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A new modified quadrature method for solving linear weakly singular integral equations

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Abstract. One kind of linear integral equation is the weakly singular integral equation; this kind of integral equation is unsolvable by using usual quadrature methods, such as the repeated trapezoid and repeated Simpson quadrature methods. In this article, we intend to present a subtraction method, which is the same as the modified quadrature methods; one can solve linear weakly singular integral equations of the second kind through these methods. Some numerical examples illustrate the theoretical results. The numerical experiments show the efficiency of our method.

Keywords: linear integral equation, weakly singular integral equation, weakly singular kernel, modified quadrature, weaken the singularity

1 Introduction

The numerical solution of linear weakly singular integral equations has attracted the interest of the authors (See [3–6, 9–11]). A general form of the linear weakly singular integral equation defined by the following:

$$u(x) = f(x) + \int_{\Omega} g(x,t)|x-t|^{-\alpha}u(t)dt, \; 0<\alpha<1, \tag{1}$$

where Ω is equal to (a,b) or (a,x) if Eq. (1) be a Fredholm or Volterra integral equation, respectively, and $g(x,t)$ is a bounded function on Ω .(Kernel in these equations is $k(x,t) = g(x,t)|x-t|^{-\alpha}$.)

One of the methods for solving this kind of equation is the modified quadrature method, which is explained in [7, 8, 14]. In this method, we weaken the singularity by subtraction as follows:

$$\begin{aligned} u(x) &= f(x) + \int_{\Omega} g(x,t)|x-t|^{-\alpha}u(t)dt, \\ &= f(x) + \int_{\Omega} g(x,t)|x-t|^{-\alpha}[u(t)-x(t)]dt + u(x)\int_{\Omega} g(x,t)|x-t|^{-\alpha}dt. \end{aligned} \tag{2}$$

Therefore, by using quadrature methods, one can solve weakly singular Volterra and Fredholm integral equations by solving the following systems (3) and (4), respectively (for further information on quadrature methods, see [1, 2, 7, 8, 13]).

$$\begin{cases} u(x_0) = f(x_0) \\ u(x_i) = f(x_i) + \sum\limits_{j=0}^{i-1} w_{ij} g(x_i,x_j)|x_i-x_j|^{-\alpha}\left[u(x_j)-u(x_i)\right] + A_i u(x_i) \\ \qquad\;\; = f(x_i) + \sum\limits_{j=0}^{i-1} w_{ij} g(x_i,x_j)|x_i-x_j|^{-\alpha}u(x_j) + w_{ii}u(x_i)\,, i=1,2,\cdots,n, \end{cases} \tag{3}$$

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where, $A_i = \int_a^{x_i} g(x_i,t)|x_i - t|^{-\alpha} dt$.

$$\begin{aligned} u(x_i) &= f(x_i) + \sum_{j=0, j\neq i}^{n} w_{ij} g(x_i,x_j)|x_i - x_j|^{-\alpha} \left[u(x_j) - u(x_i)\right] + A_i u(x_i) \\ &= f(x_i) + \sum_{j=0, j\neq i}^{n} w_{ij} g(x_i,x_j)|x_i - x_j|^{-\alpha} u(x_j) + w_{ii} u(x_i) \, , \, i = 0,1,\cdots,n, \end{aligned} \tag{4}$$

where, $A_i = \int_a^{b} g(x_i,t)|x_i - t|^{\alpha} dt$.

In this article, we intend to demonstrate a new subtraction method similar to the above method. Afterwards, by using this process in the repeated trapezoid and repeated Simpson and block-by-block quadrature methods, the weakly singular integral equations of the form Eq. (1) can be solved.

2 Theory of the method

For describing the theory of the new modified quadrature method, we consider two cases for Fredholm and Volterra integral equations and explain the situation for each case. The general form of the linear weakly singular Volterra integral equations is as follows:

$$u(x) = f(x) + \int_a^x g(x,t)|x-t|^{-\alpha} u(t)dt \, , \, 0 < \alpha < 1. \tag{5}$$

From Eq. (5), we get:

$$u(x_i) = f(x_i) + \int_a^{y_i} g(x_i,t)|x_i - t|^{-\alpha} u(t)dt + \int_{y_i}^{x_i} g(x_i,t)|x_i - t|^{-\alpha} u(t)dt, \tag{6}$$

where, x_is are points of a closed integration formula and $a \leq y_i \leq x_i$. Now we weaken the singularity in the following way:

$$\begin{aligned} u(x_i) =& f(x_i) + \int_a^{y_i} g(x_i,t)|x_i - t|^{-\alpha} u(t)dt + \int_{y_i}^{x_i} g(x_i,t)|x_i - t|^{-\alpha} \left[u(t) - u(x_i)\right] dt \\ & + u(x_i) \int_{y_i}^{x_i} g(x_i,t)|x_i - t|^{-\alpha} dt. \end{aligned} \tag{7}$$

Therefore, by choosing $a \leq y_i \leq x_i$, one can see a more general form of subtraction for weakening the singularity.

