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Synchronization of non-identical fractional order hyperchaotic systems using active control

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Abstract. The problem of synchronization between non-identical hyperchaotic fractional order systems is discussed in this article. We use the method of active control to synchronize the chaotic trajectories. To illustrate the proposed theory, we consider two particular examples of synchronization in fractional order hyperchaotic systems viz. Rossler system with new system and Chen system with Lü system. The active control terms are chosen properly so as the errors in synchronization become zero after small duration of time. It is observed that the synchronization time decreases with increase in fractional order.

Keywords: hyperchaotic systems, synchronization, active control, fractional order

1 Introduction

Fractional calculus (FC) is a branch of mathematics that deals with derivatives and integration of arbitrary order[1–3]. Although the subject is as old as the conventional calculus, it did not attract enough attention until recent decades. Nowadays FC is providing more realistic models in all the branches of applied sciences. The applications of FC can be found in control theory[4], viscoelasticity[5], diffusion[6–10], boundary layer effects in ducts[11], electromagnetic waves[12], signal processing[13, 14] and bio-engineering[15].

Chaotic dynamics of fractional order nonlinear dynamical systems is studied by many researchers. It is observed that unlike in integer order case, the fractional systems with dimension less than three can exhibit chaos. Grigorenko and Grigorenko investigated fractional ordered Lorenz system and observed that below a threshold order the chaos disappears[16]. Further Chen system[17], Lü system[18], Rossler system[19], Liu system[20] of fractional order are investigated.

In the pioneering work Pecora and Carroll[21] have shown that the chaotic systems can be synchronized by introducing appropriate coupling. The notion of synchronization of chaos has further been explored in secure communications of analog and digital signals[22] and for developing safe and reliable cryptographic systems[23]. For the synchronization of chaotic systems, a variety of approaches have been proposed which include nonlinear feed-back control[24], adaptive control[25–28], back-stepping design [29, 30] and active control[31–34].

The first attempt to study the synchronization in fractional order systems was by Deng and Li[35] (fractional Lü system). Li and Deng have summarized the theory and techniques of synchronization in [36].

Synchronization in hyperchaotic fractional order system is discussed by various researchers. Zang et al.[38] studied the nonautonomous chen's system for this purpose. The Laplace transformation theory and variational iteration method is used by Yu and Li[37] to analyze Rossler system. The activation feedback control is utilized by Wang and Song[39] to synchronize hyperchaotic Lorenz system. Wu and coworkers discussed various methods such as the observer method[40], generalized projective synchronization[41] and modified

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generalized projective synchronization[42] to synchronize hyperchaotic fractional order systems. Feedback control is used in [43–46] to synchronize such systems.

In most of the examples discussed above, the systems under consideration were identical. In this article, we use the method of active control[34] to synchronize nonidentical fractional order hyperchaotic systems. The classical stability analysis of fractional order system is used to choose control terms. The error terms in the synchronization become zero within a very short time.

The paper is organized as follows: In Section 2, we have proposed the method of active control for hyperchaotic fractional order systems. Section 3 deals with the illustrative examples. The synchronization between Xin-Ling and Rossler systems is discussed in Section 3.1 whereas the synchronization between Lü and Chen systems is discussed in Section 3.2. The conclusions are summarized in Section 4.

2 Synchronization technique

Consider the autonomous system

$$D^{\alpha}x = F(x), \tag{1}$$

where F is function from R^n to R^n, $(n > 3)$, $0 < \alpha \leq 1$ is fractional order and D^{α} is Caputo fractional derivative[1].

The system (1) will be treated as the drive (master) system and the response (slave) system is given by

$$D^{\alpha}y = G(y) + u. \tag{2}$$

The column vector u in (2) is control term to be chosen.

Let us write

$$\begin{aligned} F(x) &= L_1(x) + N_1(x), & (3)\\ G(y) &= L_2(y) + N_2(y), & (4) \end{aligned}$$

where L_i and N_i are linear and nonlinear functions of states respectively. Defining error vector by $e = y - x$ and subtracting (1) from (2) we get

$$\begin{aligned} D^{\alpha}e &= L_2(y) - L_1(x) + N_2(y) - N_1(x) + u \\ &= L_2(e) + (L_2 - L_1)(x) + N_2(y) - N_1(x) + u. \end{aligned} \tag{5}$$

We choose active control term as

$$u = V - (L_2 - L_1)(x) - N_2(y) + N_1(x), \tag{6}$$

where $V = Ae$ is linear function of e defined using $n \times n$ real matrix A. Error Eq. (5) can now be written as

$$D^{\alpha}e = (L_2 + A)(e). \tag{7}$$

By choosing appropriate A in (7), the error vector $e(t)$ can approach to zero as t approaches to infinity. The stability condition is [47]

$$|arg(\lambda)| > \alpha\pi/2, \tag{8}$$

for all eigenvalues λ of $L_2 + A$. Thus the synchronization is achieved.

