}, \quad \mathfrak{L}\{z_{k+1}\} = \frac{\beta}{s}\mathfrak{L}\{B_k\} - \frac{(\mu + \gamma)}{s}\mathfrak{L}\{z_k\}, \\
\mathfrak{L}\{w_0\} &= \frac{w_0}{s}, \quad \mathfrak{L}\{w_{k+1}\} = \frac{\gamma}{s}\mathfrak{L}\{z_k\} - \frac{\mu}{s}\mathfrak{L}\{w_k\},
\end{aligned} \tag{12}$$

$$\tag{13}$$

Applying the inverse Laplace transform to the first part of (12) gives  $x_0, y_0, z_0$  and  $w_0$ . These values will define  $A_0$  and  $B_0$  as follows:

$A_0 = F(t, x_0, y_0) = x_0 y_0$  and  $B_0 = G(t, x_0, z_0) = x_0 z_0$ . These values will enable us to compute  $x_1, y_1, z_1$  and  $w_1$ , as follows:

$$\begin{aligned}
\mathfrak{L}\{x_1\} &= \frac{\Lambda}{s} - \frac{\delta}{s}\mathfrak{L}\{A_0\} - \frac{\mu}{s}\mathfrak{L}\{x_0\} = \frac{\Lambda}{s} - \frac{\delta}{s}x_0 y_0 - \frac{\mu}{s}x_0 \\
\Rightarrow x_1 &= \mathfrak{L}^{-1} \left\{ \frac{\Lambda}{s} - \frac{\delta}{s}x_0 y_0 - \frac{\mu}{s}x_0 \right\} = \Lambda - \delta x_0 y_0 - \mu x_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
y_1 &= \delta x_0 y_0 - \beta x_0 z_0 - \mu y_0, \\
z_1 &= \beta x_0 z_0 - (\mu + \gamma) z_0, \\
w_1 &= \gamma z_0 - \mu w_0.
\end{aligned}$$

Now, using (10) we obtain  $A_1 = x_0 y_1 + x_1 y_0 + 2\lambda x_1 y_1$  and  $B_1 = x_0 z_1 + x_1 z_0 + 2\lambda x_1 z_1$  so that

$$\begin{aligned}
x_2 &= \Lambda - \delta(x_0 y_1 + x_1 y_0 + 2x_1 y_1) - \mu x_1, \\
y_2 &= \delta(x_0 y_1 + x_1 y_0 + 2x_1 y_1) - \beta(x_0 z_1 + x_1 z_0 + 2x_1 z_1) - \mu y_1, \\
z_2 &= \beta(x_0 z_1 + x_1 z_0 + 2x_1 z_1) - (\mu + \gamma) z_1, \\
w_2 &= \gamma z_1 - \mu w_1.
\end{aligned}$$

This successively will lead to the complete determination of the components of  $x_k, y_k$  and  $z_k, k \geq 0$  upon using (12). The series solution follows immediately after using equation (8).

## 5 Conclusion

The MSIR epidemic model has been presented and its analysis has been dealt with using analytic, numerical and the Adomian decomposition methods. Equilibrium points are shown to be locally and globally asymptotically stable whenever they exist. Sensitivity analysis has revealed that the spread of the infection can be reduced by increasing the rate at which infected individuals recovers. An extension of the analysis of the model by employing the Laplace Adomian Decomposition method [6] has been done. The analysis reflects that the series solution of the system (1) can be approximated by a powerful Laplace-Adomian Decomposition method.

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