For integral equations of Fredholm type, we have the following:

$$\begin{aligned} u(x_i) =& f(x_i) + \int_a^{b} g(x_i,t)|x_i - t|^{-\alpha} u(t)dt \\ =& f(x_i) + \int_a^{y_i} g(x_i,t)|x_i - t|^{-\alpha} u(t)dt + \int_{y_i}^{z_i} g(x_i,t)|x_i - t|^{\alpha} u(t)dt \\ & + \int_{z_i}^{b} g(x_i,t)|x_i - t|^{-\alpha} u(t)dt \\ =& f(x_i) + \int_a^{y_i} g(x_i,t)|x_i - t|^{-\alpha} u(t)dt + \int_{y_i}^{z_i} g(x_i,t)|x_i - t|^{-\alpha} \left[u(t) - u(x_i)\right] dt \\ & + u(x_i) \int_{y_i}^{z_i} g(x_i,t)|x_i - t|^{-\alpha} dt + \int_{z_i}^{b} g(x_i,t)|x_i - t|^{-\alpha} u(t)dt, \end{aligned} \tag{8}$$

where, $a \leq y_i \leq x_i$ and $x_i \leq z_i \leq b$.

As we described above, we present our idea for solving linear weakly singular integral equations in each quadrature method for solving each type of equation, i.e. Volterra and Fredholm integral equations. We will explain how we choose y_i and z_i when we explain the above method. We use the following notation for simplicity:

$$\begin{cases} u_i = u(x_i), \\ f_i = f(x_i), \\ k(x,t) = g(x,t)|x-t|^{-\alpha}, \\ k_{ij} = k(x_i, x_j). \end{cases} \tag{9}$$

3 Solving linear weakly singular volterra integral equations

3.1 Solving linear weakly singular volterra integral equations

In this quadrature method, for the subtraction and calculation of u_i for $i = 1, 2, \cdots, n$, we consider the y_i as follows:

$$\begin{cases} x_0 = a, \\ h = x_1 - x_0, \\ y_i = x_{i-1},\ i = 1, 2, \cdots, n. \end{cases} \tag{10}$$

Therefore, we weaken the singularity in the following form:

$$\begin{aligned} u_1 =& f_1 + \int_a^{x_1} k(x_1,t)u(t)dt + \int_{x_1}^{x_1} k(x_1,t)u(t)dt \\ =& f_1 + \int_a^{x_1} k(x_1,t)\left[u(t) - u_1\right]dt + u_1 \int_a^{x_1} k(x_1,t)dt \\ =& f_1 + \tfrac{h}{2}k_{10}(u_0 - u_1) + u_1 A_1 \\ =& f_1 + \tfrac{h}{2}k_{10}u_0 + (A_1 - \tfrac{h}{2}k_{10})u_1, \end{aligned} \tag{11}$$

where, $A_1 = \int_a^{x_1} k(x_1,t)dt$. Similarly, we obtain the following:

$$\begin{aligned} u_2 =& f_2 + \int_a^{x_1} k(x_2,t)u(t)dt + \int_{x_1}^{x_2} k(x_2,t)u(t)dt \\ =& f_2 + \int_a^{x_1} k(x_2,t)u(t)dt + \int_{x_1}^{x_2} k(x_2,t)\left[u(t) - u_2\right]dt + u_2 \int_{x_1}^{x_2} k(x_2,t)dt \\ =& f_2 + \tfrac{h}{2}k_{20}u_0 + \tfrac{h}{2}k_{21}u_1 + \tfrac{h}{2}k_{21}(u_1 - u_2) + u_2 A_2 \\ =& f_2 + \tfrac{h}{2}k_{20}u_0 + hk_{21}u_1 + \left(A_2 - \tfrac{h}{2}k_{21}\right)u_2, \end{aligned} \tag{12}$$

where $A_2 = \int_{x_1}^{x_2} k(x_2,t)dt$. The integrals in above are approximated by trapezoid formula.