3 Synchronization between different fractional order hyperchaotic systems

3.1 Synchronization between hyperchaotic fractional xin-ling and rossler systems

As a drive system we consider a fractional order hyperchaotic system proposed by Xin and Ling[48].

$$\begin{cases} D^\alpha x_1 = 10(y_1 - x_1) + w_1 \\ D^\alpha y_1 = 40x_1 + x_1 z_1 - w_1 \\ D^\alpha z_1 = -2.5z_1 - 4x_1^2 \\ D^\alpha w_1 = 2.5x_1 \end{cases} \tag{9}$$

where $\alpha \in (0,1)$ is fractional order. Further we consider fractional order version of the Rossler hyperchaotic system[49] as a response system.

$$\begin{cases} D^\alpha x_2 = -y_2 - z_2 + u_1 \\ D^\alpha y_2 = x_2 + 0.25y_2 + w_2 + u_2 \\ D^\alpha z_2 = 3 + x_2 z_2 + u_3 \\ D^\alpha w_2 = -0.5z_2 + 0.05w_2 + u_4 \end{cases} \tag{10}$$

We define error terms as $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$ and $e_4 = w_2 - w_1$. Subtracting (9) from (10) we get the error system as

$$\begin{cases} D^\alpha e_1 = -e_2 - e_3 - 11y_1 - z_1 + 10x_1 - w_1 + u_1 \\ D^\alpha e_2 = e_1 + 0.25e_2 + e_4 - 39x_1 + 0.25y_1 + 2w_1 - x_1 z_1 + u_2 \\ D^\alpha e_3 = 3 + e_1 e_3 + e_1 z_1 + x_1 e_3 + x_1 z_1 + 2.5z_1 + 4x_1^2 + u_3 \\ D^\alpha e_4 = -0.5e_3 + 0.05e_4 - 2.5x_1 - 0.5z_1 + 0.05w_1 + u_4 \end{cases} \tag{11}$$

The active control terms are defined by

$$\begin{cases} u_1 = V_1 + 11y_1 + z_1 - 10x_1 + w_1 \\ u_2 = V_2 + 39x_1 - 0.25y_1 - 2w_1 + x_1 z_1 \\ u_3 = V_3 - 3 - e_1 e_3 - e_1 z_1 - x_1 e_3 - x_1 z_1 - 2.5z_1 - 4x_1^2 \\ u_4 = V_4 + 2.5x_1 + 0.5z_1 - 0.05w_1 \end{cases} \tag{12}$$

where the linear functions V_i's are given by $V_1 = -e_1$, $V_2 = -e_1 - 1.025e_2$, $V_3 = -e_3$ and $V_4 = 0.5e_3 - 1.05e_4$. Using the values of (12) in (11) we get

$$D^\alpha e = Be, \tag{13}$$

where

$$B = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{14}$$

It can be verified that the error system (13) is asymptotically stable and hence the synchronization is achieved.

3.1.1 Numerical simulations

The initial conditions for drive and response system are taken as $(1,2,3,4)$ and $(10,11,12,13)$ respectively. Thus the initial values for error system are $(9,9,9,9)$. It is clear form the Figs. 1,2 and 3 that the chaotic trajectories of the systems get synchronized after a small time duration for the fractional derivatives $\alpha = 0.99$, $\alpha = 0.97$ and $\alpha = 0.95$. Trajectories of drive system are shown by solid lines whereas the trajectories of response system are dashed lines. The errors in synchronization are e_1 (solid lines), e_2 (dashed lines), e_3 (dot-dashed lines) and e_4 (dotted lines). It is also observed from the figures that the time taken for the synchronization of the systems decreases with increase in fractional order α.

