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Note on "Optimal control of vaccination and treatment for an SIR epidemiological model-Vol. 8(2012) No. 3, pp. 194-204"

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Abstract. In the article "Optimal control of vaccination and treatment for an SIR epidemiological model-World Journal of Modelling and Simulation, Vol. 8(2012) No. 3, pp. 194-204" authors have shown global stability for the SIR model considered. For this they constructed a Lyapunav function. We note that although this function is positive definite in the feasible region but its derivative along the solution trajectories of the model system is not negative definite everywhere in the feasible region. Thus this is not the right choice. We construct another Lyapunav function and establish that the endemic equilibrium is globally asymptotically stable in the entire feasible region whenever it exists.

Keywords: global stability, Lyapunov function.

1 Introduction

In this paper we shall establish global stability of the endemic equilibrium for the model system considered by [7].

In epidemiological modeling, study of long term behavior of disease progression in a population is very important. This is usually performed through stability analysis. Lyapunov second method is one such tool to establish the stability of equilibrium point of a dynamical system. For this we need a positive definite function whose derivative is negative definite along solution trajectories of the system. Though establishing stability via Lyapunov function seems simple but finding such a function is really a difficult task. Till today there is no general method to construct Lyapunov function. Depending on models, researchers constructed such functions for some general epidemiological model[2, 4, 5]. In these and other works researchers have constructed Lyapunov functions by using a combinations of logarithmic function, square function, square of sum of functions and integral forms[1–5].

In [7] authors have shown global stability for the SIR model considered using Lyapunav function. We found that this function is positive definite but its derivative is not negative definite in the entire feasible region and hence it can not be a correct choice. We construct another Lyapunav function and show the global stability of the endemic equilibrium.

The paper is organized as follows: in the following section we establish global stability by defining a Lyapunov function. Finally we conclude the result obtained.

2 Global stability

In [7], authors proposed and analyzed following SIR model:

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$$\begin{aligned}\frac{dS}{dt} &= b - \beta SI - dS - u_1 S,\\ \frac{dI}{dt} &= \beta SI - u_2 I - dI - \alpha I,\\ \frac{dN}{dt} &= b - Nd - \alpha I,\end{aligned}$$

with endemic equilibrium point:

$$E_2 = (S^*, I^*, N^*) \text{ which exists whenever } R_0 > 1,$$

where

$$\begin{aligned}&S^* = \frac{(u_2 + d + \alpha)}{(\beta)},\\ &N^* = \frac{b\beta(u_2 + d) + \alpha(u_1 + d)(u_2 + d + \alpha)}{d\beta(u_2 + d + \alpha)},\\ &I^* = (R_0 - 1)\frac{u_1 + d}{\beta},\\ &R_0 = \left(\frac{b}{d + u_1}\right)\left(\frac{\beta}{u_2 + d + \alpha}\right).\end{aligned}$$

The positively invariant region of the model system is

$$\Omega = \left\{(S, I, N) \in \mathbb{R}^3_+ \mid S \geq 0, I \geq 0, S + I \leq N \leq \frac{b}{d}\right\}.$$

In [7] authors proved global stability of endemic equilibrium by using the Lyapunov function defined for feasible region Ω. They considered following Lyapunov function:

$$V(S, I, N) = \frac{1}{2}(N - N^*)^2 + \frac{1}{2}(S - S^*)^2 + \epsilon\left(I - I^* - I^* \ln\frac{I}{I^*}\right) \quad \epsilon \geq 0,$$

such that

$$\begin{aligned}\dot{V} &= -d(N - N^*)^2 - (u_1 + d + \beta I)(S - S^*)^2 - \alpha(N - N^*)(I - I^*)\\ &\quad + \beta(\epsilon - S^*)(S - S^*)(I - I^*).\end{aligned}$$

Here, they choose $\epsilon = S^*$ so that

$$\dot{V} = -d(N - N^*)^2 - (u_1 + d + \beta I)(S - S^*)^2 - \alpha(N - N^*)(I - I^*).$$

Now authors claim that for $N \leq N^*$ and $I \leq I^*$, $\dot{V} \leq 0$. But note that in Ω there are points $N > N^*$ and $I > I^*$, as it can be easily verified that

$$N^* = \frac{b}{d} + (1 - R_0)\frac{\alpha(u_1 + d)}{d\beta} < \frac{b}{d}$$

for $R_0 > 1$.
Therefore, the $\dot{V} \leq 0$ is not guaranteed for every point in Ω.
To overcome this problem, we can instead define Lyapunov function [2] in the following form:

$$L(S, I, N) = \frac{1}{2}[(S - S^*) + (I - I^*)]^2 + m_1\left(I - I^* - I^* \ln\frac{I}{I^*}\right).$$

Here m_1 is positive constant and will be chosen suitably later. Clearly L is a positive definite function. Now differentiating L with respect to t along the system and also using the following relations

$$\begin{aligned} b &= \beta S^* I^* + dS^* + u_1 S^* = dN^* + \alpha I^*, \\ \beta S^* I^* &= (u_2 + d + \alpha) I^*, \\ b &= (u_1 + d) S^* + (u_2 + d + \alpha) I^*. \end{aligned}$$

$$\begin{aligned} \dot{L} &= [(S - S^*) + (I - I^*)] \frac{d(S+I)}{dt} + m_1 \frac{(I - I^*)}{I} \frac{dI}{dt} \\ &= [(S - S^*) + (I - I^*)][(d + u_1) S^* + (d + u_2 + \alpha) I^* - (d + u_1) S \\ &\quad - (d + u_2 + \alpha) I] + m_1 (I - I^*)(\beta S - \beta S^*) \\ &= [(S - S^*) + (I - I^*)][-(d + u_1)(S - S^*) - (d + u_2 + \alpha)(I - I^*)] \\ &\quad + m_1 \beta (S - S^*)(I - I^*) \\ &= -(d + u_1)(S - S^*)^2 - (d + u_2 + \alpha)(I - I^*)^2 - (2d + u_1 + u_2 + \alpha) \\ &\quad (S - S^*)(I - I^*) + m_1 \beta (S - S^*)(I - I^*). \end{aligned}$$

Choosing

$$m_1 = \frac{(2d + u_1 + u_2 + \alpha)}{\beta},$$

we get $\dot{L} \leq 0$.

So $\dot{L}$ is negative and $\dot{L} = 0$ if and only if $S = S^*, I = I^*$ and $N = N^*$. Hence the singleton set $\{E_2\}$ is the largest invariant set in

$$\{(S, I, N) \in \Omega : \dot{L} = 0\},$$

where E_2 is endemic equilibrium in Ω. Then by LaSalle's invariant principle E_2 is globally asymptotically stable in the interior of Ω [6].

3 Conclusion

We established the global asymptotic stability of endemic equilibrium by constructing a Lyapunov function and with the help of LaSalle's invariance principle in entire feasible region. In view of population N not involved directly in other equations this function is possibly the best choice.

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