Hence, in general, we get:

$$\begin{cases} u_0 = f_0, \\ u_i = f_i + \tfrac{h}{2}k_{i0}u_0 + h\sum\limits_{j=1}^{i-1} k_{ij}u_j + \left(A_i - \tfrac{h}{2}k_{i,i-1}\right)u_i\,,\ i = 1, 2, \cdots, n, \end{cases} \tag{13}$$

where

$$A_i = \int_{x_{i-1}}^{x_i} k(x_i,t)dt\,,\ i = 1, 2, \cdots, n. \tag{14}$$

3.2 Repeated simpson modified quadrature method for volterra integral equations

In this quadrature method, for subtracting and calculating u_i for $i = 2, 3, \cdots, 2n$, we consider the y_i's as follows:

$$\begin{cases} x_0 = a, \\ h = x_1 - x_0, \\ y_i = \begin{cases} x_{i-3} & i = 3, 5, \cdots, 2n-1 \\ x_{i-2} & i = 2, 4, \cdots, 2n \end{cases}. \end{cases} \tag{15}$$

As a result, we obtain:

$$\begin{aligned} u_2 =& f_2 + \int_a^{x_2} k(x_2, t) u(t) dt + \int_{x_2}^{x_2} k(x_2, t) u(t) dt \\ =& f_2 + \int_a^{x_2} k(x_2, t) \left[u(t) - u_2\right] dt + u_2 \int_a^{x_2} k(x_2, t) dt \\ =& f_2 + \tfrac{h}{3} k_{20} u_0 + \tfrac{4h}{3} k_{21} u_1 + \left(A_2 - \tfrac{h}{3} k_{20} - \tfrac{4h}{3} k_{21}\right) u_2, \end{aligned} \tag{16}$$

where, $A_2 = \int_a^{x_2} k(x_2, t) dt$.

Similarly,

$$\begin{aligned} u_3 =& f_3 + \int_a^{x_3} k(x_3, t) u(t) dt + \int_{x_3}^{x_3} k(x_3, t) u(t) dt \\ =& f_3 + \int_a^{x_3} k(x_3, t) \left[u(t) - u_3\right] dt + u_3 \int_a^{x_3} k(x_3, t) dt \\ =& f_3 + \tfrac{3h}{8} k_{30} u_0 + \tfrac{9h}{8} k_{31} u_1 + \tfrac{9h}{8} k_{32} u_2 + \left(A_3 - \tfrac{3h}{8} k_{30} - \tfrac{9h}{8} k_{31} - \tfrac{9h}{8} k_{32}\right) u_3, \end{aligned} \tag{17}$$

where, $A_3 = \int_a^{x_3} k(x_3, t) dt$. The integrals approximated by Simpson's 3/8 formula.

$$\begin{aligned} u_4 =& f_4 + \int_a^{x_2} k(x_4, t) u(t) dt + \int_{x_2}^{x_4} k(x_4, t) u(t) dt \\ =& f_4 + \int_a^{x_2} k(x_4, t) u(t) dt + \int_{x_2}^{x_4} k(x_4, t) \left[u(t) - u_4\right] dt + u_4 \int_{x_2}^{x_4} k(x_4, t) dt \\ =& f_4 + \tfrac{h}{3} k_{40} u_0 + \tfrac{4h}{3} k_{41} u_1 + \tfrac{2h}{3} k_{42} u_2 + \tfrac{4h}{3} k_{43} u_3 + \left(A_4 - \tfrac{h}{3} k_{42} - \tfrac{4h}{3} k_{43}\right) u_4, \end{aligned} \tag{18}$$

where $A_4 = \int_{x_2}^{x_4} k(x_4, t) dt$ and the integrals are approximated by Simpson's formula.

Accordingly, we can write:

$$\begin{cases} u_0 = f_0, \\ u_1 = \hat{u}_2, \\ u_{2i} = f_{2i} + \frac{h}{3} \sum\limits_{j=1}^{i-1} \left(k_{2i,2j-2} u_{2j-2} + 4 k_{2i,2j-1} u_{2j-1} + k_{2i,2j} u_{2j}\right) \\ \qquad + \frac{h}{3} k_{2i,2i-2} u_{2i-2} + \frac{4h}{3} k_{2i,2i-1} u_{2i-1} + \left(A_{2i} - \frac{h}{3} k_{2i,2i-2} - \frac{4h}{3} k_{2i,2i-1}\right) u_{2i}, i = 1, 2, \cdots, n, \\ u_{2i+1} = f_{2i+1} + \frac{h}{3} \sum\limits_{j=1}^{i-1} \left(k_{2i+1,2j-2} u_{2j-2} + 4 k_{2i+1,2j-1} u_{2j-1} + k_{2i+1,2j} u_{2j}\right) \\ \qquad + \frac{3h}{8} k_{2i+1,2i-2} u_{2i-2} + \frac{9h}{8} k_{2i+1,2i-1} u_{2i-1} + \frac{9h}{8} k_{2i+1,2i} u_{2i} \\ \qquad + \left(A_{2i+1} - \frac{3h}{8} k_{2i+1,2i-2} - \frac{9h}{8} k_{2i+1,2i-1} - \frac{9h}{8} k_{2i+1,2i}\right) u_{2i+1}, i = 1, 2, \cdots, n-1, \end{cases} \tag{19}$$