3.2 Synchronization of hyperchaotic fractional lü and chen systems

As a drive systgen, consider the fractional order hyperchaotic Lü system[50]

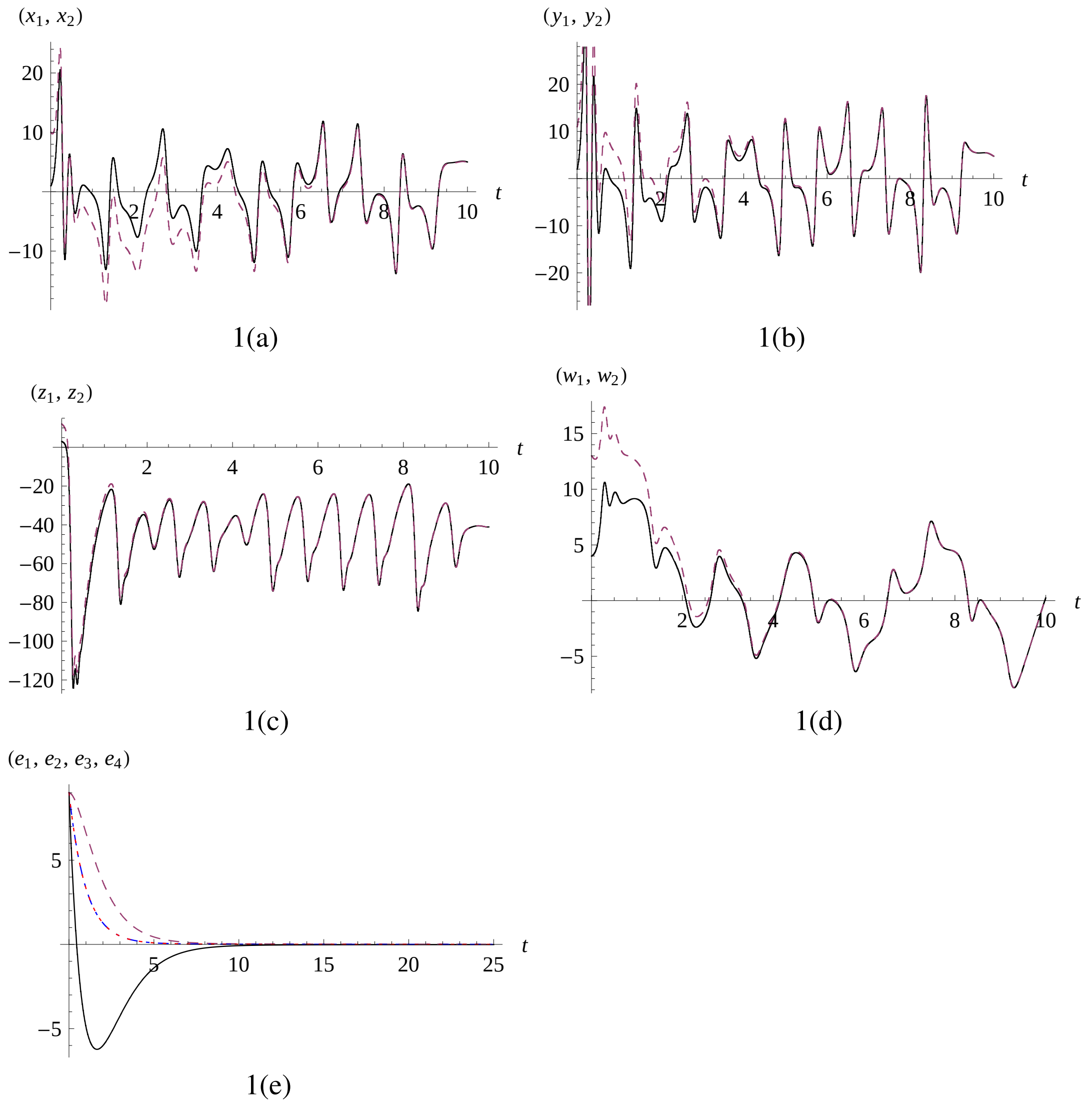


Fig. 1. Synchronization between fractional order new and Rossler systems for $\alpha = 0.99$: (a) Trajectories (x_1, x_2), (b) Trajectories (y_1, y_2), (c) Trajectories (z_1, z_2), (d) Trajectories (w_1, w_2), (e) The synchronization errors

$$\begin{cases} D^{\alpha}x_1 = 36(y_1 - x_1) + w_1 \\ D^{\alpha}y_1 = 20y_1 - x_1z_1 \\ D^{\alpha}z_1 = -3z_1 + x_1y_1\ D^{\alpha}w_1 = x_1z_1 \end{cases} \tag{15}$$

The response system is taken as a fractional order hyperchaotic Chen system[51]

$$\begin{cases} D^{\alpha}x_2 = 35(y_2 - x_2) + w_2 + u_1 \\ D^{\alpha}y_2 = 7x_2 + 12y_2 - x_2z_2 + u_2 \\ D^{\alpha}z_2 = -3z_2 + x_2y_2 + u_3 \\ D^{\alpha}w_2 = 0.5w_2 + y_2z_2 + u_4 \end{cases} \tag{16}$$

where u_i are defined by (6). With the notations discussed in Section 2

$$L_1 = \begin{pmatrix} -36 & 36 & 0 & 1 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

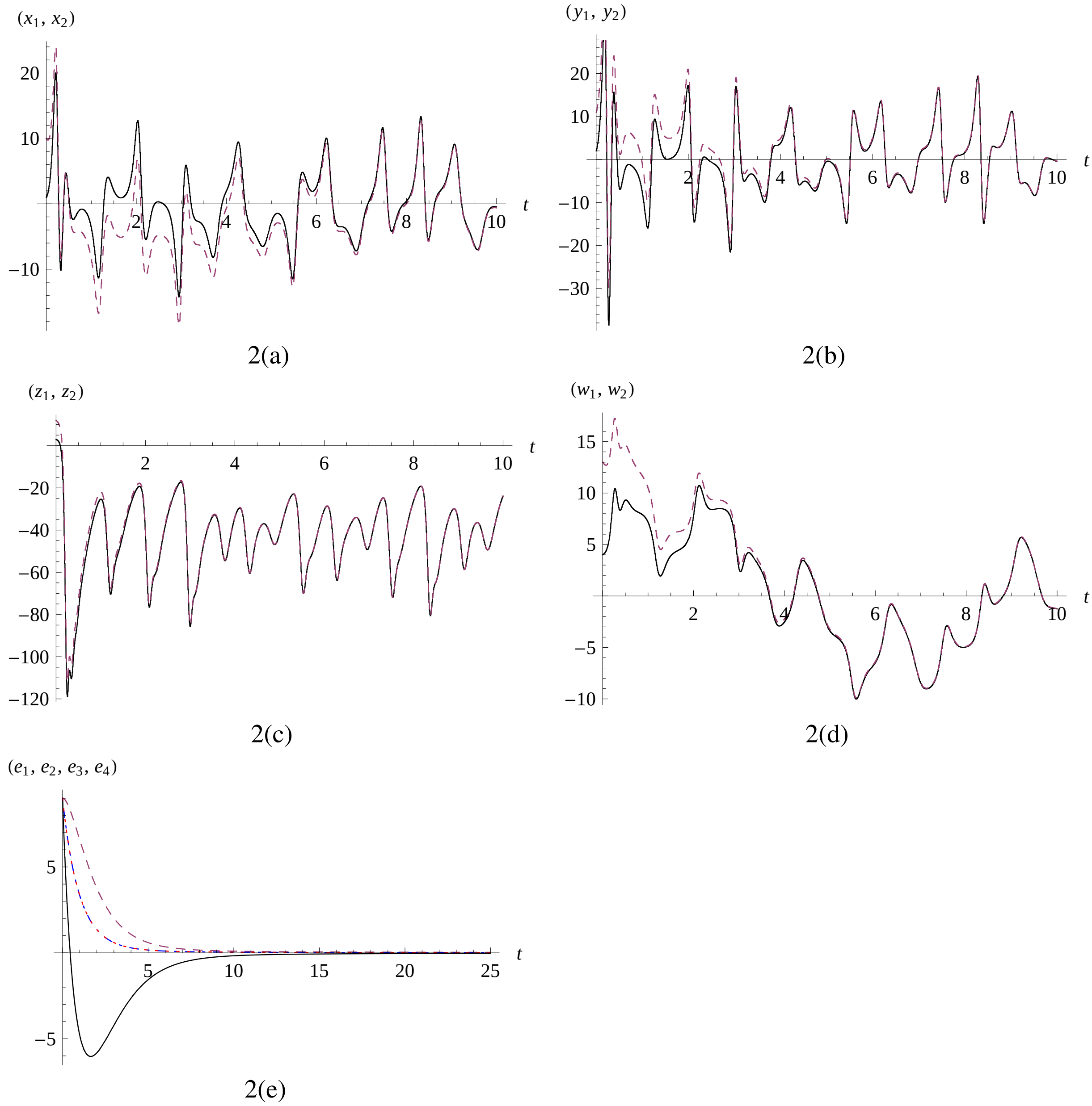


Fig. 2. Synchronization between fractional order new and Rossler systems for $\alpha = 0.97$: (a) Trajectories (x_1, x_2), (b) Trajectories (y_1, y_2), (c) Trajectories (z_1, z_2), (d) Trajectories (w_1, w_2), (e) The synchronization errors

$$L_2 = \begin{pmatrix} -35 & 35 & 0 & 1 \\ 7 & 12 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}, \tag{18}$$