where

$$\begin{cases} A_{2i} = \int_{x_{2i-2}}^{x_{2i}} k(x_{2i}, t) dt, i = 1, 2, \cdots, n, \\ A_{2i+1} = \int_{x_{2i-2}}^{x_{2i+1}} k(x_{2i+1}, t) dt, i = 1, 2, \cdots, n-1. \end{cases} \tag{20}$$

In addition, the $\hat{u}_2$ can be obtained from the following system:

$$\begin{cases} \hat{u}_0 = f(x_0), \\ \hat{u}_1 = f(\frac{x_0+x_1}{2}) + \frac{h}{4} k(\frac{x_0+x_1}{2}, x_0) \hat{u}_0 + \left[\hat{A}_1 - \frac{h}{4} k(\frac{x_0+x_1}{2}, x_0)\right] \hat{u}_1, \\ \hat{u}_2 = f(x_1) + \frac{h}{6} k(x_1, x_0) \hat{u}_0 + \frac{4h}{6} k(x_1, \frac{x_0+x_1}{2}) \hat{u}_1 + \left[\hat{A}_2 - \frac{h}{6} k(x_1, x_0) - \frac{4h}{6} k(x_1, \frac{x_0+x_1}{2})\right] \hat{u}_2,, \end{cases} \tag{21}$$

where

$$\begin{cases} \hat{A}_1 = \int_a^{\frac{x_0+x_1}{2}} k(\frac{x_0+x_1}{2}, t) dt, \\ \hat{A}_2 = \int_a^{x_1} k(x_1, t) dt. \end{cases} \tag{22}$$

3.3 Block-by-block modified quadrature method for volterra integral equations

In this quadrature method, we consider:

$$\begin{cases} x_0 = a, \\ h = x_1 - x_0, \\ y_i = \begin{cases} x_{i-1} & i = 1, 3, \cdots, 2n-1 \\ x_{i-2} & i = 2, 4, \cdots, 2n \end{cases}. \end{cases} \tag{23}$$

Doing the subtraction process similar to that of the Simpson method used in Section 3.2, (for further information on block-by-block methods see [7, 8, 14]) consequently, we obtain the following system of the equations:

$$\begin{cases} u_0 = f_0, \\ u_{2i} = f_{2i} + \frac{h}{3} \sum\limits_{j=1}^{i-1} \left(k_{2i,2j-2} u_{2j-2} + 4k_{2i,2j-1} u_{2j-1} + k_{2i,2j} u_{2j}\right) + \frac{h}{3} k_{2i,2i-2} u_{2i-2} \\ \qquad + \frac{4h}{3} k_{2i,2i-1} u_{2i-1} + \left(A_{2i} - \frac{h}{3} k_{2i,2i-2} - \frac{4h}{3} k_{2i,2i-1}\right) u_{2i}\ , i = 1, 2, \cdots, n, \\ u_{2i+1} = f_{2i+1} + \frac{h}{3} \sum\limits_{j=1}^{i} \left(k_{2i+1,2j-2} u_{2j-2} + 4k_{2i+1,2j-1} u_{2j-1} + k_{2i+1,2j} u_{2j}\right) + \frac{5h}{12} k_{2i+1,2i} u_{2i} \\ \qquad + \left(A_{2i+1} - \frac{5h}{12} k_{2i+1,2i} + \frac{h}{12} k_{2i+1,2i+2}\right) u_{2i+1} - \frac{h}{12} k_{2i+1,2i+2} u_{2i+2}\ , i = 0, 1, \cdots, n-1 \end{cases} \tag{24}$$

where

$$\begin{cases} A_{2i} = \int_{x_{2i-2}}^{x_{2i}} k(x_{2i}, t)dt\ , i = 1, 2, \cdots, n, \\ A_{2i+1} = \int_{x_{2i}}^{x_{2i+2}} k(x_{2i+1}, t)dt\ , i = 0, 1, \cdots, n-1. \end{cases} \tag{25}$$

4 Solving linear weakly singular fredholm integral equations

In this section similar to that of Volterra integral equations, we present the trapezoid and Simpson modified quadrature methods for solving linear weakly singular Fredholm integral equations.