$N_1(x_1, y_1, z_1, w_1) = (0, -x_1z_1, x_1y_1, x_1z_1)^T$ and $N_2(x_1, y_1, z_1, w_1) = (0, -x_2z_2, x_2y_2, y_2z_2)^T$.

The matrix A is chosen as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -7 & -13 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

For this choice of A the eigenvalues of $L_2 + A$ satisfy (8).

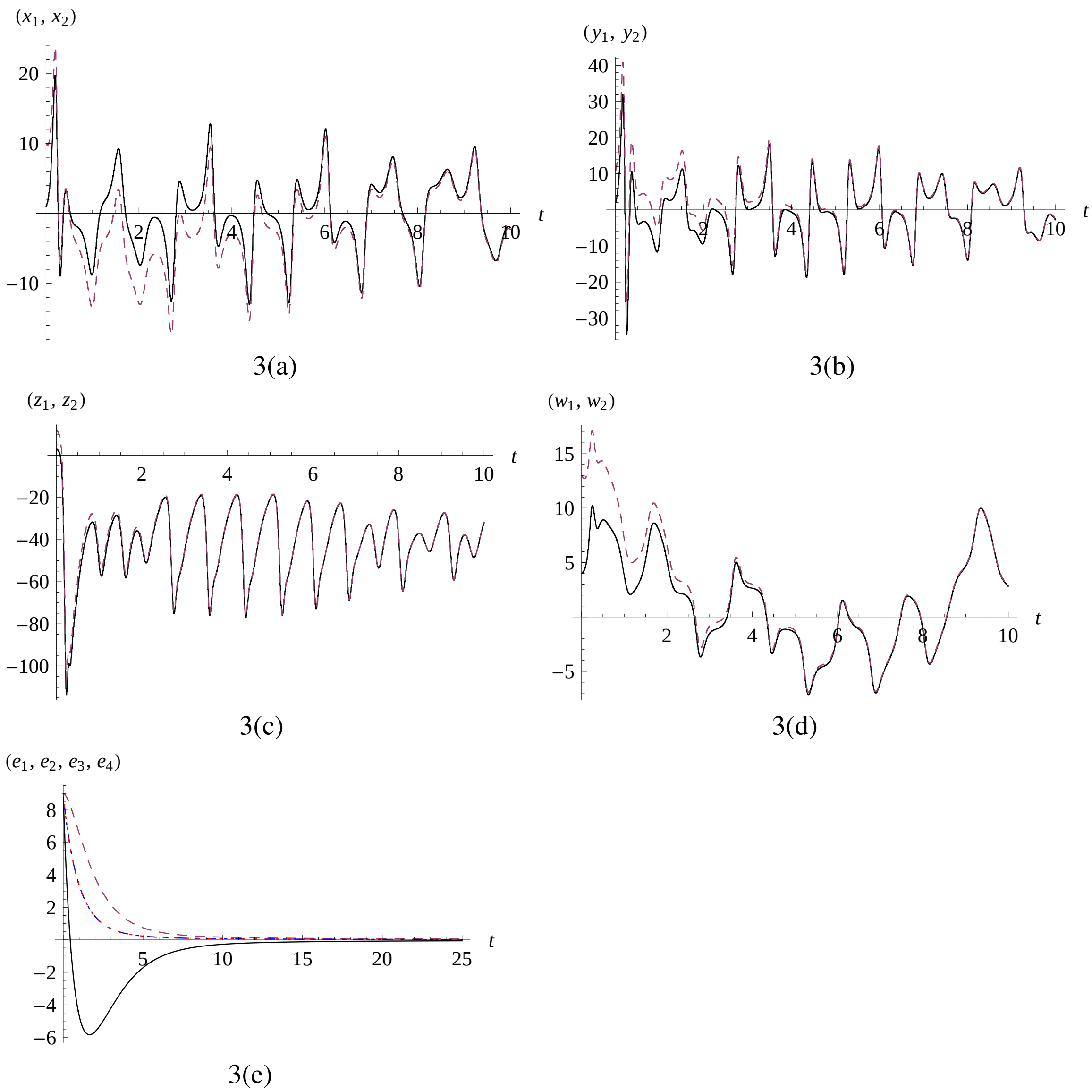


Fig. 3. Synchronization between fractional order new and Rossler systems for $\alpha = 0.95$: (a) Trajectories (x_1, x_2), (b) Trajectories (y_1, y_2), (c) Trajectories (z_1, z_2), (d) Trajectories (w_1, w_2), (e) The synchronization errors

The synchronization errors are defined by $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$, $e_4 = w_2 - w_1$, $e = (e_1, e_2, e_3, e_4)$. Using (7) we have

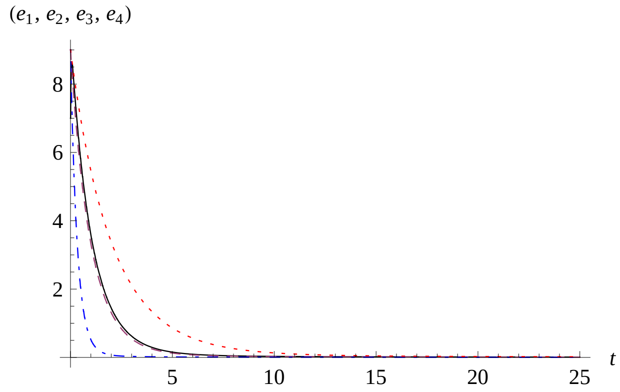
$$D^\alpha e = Be, \tag{20}$$

where

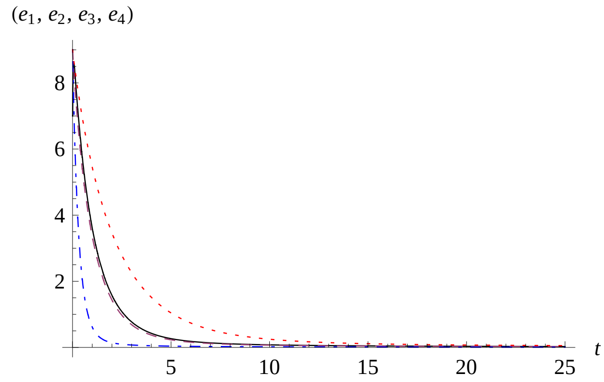
$$B = \begin{pmatrix} -35 & 35 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -0.5 \end{pmatrix}. \tag{21}$$

3.2.1 Numerical simulations

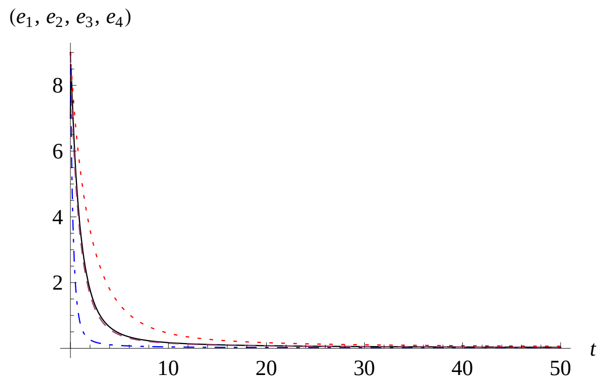
The initial conditions for drive and response system are taken as $(2, 2, 3, 4)$ and $(9, 11, 12, 13)$ respectively. The initial error are thus $(7, 9, 9, 9)$. We plot the synchronization errors e_1 (solid lines), e_2 (dashed lines), e_3 (dot-dashed lines) and e_4 (dotted lines) for $\alpha = 0.98$, $\alpha = 0.95$ and $\alpha = 0.90$ in Figs. 4. It is observed that the time taken for the synchronization of the systems decreases with increase in fractional order α.



4(a)



4(b)



4(c)

Fig. 4. Errors in the synchronization between fractional order Lü and Chen hyperchaotic systems for: (a) $\alpha = 0.98$, (b) $\alpha = 0.95$, (c) $\alpha = 0.90$

4 Conclusions

In this article, we have successfully utilized a method of active control and synchronized fractional order hyperchaotic systems. It is shown first time in the literature that the nonidentical hyperchaotic systems can also be synchronized using active control. The relation between fractional order of the systems and the time taken for the synchronization is also discussed. It is observed from Figs. 1(e), 2(e), 3(e) and 4 that the synchronization time decreases with increase in fractional order $\alpha \in (0, 1]$.

References

[1] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[2] S. G. Samko, A. A. Kilbas, O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Yverdon, 1993.

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[4] J. Sabatier, S. Poullain, P. Latteux, J. Thomas and A. Oustaloup. Robust speed control of a low damped electromechanical system based on CRONE control: Application to a four mass experimental test bench. *Nonlinear Dynamics*, 2004, **38**: 383–400.

[5] M. Caputo, F. Mainardi. A new dissipation model based on memory mechanism. *Pure Appl. Geophys.*, 1971, **91**: 134–147.

[6] F. Mainardi, Y. Luchko, G. Pagnini. The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.*, 2001, **4**(2): 153–192.

[7] O. P. Agrawal. Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynamics*, 2002, **29**: 145–155.

[8] V. Daftardar-Gejji, S. Bhalekar. Solving multi-term linear and non-linear diffusion-wave equations of fractional order by Adomian decomposition method. *Appl. Math. Comput.*, 2008, **202**: 113–120.
[9] H. G. Sun, W. Chen, Y. Q. Chen. Variable–order fractional differential operators in anomalous diffusion modeling. *Physica A*, 2009, **388**: 4586–4592.
[10] I. S. Jesus, J. A. T. Machado. Fractional control of heat diffusion systems. *Nonlinear Dynamics*, 2008, **54**(3): 263–282.
[11] N. Sugimoto. Burgers equation with a fractional derivative: Hereditary effects on nonlinear acoustic waves. *J. Fluid Mech.*, 1991, **225**: 631-653.
[12] O. Heaviside. *Electromagnetic Theory*. Chelsea, New York, 1971
[13] T. J. Anastasio. The fractional-order dynamics of Brainstem Vestibulo–Oculomotor neurons. *Biol. Cybernet.*, 1994, **72**: 69–79.
[14] M. D. Ortigueira, J. A. T. Machado. Fractional calculus applications in signals and systems. *Signal Processing*, 2006, **86**(10): 2503–2504.
[15] R. L. Magin. *Fractional Calculus in Bioengineering*. Begll House Publishers, USA, 2006.
[16] I. Grigorenko, E. Grigorenko. Chaotic dynamics of the fractional Lorenz system. *Phys. Rev. Lett.*, 2003, **91**(3): 034101.
[17] C. Li, G. Peng. Chaos in Chen's system with a fractional order. *Chaos Solitons & Fractals*, 2004, **22**: 443–450.
[18] J. G. Lü. Chaotic dynamics of the fractional order Lü system and its synchronization. *Phys. Lett. A*, 2006, **354**(4): 305–311.
[19] C. Li, G. Chen. Chaos and hyperchaos in the fractional order Rossler equations. *Phys. A: Stat. Mech. Appl.*, 2004, **341**: 55–61.
[20] V. Daftardar-Gejji, S. Bhalekar. Chaos in fractional ordered Liu system. *Comput. Math. Appl.*, 2010, **59**: 1117–1127.
[21] L. M. Pecora, T. L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.*, 1990, **64**(8): 821.
[22] R. Hilfer. *Applications of Fractional Calculus in Physics*. World Scientific, USA, 2001.
[23] R. He, P. G. Vaidya. Implementation of chaotic cryptography with chaotic synchronization. *Phys. Rev. E*, 1998, **57**(2): 1532.
[24] L. Huang, R. Feng, M. Wang. Synchronization of chaotic systems via nonlinear control. *Phys. Lett. A*, 2004, **320**: 271.
[25] T. L. Liao. Adaptive synchronization of two Lorenz systems. *Chaos, Solitons & Fractals*, 1998, **9**: 1555.
[26] M. T. Yassen. Adaptive control and synchronization of a modified Chua's circuit system. *Appl. Math. Comput.*, 2001, **135**(1): 113.
[27] Y. Wang, Z. Guan, X. Wen. Adaptive synchronization for Chen chaotic system with fully unknown parameters. *Chaos Solitons & Fractals*, 2004, **19**: 899–903.
[28] J. H. Park. Adaptive synchronization of Rossler system with uncertain parameters. *Chaos Solitons & Fractals*, 2005, **25**: 333–338.
[29] S. Bowong, F. Moukam Kakmeni. Synchronization of uncertain chaotic systems via backstepping approach. *Chaos Solitons & Fractals*, 2004, **21**: 999–1011.
[30] J. Zhang, C. Li, H. Zhang, J. Yu. Chaos synchronization using single variable feedback based on backstepping method. *Chaos Solitons & Fractals*, 2004, **21**: 1183–1193.
[31] E. W. Bai, K. E. Lonngren. Synchronization of two Lorenz systems using active control. *Chaos Solitons & Fractals*, 1997, **8**: 51–58.
[32] M. C. Ho, Y. C. Hung. Synchronization of two different systems by using generalized active control. *Phys. Lett. A*, 2002, **301**: 424.
[33] M. T. Yassen. Chaos synchronization between two different chaotic systems using active control. *Chaos Solitons & Fractals*, 2005, **23**: 131–140.
[34] S. Bhalekar, V. Daftardar-Gejji. Synchronization of different fractional order chaotic systems using active control. *Commun. Nonlinear Sci. Numer. Simulat.*, 2010, **15**(11): 3536–3546
[35] W. H. Deng, C. P. Li. Chaos synchronization of the fractional Lü system. *Physica A*, 2005, **353**: 61–72.
[36] C. P. Li, W. H. Deng. Chaos synchronization of fractional order differential system. *Int. J. Modern Phys. B*, 2006, **20**(7): 791–803.
[37] Y. Yu, H. Li. The synchronization of fractional-order Rossler hyperchaotic systems. *Physica A*, 2008, **387**: 1393–1403.
[38] H. Zhang, C. Li, G. Chen, X. Gao. Hyperchaos in the fractional-order nonautonomous chen's system and its synchronization. *Int. J. Mod. Phys. C*, 2005, **16**: 815–826.
[39] X. Wang, J. Song. Synchronization of the fractional order hyperchaos Lorenz systems with activation feedback control. *Commun. Nonlinear Sci. Numer. Simulat.*, 2009, **14(8)**: 3351-3357.