4.1 Repeated trapezoid modified quadrature method for fredholm integral equations

In this quadrature method, for solving weakly singular Fredholm integral equations, we choose y_i and z_i as follows:

$$\begin{cases} x_0 = a, \\ h = x_1 - x_0, \\ y_i = \begin{cases} a & i = 0 \\ x_{i-1} & i = 1, 2, \cdots, n \end{cases} \\ z_i = \begin{cases} x_{i+1} & i = 0, 1, \cdots, n-1 \\ b & i = n \end{cases} \end{cases} \tag{26}$$

Thus, the u_i, s can be calculated in the following way:

$$\begin{aligned} u_0 =& f_0 + \int_a^{x_1} k(x_0, t)u(t)dt + \int_{x_1}^{b} k(x_0, t)u(t)dt \\ =& f_0 + \int_a^{x_1} k(x_0, t)\left[u(t) - u_0\right]dt + u_0 \int_a^{x_1} k(x_0, t)dt + \int_{x_1}^{b} k(x_0, t)u(t)dt \\ =& f_0 + \left(A_0 - \tfrac{h}{2} k_{01}\right) u_0 + hk_{01} u_1 + hk_{02} u_2 + \cdots + hk_{0,n-1} + \tfrac{h}{2} k_{0n} u_n, \end{aligned} \tag{27}$$

where $A_0 = \int_a^{x_1} k(x_0, t)dt$.

$$
\begin{aligned}
u_1 =& f_1 + \int_a^{x_2} k(x_1,t)u(t)dt + \int_{x_2}^{b} k(x_1,t)u(t)dt \\
=& f_1 + \int_a^{x_2} k(x_1,t)\left[u(t)-u_1\right]dt + u_1\int_a^{x_2} k(x_1,t)dt + \int_{x_2}^{b} k(x_1,t)u(t)dt \\
=& f_1 + \tfrac{h}{2}k_{10}u_0 + \left(A_1 - \tfrac{h}{2}k_{10} - \tfrac{h}{2}k_{12}\right)u_1 + hk_{12}u_2 + \cdots + hk_{1,n-1} + \tfrac{h}{2}k_{1n}u_n,
\end{aligned} \tag{28}
$$

where $A_0 = \int_a^{x_1} k(x_0,t)dt$.

$$
\begin{aligned}
u_1 =& f_1 + \int_a^{x_2} k(x_1,t)u(t)dt + \int_{x_2}^{b} k(x_1,t)u(t)dt \\
=& f_1 + \int_a^{x_2} k(x_1,t)\left[u(t)-u_1\right]dt + u_1\int_a^{x_2} k(x_1,t)dt + \int_{x_2}^{b} k(x_1,t)u(t)dt \\
=& f_1 + \tfrac{h}{2}k_{10}u_0 + \left(A_1 - \tfrac{h}{2}k_{10} - \tfrac{h}{2}k_{12}\right)u_1 + hk_{12}u_2 + \cdots + hk_{1,n-1} + \tfrac{h}{2}k_{1n}u_n,
\end{aligned} \tag{29}
$$

where, $A_1 = \int_a^{x_2} k(x_1,t)dt$.

$$
\begin{aligned}
u_2 =& f_2 + \int_a^{x_1} k(x_2,t)u(t)dt + \int_{x_1}^{x_3} k(x_2,t)u(t)dt + \int_{x_3}^{b} k(x_2,t)u(t)dt \\
=& f_2 + \int_a^{x_1} k(x_2,t)u(t)dt + \int_{x_1}^{x_3} k(x_2,t)\left[u(t)-u_2\right]dt + u_2\int_{x_1}^{x_3} k(x_2,t)dt + \int_{x_3}^{b} k(x_2,t)u(t)dt \\
=& f_2 + \tfrac{h}{2}k_{20}u_0 + hk_{21}u_1 + \left(A_2 - \tfrac{h}{2}k_{21} - \tfrac{h}{2}k_{23}\right)u_2 + hk_{23}u_3 + \cdots + hk_{2,n-1}u_{n-1} + \tfrac{h}{2}k_{2n}u_n,
\end{aligned} \tag{30}
$$

where

$$
A_2 = \int_{x_1}^{x_3} k(x_2,t)dt.
$$

Therefore, by continuing this procedure, we obtain:

$$
\begin{cases}
u_0 = f_0 + h\sum\limits_{j=1}^{n-1} k_{0j}u_j + \left(A_0 - \frac{h}{2}k_{01}\right)u_0 + \frac{h}{2}k_{0n}u_n, \\
u_i = f_i + \frac{h}{2}k_{i0}u_0 + h\sum\limits_{j=1, j\neq i}^{n-1} k_{ij}u_j + \left(A_i - \frac{h}{2}k_{i,i-1} - \frac{h}{2}k_{i,i+1}\right)u_i \\
\qquad + \frac{h}{2}k_{in}u_n \ , i = 1,2,\cdots,n-1, \\
u_n = f_n + \frac{h}{2}k_{n,0}u_0 + h\sum\limits_{j=1}^{n-1} k_{nj}u_j + \left(A_n - \frac{h}{2}k_{n,n-1}\right)u_n,
\end{cases} \tag{31}
$$

where

$$
\begin{cases}
A_0 = \int_a^{x_1} k(x_0,t)dt, \\
A_i = \int_{x_{i-1}}^{x_{i+1}} k(x_i,t)dt \ , i = 1,2,\cdots,n-1, \\
A_n = \int_{x_{n-1}}^{x_n} k(x_n,t)dt.
\end{cases} \tag{32}
$$

4.2 Repeated simpson modified quadrature method for fredholm integral equations

In this quadrature method, we choose y_i and z_i in a such way that:

$$
\begin{cases}
x_0 = a, \\
h = x_1 - x_0, \\
y_i = \begin{cases} a & i = 0 \\ x_{i-1} & i = 1,3,\cdots,2n-1 \\ x_{i-2} & i = 2,4,\cdots,2n \end{cases} , z_i = \begin{cases} x_{i+1} & i = 1,3,\cdots,2n-1 \\ x_{i+2} & i = 0,2,\cdots,2n-2 \\ b & i = 2n \end{cases}
\end{cases} \tag{33}
$$

As a result, the u_is calculated as follows:

$$\begin{aligned}u_0 =& f_0 + \int_a^{x_2} k(x_0,t)u(t)dt + \int_{x_2}^{b} k(x_0,t)u(t)dt\\ =& f_0 + \int_a^{x_2} k(x_0,t)\left[u(t)-u_0\right]dt + u_0\int_a^{x_2} k(x_0,t)dt + \int_{x_2}^{b} k(x_0,t)u(t)dt\\ =& f_0 + \left(A_0 - \tfrac{4h}{3}k_{01} - \tfrac{h}{3}k_{02}\right)u_0 + \tfrac{4h}{3}k_{01}u_1 + \tfrac{2h}{3}k_{02}u_2 + \cdots + \tfrac{4h}{3}k_{0,2n-1}u_{2n-1} + \tfrac{h}{3}k_{0,2n}u_{2n},\end{aligned} \tag{34}$$

where, $A_0 = \int_a^{x_2} k(x_0,t)dt$.

$$\begin{aligned}u_1 =& f_1 + \int_a^{x_2} k(x_1,t)u(t)dt + \int_{x_2}^{b} k(x_1,t)u(t)dt\\ =& f_1 + \int_a^{x_2} k(x_1,t)\left[u(t)-u_1\right]dt + u_1\int_a^{x_2} k(x_1,t)dt + \int_{x_2}^{b} k(x_1,t)u(t)dt\\ =& f_1 + \tfrac{h}{3}k_{10}u_0 + \left(A_1 - \tfrac{h}{3}k_{10} - \tfrac{h}{3}k_{12}\right)u_1 + \tfrac{2h}{3}k_{02}u_2 + \tfrac{4h}{3}k_{03}u_3 + \cdots + \tfrac{4h}{3}k_{1,2n-1}u_{2n-1} + \tfrac{h}{3}k_{1,2n}u_{2n},\end{aligned} \tag{35}$$

where, $A_1 = \int_a^{x_2} k(x_1,t)dt$.

$$\begin{aligned}u_2 =& f_2 + \int_a^{x_4} k(x_2,t)u(t)dt + \int_{x_4}^{b} k(x_2,t)u(t)dt\\ =& f_2 + \int_a^{x_4} k(x_2,t)\left[u(t)-u_2\right]dt + u_2\int_a^{x_4} k(x_2,t)u(t)dt + \int_{x_4}^{b} k(x_2,t)u(t)dt\\ =& f_2 + \tfrac{h}{3}k_{20}u_0 + \tfrac{4h}{3}k_{21}u_1 + \left(A_2 - \tfrac{h}{3}k_{20} - \tfrac{4h}{3}k_{21} - \tfrac{4h}{3}k_{23} - \tfrac{h}{3}k_{24}\right)u_2 + \tfrac{4h}{3}k_{23}u_3 + \tfrac{2h}{3}k_{24}u_4\\ &+ \cdots + \tfrac{4h}{3}k_{2,2n-1}u_{2n-1} + \tfrac{h}{3}k_{2,2n}u_{2n},\end{aligned} \tag{36}$$

where, $A_2 = \int_a^{x_4} k(x_2,t)dt$.