[40] X. Wu, H. Lu, S. Shen. Synchronization of a new fractional-order hyperchaotic system. *Phys. Lett. A*, 2009, **373**: 2329-2337.

[41] X. Wu, Y. Lu. Generalized projective synchronization of the fractional-order Chen hyperchaotic system. *Nonlinear Dynamics*, 2009, **57**: 25–35.

[42] X. Wu, H. Wang, H. Lu. Modified generalized projective synchronization of a new fractional-order hyperchaotic system and its application to secure communication. *Nonlinear Analysis: Real World Applications*, 2012, **13**: 1441-1450.

[43] Z. Ping, W. Li-Jia, C. Xue-Feng. A novel fractional-order hyperchaotic system and its synchronization. *Chinese Physics B*, 2009, **18**: 2674–2679.

[44] H. Deng, T. Li, Q. Wang, H. Li. A fractional-order hyperchaotic system and its synchronization. *Chaos Solitons & Fractals*, 2009, **41**: 962-969.

[45] L. Pan, W. Zhou, L. Zhou, K. Sun. Chaos synchronization between two different fractional-order hyperchaotic systems. *Commun. Nonlinear Sci. Numer. Simulat.*, 2011, **16**: 2628-2640.

[46] A. S. Hegazi, A. E. Matouk. Dynamical behaviors and synchronization in the fractional order hyperchaotic Chen system. *Applied Mathematics Letters* 2011, **24**: 1938-1944.

[47] D. Matignon. Stability results for fractional differential equations with applications to control processing. **in:** *Computational Engineering in Systems and Application multiconference*, IMACS, IEEE-SMC proceedings, Lille, France, July, **2**: 1996, 963–968.

[48] L. Chong-Xin, L. Ling. Circuit implementation of a new hyperchaos in fractional-order system. *Chin. Phys. B*, 2008, **17**(8): 2829–2836.

[49] O. E. Rossler. An equation for hyperchaos. *Phys. Lett. A*, 1979, **7**; 155.

[50] A. Chen, J. Lu, J. Lü, S. Yu. Generating hyperchaotic Lü attractor via state feedback control. **Physica A**, 2006, **364**: 103–110.

[51] Y. Li, W.K.S. Tang, G. Chen. Generating hyperchaos via state feedback control. *Int. J. Bifurcation Chaos*, 2005, **15**(10): 3367–3375.