Hence, in general, we can write:

$$\begin{cases} u_0 = f_0 + \frac{h}{3}\sum_{j=1}^{n}\left(k_{0,2j-2}u_{2j-2} + 4k_{0,2j-1}u_{2j-1} + k_{0,2j}u_{2j}\right) - \frac{h}{3}k_{00}u_0 + \left(A_0 - \frac{4h}{3}k_{01} - \frac{h}{3}k_{02}\right)u_0,\\ u_{2i+1} = f_{2i+1} + \frac{h}{3}\sum_{j=1}^{n}\left(k_{2i+1,2j-2}u_{2j-2} + 4k_{2i+1,2j-1}u_{2j-1} + k_{2i+1,2j}u_{2j}\right)\\ \qquad - \frac{4h}{3}k_{2i+1,2i+1}u_{2i+1} + \left(A_{2i+1} - \frac{h}{3}k_{2i+1,2i} - \frac{h}{3}k_{2i+1,2i+2}\right)u_{2i+1}\ , i = 0,1,\cdots,n-1,\\ u_{2i} = f_{2i} + \frac{h}{3}\sum_{j=1}^{n}\left(k_{2i,2j-2}u_{2j-2} + 4k_{2i,2j-1}u_{2j-1} + k_{2i,2j}u_{2j}\right) - \frac{2h}{3}k_{2i,2i}u_{2i}\\ \qquad + \left(A_{2i} - \frac{h}{3}k_{2i,2i-2} - \frac{4h}{3}k_{2i,2i-1} - \frac{4h}{3}k_{2i,2i+1} - \frac{h}{3}k_{2i,2i+2}\right)u_{2i}\ , i = 1,2,\cdots,n-1,\\ u_{2n} = f_{2n} + \frac{h}{3}\sum_{j=1}^{n}\left(k_{2n,2j-2}u_{2j-2} + 4k_{2n,2j-1}u_{2j-1} + k_{2n,2j}u_{2j}\right) - \frac{h}{3}k_{2n,2n}u_{2n}\\ \qquad + \left(A_{2n} - \frac{h}{3}k_{2n,2n-2} - \frac{4h}{3}k_{2n,2n-1}\right)u_{2n}, \end{cases} \tag{37}$$

where,

$$\begin{cases} A_0 = \int_{x_0}^{x_2} k(x_0,t)dt,\\ A_{2i} = \int_{x_{2i-2}}^{x_{2i+2}} k(x_{2i},t)dt\ , i = 1,2,\cdots,n-1,\\ A_{2i+1} = \int_{x_{2i}}^{x_{2i+2}} k(x_{2i+1},t)dt\ , i = 0,1,\cdots,n-1,\\ A_{2n} = \int_{x_{2n-2}}^{x_{2n}} k(x_{2n},t)dt. \end{cases} \tag{38}$$

The above integrals are approximated by Simpson's formula.

5 Numerical examples

In this section, we clearly illustrate the repeated trapezoid modified quadrature methods that explained in Sections 3.1 and 4.1 considering the following two examples. We solve these examples by using MATLAB v6.5.

Example 1. Consider the following linear weakly singular Volterra integral equation:

$$u(x) = \sqrt{x} + \tfrac{1}{2}\pi x - \int_0^x \tfrac{1}{\sqrt{x-t}}u(t)dt\ ,\ 0 \le x \le 2. \quad (39)$$

We intend to solve this integral equation by using the repeated trapezoid modified quadrature method that we used for Volterra integral equations and that explained in Section 3.1. The result listed in the following table. The exact solution to this equation is $u(x) = \sqrt{x}$.

Table 1. The solution to Eq. (39) by using trapezoid modified quadrature method

Nodes	Exact Solution	Approximate Solution	Error
0	0	0	0
0.1	0.3162277660	0.3210296603	0.004801894
0.2	0.4472135955	0.4475586571	0.000345062
0.3	0.5477225575	0.5464449624	−0.001277595
0.4	0.6324555320	0.6303044526	−0.002151079
0.5	0.7071067812	0.7043987834	−0.002707998
0.6	0.7745966692	0.7714977196	−0.003098950
0.7	0.8366600265	0.8332689639	−0.003391063
0.8	0.8944271910	0.8908080916	−0.003619099
0.9	0.9486832980	0.9448803069	−0.003802991
1	1	0.9960449558	−0.003955044
1.1	1.0488088482	1.0447255507	−0.004083298
1.2	1.0954451150	1.0912518733	−0.004193242
1.3	1.1401754251	1.1358866608	−0.004288764
1.4	1.1832159566	1.1788432555	−0.004372701
1.5	1.2247448714	1.2202976983	−0.004447173
1.6	1.2649110641	1.2603972631	−0.004513801
1.7	1.3038404810	1.2992666349	−0.004573846
1.8	1.3416407865	1.3370124800	−0.004628306
1.9	1.3784048752	1.3737268930	−0.004677982
2.0	1.4142135624	1.4094900387	−0.004723524

Example 2. Consider the following linear weakly singular Fredholm integral equation:

$$u(x) = x + \int_0^1 \ln|x-t|\, u(t)dt\ ,\ 0 \le x \le 1. \quad (40)$$

We would solve this equation by using the repeated trapezoid modified quadrature method, which explained, in Section 4.1 for $n = 8, 16, 32, 64$. The numerical solutions at points $x = 0, \frac{1}{8}, \frac{2}{8}, \cdots, \frac{8}{8}$ given in the following table:

6 Conclusion

Usually, the weakly singular integral equations by using the quadrature methods such as the repeated trapezoid and the repeated Simpson quadrature methods are unsolvable, but the modified quadrature methods that explained in this paper can solve these kinds of integral equations. The method explained in Section 2 is better than the one explained in Section 1 because of the following reasons:

Table 2. The solution to Eq. (20) by using trapezoid modified quadrature method

Nodes	$n = 8$	$n = 16$	$n = 32$	$n = 64$	Error for
0	-0.063548	-0.064812	-0.065060	-0.065047	-0.000013
0.125	0.008864	0.008332	0.008208	0.008188	0.000020
0.25	0.067351	0.066180	0.065731	0.065543	0.000188
0.375	0.123611	0.121476	0.120552	0.120128	0.000424
0.5	0.183109	0.179469	0.177827	0.177054	0.000773
0.625	0.251528	0.245377	0.242570	0.241236	0.001334
0.75	0.338353	0.327431	0.322466	0.320109	0.002357
0.875	0.467942	0.444938	0.434770	0.430005	0.004765
1	0.608145	0.621313	0.629216	0.633750	-0.004534

1. For calculating A_i in each quadrature method we explained, we approximate the integral on an interval of the length $h, 2h, 3h$ or $4h$, but in the first method, we must approximate the integrals on an interval of the length (a, b) or (a, x). This increases the error of approximation in the first method.
2. For calculating the values of weights w_{ii}s in the first method, we must calculate $w_{ii} = (A_i - \sum_{j\neq i} w_{ij}k_{ij})$ that contains $(i + 1)$ statements in Volterra integral equations and $(n + 1)$ statements in Fredholm integral equations (see systems (3) and (4)). However, the principal advantage of our method is that we only need to calculate at most five statements. For instance, in the repeated Simpson modified quadrature method, for solving weakly singular Fredholm integral equations explained in Section 4.2, the weights w_{ii}s calculated as follows:

$$\begin{cases} w_{00} = A_0 - \frac{h}{3}(4k_{01} + k_{02}), \\ w_{2i,2i} = A_{2i} - \frac{h}{3}(k_{2i,2i-2} + 4k_{2i,2i-1} + 4k_{2i,2i+1} + k_{2i,2i+2})\,, i = 1, 2, \cdots, n-1, \\ w_{2i+1,2i+1} = A_{2i+1} - \frac{h}{3}(k_{2i+1,2i} + k_{2i+1,2i+2}\,), i = 0, 1, \cdots, n-1, \\ w_{2n,2n} = A_{2n} - \frac{h}{3}(k_{2n,2n-2} + 4k_{2n,2n-1}). \end{cases} \tag{41}$$

In the other methods that we explained before in Sections 3.1, 3.2, 3.3 and 4.1, the number of statements for calculating w_{ii} is less than the above and is independent of the values of n. Hence, number of calculation in our methods is less than the number of calculation in methods given in Eqs. (3) and (4).

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